Math 4181H **Solutions to Midterm 1**

10% **1.** *If $a, b \in \mathbb{R}$ and $b < a + 1/n$ for all $n \in \mathbb{N}$, prove that $b \leq a$.*

Solution. Proof by contraposition: Suppose $b > a$, then $b - a > 0$, so, by a property of natural numbers, there exists $n \in \mathbb{N}$ such that $1/n < b - a$, which means that $b > a + 1/n$ for this n.

10% **2.** (a) *Let A and B be nonempty sets of real numbers such that $a < b$ for all $a \in A$ and all $b \in B$. Prove that there is $x \in \mathbb{R}$ such that $a \leq x \leq b$ for all $a \in A$ and all $b \in B$.*

Solution. A is bounded above (by any element of B). So, $x = \sup A$ exists (by the axiom of completeness). By definition, we have $x \geq a$ for all $a \in A$, and for every $b \in B$, since b is an upper boumd of A, $x \leq b$.

10% (b) *Give an example of nonepty sets A and B of rational numbers such that $a < b$ for all $a \in A$ and all $b \in B$ and there are no $x \in \mathbb{Q}$ such that $a \leq x \leq b$ for all $a \in A$ and all $b \in B$. (And justify your answer, of course.)*

Solution. Put $A = \{a \in \mathbb{Q} : a < \sqrt{2}\}$ and $B = \{b \in \mathbb{Q} : b > \sqrt{2}\}$, then $b > a$ for all $a \in A$ and $b \in B$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\sup A = \inf B = \sqrt{2}$, so $x = \sqrt{2}$ is the only real number satisfying $a \leq x \leq b$ for all $a \in A$ and $b \in B$. However, $\sqrt{2} \notin \mathbb{Q}$, so there is no $x \in \mathbb{Q}$ with this property.

15% **3.** *Let $A \subseteq \mathbb{R}$ be bounded above and have no maximal element. Prove that $\sup A$ is a limit point of A.*

Solution. Let $b = \sup A$. Since A has no maximal element, $b \notin A$. For every $\varepsilon > 0$ there is $a \in A$ such that $a > b - \varepsilon$, and $a \leq b$ by definition, so $|a - b| < \varepsilon$ with $a \neq b$. Hence, b is a limit point of A.

15% **4.** *Let $a \in \mathbb{R} \setminus \{0\}$ and let $\varepsilon > 0$. If $|x - a| < \min\{|a|, \varepsilon/(3|a|)\}$, prove that $|x^2 - a^2| < \varepsilon$.*

Solution. If $|x - a| < |a|$, then $|x + a| \leq |x - a| + |2a| < 3|a|$; if also $|x - a| < \varepsilon/(3|a|)$, then $|x^2 - a^2| = |x - a| \cdot |x + a| < \big(\varepsilon/(3|a|)\big)(3|a|) = \varepsilon$.

15% **5.** *Let $a > 1$, let $n, m \in \mathbb{N}$, $n < m$, and assume that $b = \sqrt[n]{a}$ and $c = \sqrt[m]{a}$ exist (so that $b^n = c^m = a$). Prove that $b > c$.*

Solution. First of all, if $x > y > 0$ then $x^n > y^n$ for all $n \in \mathbb{N}$ by induction. We now have $c > 1$, since otherwise $c^m \leq 1^m = 1 < a$, and, since $m - n \in \mathbb{N}$, also $c^{m-n} > 1$. By the way of contradiction, assume that $c \geq b$; then $a = c^m = c^n c^{m-n} \geq b^n c^{m-n} > a \cdot 1 = a$.

10% **6.** *Prove that for every even $n \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and an odd $m \in \mathbb{N}$ such that $n = 2^r m$.*

Solution. I'll use complete induction. For $n = 2$ the statement is true, $n = 2^1 \cdot 1$. Suppose $n \in \mathbb{N}$ is even and the statement holds for all even natural numbers less than n. n is even, so $n = 2k$ for some $k \in \mathbb{Z}$, and then $1 \leq k < n$. If k is odd, we are done, $n = 2^1 k$. If k is even, by induction hypothesis $k = 2^r m$ for some $r \in \mathbb{N}$ and an odd $m \in \mathbb{N}$, and so $n = 2^{r+1} m$.

Another solution. Let $n \in \mathbb{N}$ be even, let $S = \big\{r \in \mathbb{N}"2^r \bigm| n\big\}$. $S \neq \emptyset$ since $2^1 \bigm| n$. S is bounded above since there is $r_0 \in \mathbb{N}$ such that $2^{r_0} > n$, then $2^r > n$ for all $r \geq r_0$, and then $2^r \nmid n$ for all $r \geq r_0$. Hence, S has a maximal element r; let $m = n/2^r$, then $m \in \mathbb{N}$. If m is even, then $m = 2k$ for some $k \in \mathbb{N}$, so $n = 2^r m = 2^{r+1} k$, so $2^{r+1} \bigm| n$, which contradicts the choice of r. Hence, m is odd, and $n = 2^r m$.

10% **7.** (a) *Prove that the set of transcendental (that is, non-algebraic) numbers is dense in $\mathbb{R}$.*

Solution. The set of algebraic numbers is countable whereas every interval I in $\mathbb{R}$ is uncountable, so I cannot consist of algebraic numbers only.

10% (b) *Prove that Cantor's set contains a transcendental number.*

Solution. (I know that this problem was in the list of "review problems", but I like it.) Cantor's set cannot consist of algebraic numbers only since it is uncountable whereas the set of algebraic numbers is countable.