Math 4181H **Solutions to Final exam review problems**

1. *Let $\sum a_i$ be a converging series. Prove or disprove:*

(i) *If $b_i \longrightarrow 0$, then the series $\sum a_i b_i$ converges.*

Solution. False: Let $a_i = b_i = \frac{(-1)^{i-1}}{\sqrt{i}}$, $i \in \mathbb{N}$, then $\sum a_i$ converges by Leibniz's test, but $\sum a_i b_i = \sum \frac{1}{i}$ diverges.

(ii) *If $b_i \longrightarrow 0$ and $b_i \geq 0$ for all i, then $\sum a_i b_i$ converges.*

Solution. False: Let $a_i = \frac{(-1)^{i-1}}{\sqrt{i}}$ for all i and $b_i = \frac{1}{\sqrt{i}}$ for odd i and $= 0$ for even i. Then $\sum a_i$ converges by Leibniz's test, but $\sum a_i b_i = 1 + 0 + \frac{1}{2} + 0 + \frac{1}{4} + \cdots$ diverges.

(iii) *If $b_i \searrow 0$ (decreases and tends to 0), then $\sum a_i b_i$ converges.*

Solution. This is true by Abel's test. (In fact, (b_i) could converge to any limit, not necessarily 0.)

(iv) *If $\sum a_i$ converges absolutely and $b_i \longrightarrow 0$, then $\sum a_i b_i$ converges.*

Solution. This is true: the sequence (b_i) is bounded, there is b such that $|b_i| \leq b$ for all i, so $|a_i b_i| \leq b|a_i|$ for all i, and the series $\sum b|a_i| = b \sum |a_i| < \infty$.

2. *Suppose f is differentiable on an interval I. Prove that f' is a pointwise limit of a sequence of continuous functions.*

Solution. For any $x \in I$, $f'(x) = \lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n} = \lim_{n\to\infty} n\big(f(x+1/n) - f(x)\big)$, where the functions $f_n(x) = n\big(f(x+1/n) - f(x)\big)$ are continuous (moreover, differentiable) for all n.

3. *Prove Dini's theorem: if (f_n) is a monotone sequence of continuous functions on a closed bounded interval I that converges pointwise to a continuous function f, then $f_n \rightrightarrows f$.*

Solution. Replacing f_n by $-f_n$ for all n if needed, we may assume that f_n decrease to f, $f_1(x) \geq f_2(x) \geq \cdots \geq f(x)$ and $f_n(x) \longrightarrow f(x)$ for all $x \in I$. Assume that f_n do not converge to f uniformly, then there is $\varepsilon > 0$ and a subsequence (f_{n_k}) of (f_n) such that $\|f_{n_k} - f\| > \varepsilon$ for all k. Then for every k there is $x_k \in I$ such that $f_{n_k}(x_k) > f(x_k) + \varepsilon$. By Bolzano-Weierstrass's theorem, there exists a subsequence (x_{k_i}) of (x_k) that converges to a point $a \in I$. Now, for every n, $f_n(x_{k_i}) \geq f_{n_{k_i}}(x_{k_i}) > f(x_{k_i}) + \varepsilon$ for all i such that $n_{k_i} \geq n$, so, by continuity of f_n and f, $f_n(a) = \lim_{i\to\infty} f_n(x_{k_i}) \geq \lim_{i\to\infty} f(x_{k_i}) + \varepsilon = f(a) + \varepsilon$. Hence, $f_n(a) \not\longrightarrow f(a)$, contradiction.

4. *Let $[a,b]$ be a (closed bounded) interval and let (c_n) be a sequence diverging to $+\infty$.*

(a) Prove that $\int_a^b \sin(c_n x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$.

Solution. For n such that $c_n \neq 0$,

$$\Big|\int_a^b \sin(c_n x)\,dx\Big| = \frac{1}{|c_n|}\big|\cos(c_n b) - \cos(c_n a)\big| \leq \frac{2}{c_n} \longrightarrow 0.$$

(b) *Prove the Riemann-Lebesgue's lemma: For any continuous function f on a closed bounded interval $[a,b]$, $\int_a^b f(x)\sin(c_n x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$.*

Solution. First, let's show that $\int_a^b h(x)\sin(c_n x)\,dx \longrightarrow 0$ for every step function h. Let $a = x_0 < x_1 < \cdots x_m = b$ and $d_1,\dots,d_m$ be such that $h(x) = d_i$ on (x_{i-1}, x_i), $i = 1,\dots,m$. By (a) for every i, $\int_{x_{i-1}}^{x_i} \sin(c_n x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$, hence $\int_a^b h(x)\sin(c_n x)\,dx = \sum_{i=1}^m d_i \int_{x_{i-1}}^{x_i} \sin(c_n x)\,dx \longrightarrow 0$.

Let $\varepsilon > 0$; find a step function h such that $\|f - h\| < \varepsilon$. Then for every n, $\int_a^b \big|\big(f(x) - h(x)\big)\sin(c_n x)\big|\,dx < \varepsilon(b-a)$, so

$$\Big|\int_a^b f(x)\sin(c_n x)\,dx\Big| \leq \Big|\int_a^b h(x)\sin(c_n x)\,dx\Big| + \Big|\int_a^b \big(f(x)-h(x)\big)\sin(c_n x)\,dx\Big| \leq \Big|\int_a^b h(x)\sin(c_n x)\,dx\Big| + \varepsilon(b-a).$$

Since $\lim_{n\to\infty} \int_a^b h(x)\sin(c_n x)\,dx = 0$, $\limsup_{n\to\infty}\big|\int_a^b f(x)\sin(c_n x)\,dx\big| \leq \varepsilon(b-a)$. Since this is true for every $\varepsilon > 0$, $\limsup_{n\to\infty}\big|\int_a^b f(x)\sin(c_n x)\,dx\big| = 0$, so $\lim_{n\to\infty}\big|\int_a^b f(x)\sin(c_n x)\,dx\big| = 0$.

5. *Prove that the series* $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ *converges uniformly on* $\mathbb{R}$.

Solution. For each n and any x, the function $f_n(x) = \frac{x}{n(1+nx^2)}$ tends to 0 as $x \longrightarrow \infty$ and $f_n'(x) = 0$ iff $x = \pm\frac{1}{\sqrt{n}}$, so $\|f_n\| = \sup|f_n(x)| = \left|f_n\left(\pm\frac{1}{\sqrt{n}}\right)\right| = \frac{1}{2n\sqrt{n}}$. Since the series $\sum \frac{1}{2n\sqrt{n}}$ converges, $\sum f_n$ converges uniformly by the Weierstrass M-test.

6. *Find the set of* x *for which the series* $\sum_{n=0}^{\infty} 2^n \sin^n x$ *converges, and find the sum of this series on this set.*

Solution. For every x, this is a geometric progression, which converges iff $|2\sin x| < 1$, that is, iff $|\sin x| < 1/2$, that is, iff $x \in \bigcup_{n\in\mathbb{Z}}\big(n\pi - \pi/6, n\pi + \pi/6\big)$. The sum of the series on this set is $\frac{1}{1-2\sin x}$.

7. *Prove that the zeta function* $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, $x > 1$, *is infinitely differentiable on* $(1, +\infty)$.

Solution. First of all notice that for any $a > 1$ and $k \in \mathbb{N}$ the series $\sum(\log n)^k n^{-a}$ converges. Indeed, let $1 < b < a$, then $\lim_{n\to\infty}(\log n)^k/n^{a-b} = 0$ (because $\lim_{t\to+\infty} t^k/e^{(a-b)t} = 0$), so $(\log n)^k n^{-a} < n^{-b}$ for all n large enough, and $\sum n^{-b} < \infty$, so $\sum(\log n)^k n^{-a} < \infty$ by comparison.

The series $\sum_{n=1}^{\infty} n^{-x}$ converges (by the integral test) for every $x > 1$, and so, ζ is defined on $(1, +\infty)$. The convergence is locally uniform: for every $a > 1$, for every $x \geq a$ we have $n^{-x} \leq n^{-a}$ and $\sum n^{-a} < \infty$, so $\sum n^{-x}$ converges uniformly on $[a, +\infty)$ by the M-test. Hence, ζ is continuous on $(1, +\infty)$.

Consider the series $\sum(n^{-x}) = -\sum(\log n)n^{-x}$. It also converges locally uniformly: for every $a > 1$, for every $x \geq a$ we have $(\log n)n^{-x} \leq (\log n)n^{-a}$ and $\sum(\log n)n^{-a} < \infty$, so $\sum(\log n)n^{-x}$ converges uniformly on $[a, +\infty)$ by the M-test. Hence, ζ is differentiable with $\zeta'(x) = -\sum_{n=1}^{\infty}(\log n)n^{-x}$.

Now assume by induction on k that, for some $k \in \mathbb{N}$, ζ is k-times differentiable with $\zeta^{(k)}(x) = (-1)^k \sum_{n=1}^{\infty}(\log n)^k n^{-x}$, $x > 1$. Then, since the series $\sum\big((\log n)^k n^{-x}\big)' = -\sum(\log n)^{k+1}n^{-x}$ converges locally uniformly on $(1, +\infty)$, $\zeta^{(k)}$ is differentiable and $\zeta^{(k+1)}(x) = (-1)^{k+1}\sum_{n=1}^{\infty}(\log n)^{k+1}n^{-x}$. So, by induction, ζ is infinitely differentiable on $(1, +\infty)$, with $\zeta^{(k)}(x) = (-1)^k \sum_{n=1}^{\infty}(\log n)^k n^{-x}$, $x > 1$, for all k.

8. *Prove that if* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is an even function, then* $a_n = 0$ *for all odd* n*, and if* f *is an odd function, then* $a_n = 0$ *for all even* n.

Solution. If f is even, then $f(x) = f(-x) = \sum_{n=0}^{\infty} a_n(-1)^n x^n$. Since the power series defining f is unique, we must have $a_n = (-1)^n a_n$ for all n, so $a_n = 0$ for all odd n.

If f is odd, then $f(x) = -f(-x) = -\sum_{n=0}^{\infty} a_n(-1)^n x^n = \sum_{n=0}^{\infty} a_n(-1)^{n+1}x^n$. Since the power series defining f is unique, we must have $a_n = (-1)^{n+1}a_n$ for all n, so $a_n = 0$ for all even n.

Another solution. If f is even, then $f^{(n)}$ is an odd function for every odd n, so $f^{(n)}(0) = 0$, so $a_n = \frac{f^{(n)}(0)}{n!} = 0$ for all odd n. If f is odd, then $f^{(n)}$ is an odd function for every even n, so $f^{(n)}(0) = 0$, so $a_n = \frac{f^{(n)}(0)}{n!} = 0$ for all even n.

9. *Find each of the following sums.*

(i) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

Solution. This is e^{-x}. (The series is obtained by substituting x by $-x$ in $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$)

(ii) $1 - x^3 + x^6 - x^9 + \ldots$, $|x| < 1$.

Solution. This is $\frac{1}{1+x^3}$. (The series is obtained by substituting x by x^3 in $1 - x + x^2 - x^3 + \ldots$)

(iii) $\frac{x^2}{2} - \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} - \frac{x^5}{5\cdot 4} + \ldots$, $|x| < 1$.

Solution. Let $f(x) = \frac{x^2}{2} - \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} - \frac{x^5}{5\cdot 4} + \ldots$, $|x| < 1$. The radius of convergence of this series is 1 (by the ratio test), so the function is defined on $(-1, 1)$. Then $f'(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \log(1+x)$, $|x| < 1$. So, $f(x) = \int \log(1+x) = (1+x)(\log(1+x)-1)+C$. Since $f(0) = 0$ we have $C = 1$, so $f(x) = (1+x)\log(1+x) - x$. (And indeed, we can check that $(1+x)\big(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\big) - x = \frac{x^2}{2} - \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} - \frac{x^5}{5\cdot 4} + \ldots$.)

10. *Evaluate the following sums:*

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}\pi^{2n}}{(2n)!}$.

Solution. This is the value at the point 2π of the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$. So, the sum is equal to $\cos(2\pi) = 1$.

(ii) $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

Solution. This is the value at the point 1 of the function $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x = \frac{e^x+e^{-x}}{2}$, so the sum is $\cosh 1 = \frac{e^1+e^{-1}}{2} = \frac{e^2+1}{2e}$.

(iii) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)2^n}$.

Solution. This is the value at the point $\frac{1}{\sqrt{2}}$ of the function $f(x) = \sqrt{2}\sum_{n=0}^{\infty} \frac{1}{2n+1}x^{2n+1}$. We have $f'(x) = \sqrt{2}\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$, so $f(x) = \sqrt{2}\int \frac{dx}{1-x^2} = \frac{1}{\sqrt{2}}\big(\log(1+x) - \log(1-x)\big) = \frac{1}{\sqrt{2}}\log\frac{1+x}{1-x}$, $|x| < 1$. So, the sum is $\frac{1}{\sqrt{2}}\log\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\log(3+2\sqrt{2})$.

(iv) $\sum_{n=0}^{\infty} \frac{1}{3^n(n+1)}$.

Solution. This is the value at the point $\frac{1}{3}$ of the function $f(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)}x^n = \frac{-1}{x}\log(1-x)$. So, the sum is $-3\log(2/3) = 3\log(3/2)$.

11. *If $f(x) = (\sin x)/x$ and $f(0) = 1$, find $f^{(k)}(0)$, $k \in \mathbb{N}$.*

Solution. For all x, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$, so for all $x \neq 0$, $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n}$. For $x = 0$ this formula also works, since at 0 both f and the series are equal to 1. Hence, f is an analytic function (it is given by a power series on the whole $\mathbb{R}$), and so, $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n}$ is the Taylor series of f. Hence, for any k, $f^{(k)}(0) = (2n)!\frac{(-1)^n}{(2n+1)!} = \frac{(-1)^n}{2n+1}$ if $k = 2n$ for some n (k is even), and $f^{(k)}(0) = 0$ if k is odd.

12. *Let $\alpha \in \mathbb{R}$.*

(a) *Let $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n}x^n$, $|x| < 1$. Prove that $(1+x)f'(x) = \alpha f(x)$.*

Solution. The radius of convergence of the series is 1 by the "ratio test", since $\big|\binom{\alpha}{n}/\binom{\alpha}{n+1}\big| = \big|(n+1)/(\alpha - n)\big| \longrightarrow 1$. I'll use the identities $n\binom{\alpha}{n} = \alpha\binom{\alpha-1}{n-1}$ and $\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \binom{\alpha}{n}$, $\alpha \in \mathbb{R}$, $n \geq 1$. We have $f'(x) = \sum_{n=1}^{\infty} n\binom{\alpha}{n}x^{n-1} = \alpha\sum_{n=1}^{\infty}\binom{\alpha-1}{n-1}x^{n-1} = \alpha\sum_{n=0}^{\infty}\binom{\alpha-1}{n}x^n$, so

$$(1+x)f'(x) = \alpha\Big(\sum_{n=0}^{\infty}\tbinom{\alpha-1}{n}x^n + \sum_{n=0}^{\infty}\tbinom{\alpha-1}{n}x^{n+1}\Big) = \alpha\Big(\sum_{n=0}^{\infty}\tbinom{\alpha-1}{n}x^n + \sum_{n=1}^{\infty}\tbinom{\alpha-1}{n-1}x^n\Big)$$
$$= \alpha\Big(1 + \sum_{n=1}^{\infty}\big(\big(\tbinom{\alpha-1}{n} + \tbinom{\alpha-1}{n-1}\big)x^n\Big) = \alpha\sum_{n=0}^{\infty}\tbinom{\alpha}{n}x^n = \alpha f(x).$$

(b) *Prove that any function f satisfying the differential equation $(1+x)f'(x) = \alpha f(x)$ has form $f(x) = c(1+x)^\alpha$ for some $c \in \mathbb{R}$, and deduce "the binomial formula" $(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$, $|x| < 1$.*

Solution. Consider the function $g(x) = f(x)(1+x)^{-\alpha}$. We have $g'(x) = f'(x)(1+x)^{-\alpha} - \alpha f(x)(1+x)^{-\alpha-1} = (1+x)^{-\alpha-1}\big(f'(x)(1+x) - \alpha f(x)\big) = 0$, so $g = \text{const} = c$. Hence, $f(x) = c(1+x)^\alpha$. Now, if $f(x) = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$, then from (a), $(1+x)f'(x) = \alpha f(x)$, so $f(x) = c(1+x)^\alpha$ for some c. Since $f(0) = 1$, $c = 1$.

13. *The Fibbonaci sequence is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for all $n \in \mathbb{N}$.*

(a) *Show that $a_{n+1}/a_n \leq 2$.*

Solution. Clearly, (a_n) is an increasing sequence of positive integers. Since for any $n \geq 2$, $a_{n+1} = a_n + a_{n-1} \leq 2a_n$, we get that $a_{n+1}/a_n \leq 2$.

(b) *Let $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$. Prove that f is defined on $\big(\frac{-1}{2}, \frac{1}{2}\big)$.*

Solution. For any x with $|x| < 1/2$ we have $\limsup\big|\frac{a_{n+1}x^n}{a_n x^{n-1}}\big| = \limsup\big|\frac{a_{n+1}}{a_n}\big| \cdot |x| < 1$, so the series $\sum a_n x^{n-1}$ converges (absolutely) by the ratio test.

(c) *Prove that if $|x| < 1/2$, then $f(x) = \frac{1}{1-x-x^2}$.*

Solution. Adding the series $xf(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1}x^{n-1}$ and $x^2 f(x) = \sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=3}^{\infty} a_{n-2}x^{n-1}$, we obtain

$$xf(x) + x^2 f(x) = a_1 x + \sum_{n=3}^{\infty}(a_{n-1} + a_{n-2})x^{n-1} = a_1 x + \sum_{n=3}^{\infty} a_n x^{n-1} = \sum_{n=2}^{\infty} a_n x^{n-1} = f(x) - 1,$$

so $f(x) - xf(x) - x^2 f(x) = 1$, so $f(x) = \frac{1}{1-x-x^2}$.

(d) *Decompose* $\frac{1}{1-x-x^2}$ *as* $\frac{b_1}{c_1-x}+\frac{b_2}{c_2-x}$ *to obtain another power series for* f *and prove that* $a_n=\frac{1}{\sqrt5}\big(\frac{1+\sqrt5}{2}\big)^n-\frac{1}{\sqrt5}\big(\frac{1-\sqrt5}{2}\big)^n$, $n\in\mathbb{N}$.

Solution. We have $\frac{1}{1-x-x^2}=\frac{1}{\sqrt5}\big(\frac{1}{c_1-x}-\frac{1}{c_2-x}\big)$, where $c_1=\frac{-1+\sqrt5}{2}$ and $c_2=\frac{-1-\sqrt5}{2}$. So,

$$f(x)=\tfrac{1}{\sqrt5}\big(\tfrac{1}{c_1}\textstyle\sum_{n=0}^\infty\big(\tfrac{x}{c_1}\big)^n-\tfrac{1}{c_2}\sum_{n=0}^\infty\big(\tfrac{x}{c_2}\big)^n\big)=\tfrac{1}{\sqrt5}\sum_{n=1}^\infty\big(\tfrac{1}{c_1^n}-\tfrac{1}{c_2^n}\big)x^{n-1}.$$

Note that $\frac{1}{c_1}=\frac{1+\sqrt5}{2}$ and $\frac{1}{c_2}=\frac{1-\sqrt5}{2}$, so $f(x)=\sum_{n=1}^\infty\frac{1}{\sqrt5}\big(\big(\frac{1+\sqrt5}{2}\big)^n-\big(\frac{1-\sqrt5}{2}\big)^n\big)x^{n-1}$. Comparing the coefficients of the two power series for f, we see that $a_n=\frac{1}{\sqrt5}\big(\frac{1+\sqrt5}{2}\big)^n-\frac{1}{\sqrt5}\big(\frac{1-\sqrt5}{2}\big)^n$ for all n.

14. (a) *Prove that the series* $\sum 2^n\sin\frac{1}{3^nx}$ *converges uniformly on* $[a,+\infty)$ *for any* $a>0$.

Solution. Let $a>0$. We have $\frac{1}{3^na}\longrightarrow0$ as $n\longrightarrow\infty$, so there is k such that for all $n\ge k$, $0<\frac{1}{3^na}<\frac{\pi}{2}$. For any $x\ge a$ we have $0<\frac{1}{3^nx}\le\frac{1}{3^na}$, so $0<\frac{1}{3^nx}<\frac{\pi}{2}$ for all $n\ge k$. sin is increasing on the interval $\big[0,\frac{\pi}{2}\big]$, so $\sin\frac{1}{3^nx}\le\sin\frac{1}{3^na}$, and $2^n\sin\frac{1}{3^nx}\le2^n\sin\frac{1}{3^na}$, for all $x\ge a$ and $n\ge k$.

The series $\sum 2^n\sin\frac{1}{3^na}$ converges by the limit comparison test: since $\lim_{x\to0}\frac{\sin x}{x}=1$, we have $\lim_{n\to\infty}\big(2^n\sin\frac{1}{3^na}\big)/\big(\frac{2^n}{3^na}\big)=1$ and $\sum\frac{2^n}{3^na}=\frac{1}{a}\sum\big(\frac{2}{3}\big)^n<\infty$. Hence, $\sum2^n\sin\frac{1}{3^nx}$ converges absolutely uniformly on $[a,\infty)$ by the M-test.

(b) *By considering* $\sum 2^n\sin\frac{1}{3^nx}$ *for* $x=\frac{2}{3^n\pi}$, *show that the series doesn't converge uniformly on* $(0,\infty)$.

Solution. For any n, for $x=\frac{2}{3^n\pi}$ we have $2^n\sin\frac{1}{3^nx}=2^n\sin(\pi/2)=2^n$, hence $\big\|2^n\sin\frac{1}{3^nx}\big\|\ge2^n\not\to0$.

(c) *For* $f(x)=\sum2^n\sin\frac{1}{3^nx}$, $x>0$, *find (that is, express in the form of a series)* f'.

Solution. To learn if the series for f can be differentiated term-by-term, consider the series $\sum\big(2^n\sin\frac{1}{3^nx}\big)'=\sum2^n\cos\frac{1}{3^nx}\cdot\frac{-1}{(3^nx)^2}3^n=-\sum\big(\frac{2}{3}\big)^n\frac{1}{x^2}\cos\frac{1}{3^nx}$. For any $a>0$, for any n, for any $x\ge a$ we have $\big|\big(\frac{2}{3}\big)^n\frac{1}{x^2}\cos\frac{1}{3^nx}\big|\le\big(\frac{2}{3}\big)^n\frac{1}{a^2}$, and $\sum\big(\frac{2}{3}\big)^n\frac{1}{a^2}<\infty$, so the series $\sum\big(\frac{2}{3}\big)^n\frac{1}{x^2}\cos\frac{1}{3^nx}$ converges uniformly on $[a,\infty)$ by M-test, so converges locally uniformly on $(0,\infty)$. Hence, $f'(x)=-\sum\big(\frac{2}{3}\big)^n\frac{1}{x^2}\cos\frac{1}{3^nx}$ on $(0,\infty)$.

15. *Find* $\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n(n+1)}$.

Solution. The series $\sum\frac{(-1)^{n-1}}{n(n+1)}$ converges absolutely (by comparison with $\sum\frac{1}{n^2}$). So, by Abel's theorem, the function $f(x)=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n(n+1)}x^{n+1}$ is continuous on $[0,1]$. Since power series can be differentiated term-by-term, we have $f'(x)=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}x^n=\log(1+x)$, $|x|<1$. So, $f(x)=\int\log(1+x)=(1+x)\log(1+x)-x+C$ on $(-1,1)$ and therefore on $(-1,1]$; since $f(0)=0$, $C=0$. Thus, $\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n(n+1)}=f(1)=2\log2-1$.

16. (a) *Show that the series* $\sum_{n=0}^\infty\big(\frac{x^{2n+1}}{2n+1}-\frac{x^{n+1}}{2n+2}\big)$ *converges to* $\frac12\log(1+x)$ *locally uniformly on* $(-1,1)$, *but converges to* $\log2$ *at* 1.

Solution. The power series $\sum\frac{x^{2n+1}}{2n+1}$ and $\sum\frac{x^{n+1}}{2n+2}$ converge locally uniformly on $(-1,1)$, and so does their sum. Both series converge absolutely on $(-1,1)$, so their sums can be computed in any order. So,

$$\sum_{n=0}^\infty\frac{x^{n+1}}{2n+2}=\frac12\sum_{n=0}^\infty\frac{x^{n+1}}{n+1}=\frac12\sum_{n=0}^\infty\frac{x^{2n+1}}{2n+1}+\frac12\sum_{n=0}^\infty\frac{x^{2n+2}}{2n+2}$$

and

$$\sum_{n=0}^\infty\Big(\frac{x^{2n+1}}{2n+1}-\frac{x^{n+1}}{2n+2}\Big)=\sum_{n=0}^\infty\frac{x^{2n+1}}{2n+1}-\frac12\sum_{n=0}^\infty\frac{x^{2n+1}}{2n+1}-\frac12\sum_{n=0}^\infty\frac{x^{2n+2}}{2n+2}=\frac12\sum_{n=0}^\infty\frac{x^{2n+1}}{2n+1}-\frac12\sum_{n=0}^\infty\frac{x^{2n+2}}{2n+2}$$
$$=\frac12\sum_{n=1}^\infty(-1)^{n-1}\frac{x^n}{n}=\frac12\log(1+x).$$

At $x=1$ the two power series do not converge, so this argument isn't applicable. The series $\sum_{n=0}^\infty\big(\frac{1}{2n+1}-\frac{1}{2n+2}\big)$ is a grouping of the series $\sum_{n=1}^\infty(-1)^{n-1}\frac1n$, so its sum is equal to $\sum_{n=1}^\infty(-1)^{n-1}\frac1n=\log2$.

(b) *Why doesn't this contradict Abel's theorem?*

Solution. Because this series is not a power series, – it is a sum of two power series, but not a "term-by-same-degree-term" sum.

17. (a) *Prove that for every $n \in \mathbb{N}$, $\int_0^\pi x\cos(nx)\,dx = \frac{-2}{n^2}$ if n is odd and 0 if n is even.*
Solution.

$$\int_0 \pi x\cos(nx)\,dx = \frac{1}{n}\int_0 \pi x\,d\sin(nx) = \frac{1}{n}x\sin(nx)\,\Big|_0^\pi - \frac{1}{n}\int_0 \pi\sin(nx)\,dx = 0 + \frac{1}{n^2}\cos(nx)\,\Big|_0^\pi = \frac{(-1)^n-1}{n^2}.$$

(b) *Prove that for every $n \in \mathbb{N}$, $f_n(x) = 1 + 2\sum_{i=1}^n \cos(ix) = \sin\big((n+1/2)x\big)/\sin(x/2)$. Prove that the function $x/\sin(x/2)$, $x \neq 0$, can be extended to 0 by continuity. Deduce that $\int_0^\pi xf_n(x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$.*
Solution. For any n,

$$2\sin(x/2)\sum_{i=1}^n\cos(ix) = \sum_{i=1}^n 2\sin(x/2)\cos(ix) = \sum_{i=1}^n\big(\sin((i+1/2)x)-\sin((i-1/2)x)\big) = \sin((n+1/2)x)-\sin(x/2).$$

So, $2\sum_{i=1}^n\cos(ix) = \sin\big((n+1/2)x\big)/\sin(x/2)-1$.

As $\sin(x/2) \neq 0$ for all $x \in (0,2\pi)$, the function $g(x) = \frac{x}{\sin(x/2)}$ is continuous on $(0,2\pi)$. Since $\lim_{x\to 0}\frac{x}{\sin(x/2)} = 2\lim_{x\to 0}\frac{x/2}{\sin(x/2)} = 2$, if we define $g(0) = 2$ the function g is continuous in $[0,2\pi)$. By the Riemann-Lebesgue lemma,

$$\int_0^\pi xf_n(x)\,dx = \int_0^\pi x\sin\big((n+1/2)x\big)/\sin(x/2)\,dx = \int_0^\pi g(x)\sin\big((n+1/2)x\big)\,dx \longrightarrow 0$$

as $n \longrightarrow \infty$.

(c) *Combine (a) and (b) to prove that $\sum_{\text{odd } n\in\mathbb{N}}\frac{1}{n^2} = \frac{\pi^2}{8}$. Notice that $\sum_{\text{even } n\in\mathbb{N}}\frac{1}{n^2} = \frac{1}{4}\sum_{\text{all } n\in\mathbb{N}}\frac{1}{n^2}$ and deduce that $\sum_{n=1}^\infty\frac{1}{n^2} = \frac{\pi^2}{6}$.*
Solution. For any n we have

$$\int_0^\pi xf_n(x)\,dx = \int_0^\pi x dx + 2\sum_{i=1}^n\int_0^\pi x\cos(ix)\,dx = \frac{\pi^2}{2} - 4\sum_{\substack{i\le n\\ i \text{ is odd}}}\frac{1}{i^2}.$$

Since $\int_0^\pi xf_n(x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain that $\frac{\pi^2}{2} - 4\lim_{n\to\infty}\sum_{\substack{i\le n\\ i \text{ is odd}}}\frac{1}{i^2} = 0$, that is, $\sum_{\text{odd } i\in\mathbb{N}}\frac{1}{i^2} = \frac{\pi^2}{8}$.

Let $s = \sum_{i=1}^\infty\frac{1}{i^2}$. Then

$$\sum_{\text{even } i\in\mathbb{N}}\frac{1}{i^2} = \sum_{k=1}^\infty\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^\infty\frac{1}{k^2} = \tfrac{1}{4}s.$$

Hence, $\frac{\pi^2}{8} = \sum_{\text{odd } i\in\mathbb{N}}\frac{1}{i^2} = s - \frac{1}{4}s = \frac{3}{4}s$, and $s = \frac{\pi^2}{6}$.