Math 4181H **Solutions to Homework 9**

10pt **A1.** *Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that f is infinitely differentiable on $\mathbb{R}$ with $f^{(n)}(0) = 0$ for all n.*

Solution. First, let's prove that for any polynomial p, $\lim_{x\to 0} e^{-1/x^2} p(1/x) = 0$. It suffices to prove that $\lim_{x\to 0} e^{-1/x^2}(1/x)^n = 0$ for all $n \in \mathbb{N}$; and indeed, $\lim_{x\to 0} e^{-1/x^2}(1/x)^n = \lim_{y\to\infty} \frac{y^n}{e^{y^2}} = 0$ since $\lim_{y\to+\infty} \frac{y^n}{e^y} = 0$ and $e^{y^2} > e^{|y|}$ for $y > 1$.

Now let's prove that for every $n = 0, 1, 2, \ldots$, on $\mathbb{R}\setminus\{0\}$ we have $f^{(n)}(x) = e^{-1/x^2} p_n(1/x)$ for a polynomial p_n. Indeed, this is true for $n = 0$, and if this is true for some n then

$$f^{(n+1)}(x) = e^{-1/x^2} \tfrac{2}{x^3} p_n(1/x) - e^{-1/x^2} p_n'(1/x) \tfrac{1}{x^2} = e^{-1/x^2} p_{n+1}(x)$$

where $p_{n+1}(x) = p_n(1/x) \cdot (2/x^3) - p_n'(1/x) \cdot (1/x^2)$.

Now if, by induction, $f^{(n)}(0) = 0$ for some n, then

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{e^{-1/x^2} p_n(1/x)}{x} = \lim_{x\to 0} e^{-1/x^2} p_n(1/x) \cdot (1/x) = 0.$$

Chapter 23, pp. 489-498:

10pt **2.** *Prove that the series $\sum a^n n!/n^n$ converges for $0 < a < e$ and diverges for $a > e$.*

Solution. To use the ratio test, we compute

$$\frac{a^{n+1}(n+1)!/(n+1)^{n+1}}{a^n n!/n^n} = \frac{a(n+1)}{(n+1)^{n+1}/n^n} = \frac{a}{(n+1)^n/n^n} = \frac{a}{(1+1/n)^n},$$

which converges to a/e as $n \longrightarrow \infty$. Thus by the ratio test, the series converges if $a/e < 1$ and diverges if $a/e > 1$.

5pt **5.** (a) *Prove that if the series $\sum x_i$ converges absolutely, then so does $\sum x_i^3$.*

Solution. Since $\sum x_i$ converges, $x_i \longrightarrow 0$ as $i \longrightarrow \infty$, so $|x_i|^3 < |x_i|$ for all n large enough, and since $\sum |x_i|$ converges, $\sum |x_i|^3$ converges by the comparison test.

10pt (b) *Show that the series $\sum_{i=1}^\infty x_i = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}} + \cdots$ converges, but $\sum x_i^3$ diverges.*

Solution. We have $x_i \longrightarrow 0$, and the grouping $\big(1 - \frac{1}{2} - \frac{1}{2}\big) + \big(\frac{1}{\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}}\big) + \big(\frac{1}{\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}}\big) + \cdots = 0 + 0 + 0 + \cdots$ of $\sum x_i$, with bounded size of groups, converges, so the series $\sum x_i$ converges.

Now, $\sum x_i^3 = 1 - \frac{1}{8} - \frac{1}{8} + \frac{1}{2} - \frac{1}{8}\cdot\frac{1}{2} - \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{3} - \frac{1}{8}\cdot\frac{1}{3} - \frac{1}{8}\cdot\frac{1}{3} + \cdots$, its grouping

$$\big(1 - \tfrac{1}{8} - \tfrac{1}{8}\big) + \big(\tfrac{1}{2} - \tfrac{1}{8}\cdot\tfrac{1}{2} - \tfrac{1}{8}\cdot\tfrac{1}{2}\big) + \big(\tfrac{1}{3} - \tfrac{1}{8}\cdot\tfrac{1}{3} - \tfrac{1}{8}\cdot\tfrac{1}{3}\big) + \cdots = \tfrac{3}{4}\cdot 1 + \tfrac{3}{4}\cdot\tfrac{1}{2} + \tfrac{3}{4}\cdot\tfrac{1}{3} + \cdots = \tfrac{3}{4}\big(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots\big)$$

diverges, so $\sum x_i^3$ also diverges.

10pt **A2.** (a) *Let $f\colon [1, +\infty) \longrightarrow \mathbb{R}$ be a decreasing nonnegiative function. For every $i \in \mathbb{N}$, let $a_i = f(i)$. Prove that a finite limit $l = \lim_{n\to\infty}\big(\sum_{i=1}^n a_i - \int_1^n f\big)$ exists and satisfies $0 \leq l \leq a_1$.*

Solution. For any i, $a_i \geq f(x) \geq a_{i+1}$ for any $x \in [a_i, a_{i+1}]$, so $a_i = a_i \cdot 1 \geq \int_i^{i+1} f \geq a_{i+1} \cdot 1 = a_{i+1}$. Put $\gamma_n = \sum_{i=1}^n a_i - \int_1^n f$, $n \in \mathbb{N}$. For any n, $\gamma_{n+1} = \gamma_n + a_{n+1} - \int_n^{n+1} f \leq \gamma_n$, so the sequence (γ_n) is decreasing. Also for any n, $\gamma_n = \sum_{i=1}^{n-1}\big(a_i - \int_i^{i+1} f\big) + a_n \geq 0$, so $l = \lim \gamma_n \geq 0$ exists. And since $\gamma_1 = a_1$, $l \leq a_1$.

5pt (b) *Prove that a finite limit $\gamma = \lim_{n\to\infty}\big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\big)$ exists. (This $\gamma = 0.5772\ldots$ is called Euler-Mascheroni constant.)*

Solution. For the decreasing function $f(x) = 1/x$, $x \geq 1$, we have $a_i = f(i) = \frac{1}{i}$, $i \in \mathbb{N}$, and $\int_1^n f = \log n$, $n \in \mathbb{N}$. So by (a), $\gamma = \lim_{n\to\infty}\big(\sum_{i=1}^n \frac{1}{i} - \log n\big)$ exists, and is $\leq a_1 = 1$.