Math 4181H **Solutions to Midterm 4**

20% **1.** *If f is an integrable function on $[a,b]$ with the property that the set $\{x : f(x) = 0\}$ is dense in $[a,b]$, prove that $\int_a^b f = 0$.*

Solution. For any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ for every i there is a point $z_i \in [x_{i-1}, x_i]$ such that $f(z_i) = 0$; so, for the selection $\sigma = \{z_1, \dots, z_n\}$ subordinate to P the Riemann sum $S(f, P, \sigma) = \sum_{i=1}^n f(z_i)\Delta x_i = 0$. Choose a sequence (P_m) of partitions with $\Delta(f, P_m) \longrightarrow 0$ (which exists since f is integrable) and a sequence (σ_m) of selections subordinate to P_m for every m such that $S(f, P_m, \sigma_m) = 0$ for every m. Since $S(f, P_m, \sigma_m) \longrightarrow \int_a^b f$, we obtain that $\int_a^b f = 0$.

20% **2.** *Derive the mean value theorem for integrals (*if f is continuous on $[a,b]$ then $\int_a^b f = f(c)(b-a)$ for some $c \in [a,b]$*) from the (Lagrange's) mean value theorem for derivatives.*

Solution. Let f be continuous on $[a,b]$, then, by the F.T.C., the integral function F is differentiable with $F' = f$. By Lagrange's M.V.T. for derivatives, $\int_a^b f = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a)$ for some $c \in (a,b)$.

20% **3.** *Find a primitive (antiderivative) function F on $\mathbb{R}$ of the function* $f(x) = \begin{cases} x, & x \le 0 \\ \sin x, & x \ge 0 \end{cases}$. (Notice that F must be differentiable.)

Solution. Since $\sin 0 = 0$, f is continuous on $\mathbb{R}$, and so, has a primitive F. On $(-\infty, 0)$, $F(x) = x^2/2 + C_1$ for some $C_1 \in \mathbb{R}$, on $(0, +\infty)$, $F(x) = -\cos x + C_2$ for some $C_2 \in \mathbb{R}$. For F to be continuous we need $F(0) = (x^2/2 + C_1)|_{x=0} = (-\cos x + C_2)|_{x=0}$, so $C_1 = C_2 - 1$. Since F is defined up to constant, we can put $C_1 = 0$ and thus $C_2 = 1$; then $F(x) = \begin{cases} x^2/2, & x \le 0 \\ 1 - \cos x, & x \ge 0. \end{cases}$

20% **4.** *Prove that the improper integral $\int_0^{+\infty} e^{-x^2} \sin x\, dx$ converges.*

Solution. For all $x \ge 1$, $\left|e^{-x^2} \sin x\right| \le e^{-x}$ and the integral $\int_1^{+\infty} e^{-x} dx$ converges, so the improper integral $\int_1^{+\infty} e^{-x^2} \sin x\, dx$ converges absolutely by comparison. The integral $\int_0^1 e^{-x^2} \sin x\, dx$ is proper. So, $\int_0^{+\infty} e^{-x^2} \sin x\, dx$ converges.

20% **5.** *Find* $\lim\limits_{x \to 0} \dfrac{\sin(x^2) - x^2}{(\cos x - 1)^3}$.

Solution. $$\frac{\sin(x^2) - x^2}{(\cos x - 1)^3} = \frac{x^2 - \frac{1}{3!}x^6 + o(x^6) - x^2}{(1 - \frac{1}{2}x^2 + o(x^2) - 1)^3} = \frac{-\frac{1}{3!}x^6 + o(x^6)}{(-\frac{1}{2}x^2 + o(x^2))^3} = \frac{-\frac{1}{6}x^6 + o(x^6)}{-\frac{1}{8}x^6 + o(x^6)} \underset{x\to 0}{\longrightarrow} 8/6 = 4/3.$$

20% **6.** *Assuming that the function $f(x) = (\sin x)/x$ for $x \ne 0$, $f(0) = 1$, is 100 times differentiable at 0 (don't prove this), find $f^{(100)}(0)$.*

Solution. The 100th Taylor polynomial of f at 0 is $(x - \frac{x^3}{3!} + \cdots + \frac{x^{101}}{101!})/x = 1 - \frac{x^2}{3!} + \cdots + \frac{x^{100}}{101!}$. Hence, $f^{(100)}(0) = 100!/101! = 1/101$.