Math 4181H **Solutions to Midterm 2**

20% **1.** *Find* $\lim \sqrt[n]{100n^{100}+2^n}$ (and justify your answer, of course).

Solution. Since $100n^{100}/2^n \longrightarrow 0$, we have $100n^{100} < 2^n$ for all n large enough, so $2^n < 100n^{100}+2^n < 2\cdot 2^n$ for large n, so $2 < \sqrt[n]{100n^{100}+2^n} < 2\sqrt[n]{2}$ for large n. Since $\lim 2\sqrt[n]{2} = 2$, $\lim \sqrt[n]{100n^{100}+2^n} = 2$ by the squeeze theorem.

Another solution. For all n, $\sqrt[n]{100n^{100}+2^n} = 2\sqrt[n]{100n^{100}/2^n+1} > 2$. Since $100n^{100}/2^n \longrightarrow 0$ we have $100n^{100}/2^n < 1$ for all n large enough, so $\limsup 2\sqrt[n]{100n^{100}/2^n+1} \le 2\limsup\sqrt[n]{2} = 2$. Hence, $\lim \sqrt[n]{100n^{100}+2^n} = 2$.

Yet another solution. We have $\frac{100(n+1)^{100}+2^{n+1}}{100n^{100}+2^n} = \frac{100(n+1)^{100}/2^n+2}{100n^{100}/2^n+1} \longrightarrow \frac{0+2}{0+1} = 2$, so $\lim \sqrt[n]{100n^{100}+2^n} = 2$ by one of the "review problems".

20% **2.** *Let* (x_n) *be a sequence satisfying* $|x_n - x_m| < 1/\min\{n,m\}$ *for all* n,m. *Prove that this sequence converges.*

Solution. The sequence is Cauchy: given $\varepsilon > 0$, find k such that $1/k < \varepsilon$, then for any $n,m \ge k$ we have $|x_n - x_m| < 1/\min\{n,m\} \le 1/k < \varepsilon$. Hence, it converges.

20% **3.** *If* $\lim_{x\to 0^+} f(x) = \infty$, *prove that* $\lim_{x\to+\infty} \frac{1}{f(1/x)} = 0$. (Proofs like $\frac{1}{0^+} = +\infty$ are not accepted.)

Solution. Let $A = \mathrm{Rng}(f)\cap(0,+\infty)$, let $B = \{1/x : x\in A\} = \mathrm{Rng}(1/f)\cap(0,+\infty)$. Let $\varepsilon > 0$. Find $\delta > 0$ such that for every $x \in A$ with $x < \delta$ one has $|f(x)| > 1/\varepsilon$. Then for any $x \in B$ with $x > 1/\delta$ we have $1/x \in A$ and $0 < 1/x < \delta$, so $|f(1/x)| > 1/\varepsilon$, so $\left|\frac{1}{f(1/x)}\right| < \varepsilon$. Hence, $\lim_{x\to+\infty}\frac{1}{f(1/x)} = 0$.

Another solution. Let $A = \mathrm{Rng}(f)\cap(0,+\infty)$, let $B = \{1/x : x\in A\} = \mathrm{Rng}(1/f)\cap(0,+\infty)$. Let (x_n) be a sequence in B such that $x_n \longrightarrow +\infty$, then $1/x_n \in A$ and $1/x_n \longrightarrow 0^+$ ($1/x_n \longrightarrow 0$ with $1/x_n > 0$ for all n), so $f(1/x_n) \longrightarrow \infty$, so $\frac{1}{f(1/x_n)} \longrightarrow 0$.

Yet another solution. The function $g(x) = \frac{1}{f(1/x)}$ is the composition of three functions: $x \mapsto 1/x$, $y \mapsto f(y)$, and $z \mapsto 1/z$. As $x \longrightarrow +\infty$, $1/x \longrightarrow 0^+$; as $y \mapsto 0^+$, $f(y) \longrightarrow \infty$; as $z \longrightarrow \infty$, $1/z \longrightarrow 0$. Also, the function $1/x$ doesn't take the "forbidden" (for f) value 0 in a neighborhood of $+\infty$, and f doesn't take the "forbidden" (for $1/z$) value ∞. Hence, by the theorem about the limit of the composition, $\lim_{x\to+\infty} g(x) = 0$.

20% **4.** *Let* $f\colon A \longrightarrow \mathbb{R}$ *be a monotone function and let* $a \in A$ *be a limit point of both* $A\cap(-\infty,a)$ *and* $A\cap(a,+\infty)$. *Suppose there are sequences* (x_n) *and* (y_n) *in* $A\setminus\{a\}$ *such that* (x_n) *is increasing to* a, (y_n) *is decreasing to* a, *and* $\lim f(x_n) = \lim f(y_n)$. *Prove that* f *is continuous at* a.

Solution. Since f is monotone, both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and satisfy $\lim_{x\to a^-} f(x) \le f(a) \lim_{x\to a^+} f(x)$. Since these limits exist, they can be found along any corresponding sequence: we have $\lim_{x\to a^-} f(x) = \lim f(x_n)$ and $\lim_{x\to a^+} f(x) = \lim f(y_n)$. Since $\lim f(x_n) = \lim f(y_n)$, we obtain that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, and so, $\lim_{x\to a} f(x)$ exists and equals $f(a)$.

5. *Let* $f\colon[0,+\infty) \longrightarrow \mathbb{R}$ *be a continuous function with the property that a finite* $\lim_{x\to+\infty} f(x)$ *exists.*

20% (a) *Prove that* f *is bounded.*

Solution. Let $b = \lim_{x\to+\infty} f(x)$. Find M such that $|f(x)-b| < 1$ for all $x \ge M$, then $|f(x)| \le |f(x)-b|+|b| < 1+|b|$ on $[M,+\infty)$. Since f is continuous on the closed bounded interval $[0,M]$, f is bounded on $[0,M]$. Hence, f is bounded on $[0,+\infty)$.

5% (b) *Does* f *have to attain its maximal value?*

Solution. No, as the example $f(x) = \frac{x}{1+x}$, $x > 0$, shows: $\lim_{x\to+\infty} f(x) = 1 = \sup\mathrm{Rng}(f)$, but $f(x) \ne 1$ for all x.