Math 4181H **Solutions to Midterm 3**

1. *Let f be a function differentiable in a neighborhood of a point a with $f'(a) > 0$.*

10% (a) *Prove that f is strictly increasing at a (that is, for all x in a neighborhood of a, $f(x) < f(a)$ if $x < a$ and $f(x) > f(a)$ if $x > a$).*

Solution. Since $f'(a) = \lim_{x\to 0} \frac{f(x)-f(a)}{x-a} > 0$, we have $\frac{f(x)-f(a)}{x-a} > 0$ for all $x \neq a$ in a neighborhood I of a. This imlies that $f(x) - f(a) > 0$ for all $x \in I$ with $x > a$ and $f(x) - f(a) < 0$ for all $x \in I$ with $x < a$, that is, $f(x) > f(a)$ for all $x \in I$ with $x > a$ and $f(x) < f(a)$ for all $x \in I$ with $x < a$.

20% (b) *If f' is continuous at a, prove that f is strictly increasing in a neighborhood of a.*

Solution. If f' is continuous at a and $f'(a) > 0$ then $f'(x) > 0$ for all x in some neighborhood I of a, so, f is strictly increasing on I.

20% **2.** *Prove that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n > 0$, $\log\bigl(\frac{x_1+\cdots+x_n}{n}\bigr) \geq \frac{\log x_1+\cdots+\log x_n}{n}$.*

Solution. The function log is concave (as the inverse of the increasing convex function exp, or since $\log' x = 1/x$ is decreasing). So by Jensen's inequality for concave functions, $\log\bigl(\frac{x_1+\cdots+x_n}{n}\bigr) \geq \frac{\log x_1+\cdots+\log x_n}{n}$.

(Ok, if we only have Jensen's inequality for convex functions: Given $x_1, \ldots, x_n > 0$, put $y_i = \log x_i$, $i = 1, \ldots, n$, then by (the convex) Jensen's inequality $\exp\bigl(\frac{y_1+\cdots+y_n}{n}\bigr) \leq \frac{e^{y_1}+\cdots+e^{y_n}}{n} = \frac{x_1+\cdots+x_n}{n}$. Since log is increasing, this implies that $\frac{y_1+\cdots+y_n}{n} \leq \log\bigl(\frac{x_1+\cdots+x_n}{n}\bigr)$.)

20% **3.** *Suppose f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$; prove that f is strictly monotone on I.*

Solution. It cannot be that $f'(x) > 0$ and $f'(y) < 0$ for some $x, y \in I$, since then, by Darboux's theorem, we would have $f'(c) = 0$ for some c between x and y. Hence, either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$; so either f is strictly increasing on I, or f is strictly decreasing on I.

Another solution. Suppose f is not strictly monotone on I. Then f is not strictly monotone on a three-point subset of I: there are $x, y, z \in I$ with $x < y < z$ such that $f(x) \leq f(y) \geq f(z)$ or $f(x) \geq f(y) \leq f(z)$. Then on the closed bounded interval $[x, z]$, f attains its maximal or minimal value at an inner point u (that is, a point $u \in (x, z)$), and then $f'(u) = 0$.

15% **4.** (a) *Prove that for any $x > 0$, $\sin x < x$.*

Solution. I'll use the fact that if $f(0) = 0$ and $f' > c$ on an interval $(0, a)$, then $f(x) > cx$ on $(0, a)$. Since $\sin(0) = 0$ and $\sin' = \cos < 1$ on the interval $(0, \pi/2)$, $\sin x < x$ on this interval. For $x \geq \pi/2$, $\sin x \leq 1 < \pi/2 \leq x$.

15% (b) *Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sin x > (1-\varepsilon)x$ for all $x \in (0, \delta)$.*

Solution. We have $\lim_{x\to 0} \frac{\sin x}{x} = \sin' 0 = 1$, so for any $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{\sin x}{x} > 1 - \varepsilon$ for all $x \in (0, \delta)$, so $\sin x > (1-\varepsilon)x$ for all $x \in (0, \delta)$.

20% **5.** *Find $(f^{-1})''(f(a))$ in terms of the derivatives of f at a.*

Solution. For any x in a neighborhood of $b = f(a)$, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. So, by the formula $(1/h)' = -h'/h^2$ and by the chain rule,

$$(f^{-1})''(b) = \frac{-1}{f'(f^{-1}(b))^2} f''(f^{-1}(b))(f^{-1})'(b) = \frac{-1}{f'(a)^2} f''(a) \frac{1}{f'(a)} = \frac{-f''(a)}{f'(a)^3}.$$