Math 4181H **Solutions to Midterm 2 review problems**

1. *What can be said about the sequence (x_n) if it converges and $x_n \in \mathbb{Z}$ for all n?*

Solution. This sequence is eventually constant: there is k such that $x_n = x_k$ for all $n \geq k$. Indeed, (x_n) is Cauchy, thus there exists k such that $|x_n - x_m| < 1$ for all $n, m \geq k$; since $x_n, x_m \in \mathbb{Z}$, this implies that $x_n = x_m$. Hence, $x_n = x_k$ for all $n \geq k$.

2. *Find* $\lim \dfrac{n^n + 3n! + 10n^{10} + 2 \cdot 3^n}{5n! + 10^n - 2n^n + 4}$.

Solution. We have

$$x_n = \frac{n^n + 3n! + 10n^{10} + 2 \cdot 3^n}{5n! + 10^n - 2n^n + 4} = \frac{1 + 3n!/n^n + 10n^{10}/n^n + 2 \cdot 3^n/n^n}{5n!/n^n + 10^n/n^n - 2 + 4/n^n}.$$

Since $\lim(3n!/n^n) = \lim(10n^{10}/n^n) = \lim(2 \cdot 3^n/n^n) = \lim(5n!/n^n) = \lim(10^n/n^n = \lim(4/n^n) = 0$, we get that $\lim x_n = -1/2$.

3. *If (x_n) is a sequence with $x_n \geq 0$ for all n and $x_n \longrightarrow a$, prove that $\sqrt{x_n} \longrightarrow \sqrt{a}$.*

Solution. Let $a = 0$. Given $\varepsilon > 0$, find k such that $x_n < \varepsilon^2$ for all $n \geq k$, then $\sqrt{x_n} < \varepsilon$ for all $n \geq k$. Hence, $\sqrt{x_n} \longrightarrow 0 = \sqrt{a}$.

Now assume that $a > 0$. For any n we have

$$\left|\sqrt{x_n} - \sqrt{a}\right| = \left|\frac{x_n - a}{\sqrt{x_n} + \sqrt{a}}\right| \leq \frac{|x_n - a|}{\sqrt{a}}.$$

Since $|x_n - a| \longrightarrow 0$, $0 \leq \left|\sqrt{x_n} - \sqrt{a}\right| \longrightarrow 0$ by the squeeze theorem, so $\sqrt{x_n} \longrightarrow \sqrt{a}$.

Another solution. The function $f(x) = \sqrt{x}$ is continuous (it is the inverse of the continuous function $g(x) = x^2$), so $x_n \longrightarrow a$ implies $\sqrt{x_n} = f(x_n) \longrightarrow f(a)$.

4. (a) *If the sequence (x_n) diverges to ∞ and the sequence (y_n) is bounded, prove that the sequence $(x_n + y_n)$ diverges to ∞.*

Solution. Let N be such that $|y_n| \leq N$ for all n. Let $M \in \mathbb{R}$; find k such that $|x_n| > M + N$ for all $n \geq k$. Then for any $n \geq k$, $|x_n + y_n| \geq |x_n| - |y_n| > M + N - N = M$.

(b) *If the sequence (x_n) converges to 0 and the sequence (y_n) is bounded, prove that the sequence $(x_n y_n)$ converges to 0.*

Solution. Let $|y_n| \leq M$ for all n; then $0 \leq |x_n y_n| \leq M|x_n|$ for all n and $M|x_n| \longrightarrow 0$, so $x_n y_n \longrightarrow 0$ by the squeeze theorem.

5. *Let (x_n) be a sequence of real numbers. For each of the following statements, if it is true, say so; if false, give an example demonstrating this:*

(i) *If (x_n) converges it is bounded.*

Solution. True.

(ii) *If (x_n) is bounded it converges.*

Solution. False: the sequence $x_n = (-1)^n$, $n \in \mathbb{N}$, is bounded, but diverges.

(iii) *If (x_n) is monotone it converges.*

Solution. False: the sequence $x_n = n$, $n \in \mathbb{N}$, is increasing but diverges (to $+\infty$).

(iv) *If (x_n) is monotone and bounded it converges.*

Solution. True.

(v) If (x_n) *diverges to ∞ it is unbounded.*

Solution. True.

(vi) If (x_n) *is unbounded it diverges to ∞.*

Solution. False: the sequence $(0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots)$ is unbounded but doesn't diverge to ∞.

(vii) *If (x_n) is monotone and unbounded it diverges to ∞.*

Solution. True.

6. *If the sequence* (x_n) *satisfies* $1/n \le x_n \le n$ *for all* n, *prove that* $\lim \sqrt[n]{x_n} = 1$.

Solution. Both $\lim \sqrt[n]{1/n} = \lim \sqrt[n]{n} = 1$, so $\lim \sqrt[n]{x_n} = 1$ by the squeeze theorem.

7. *Prove that* $\lim \sqrt[n]{n!} = +\infty$.

Solution. For any $M > 0$, since $n!/M^n \longrightarrow= +\infty$, there exists k such that $n! > M^n$ for all $n \ge k$, and so, $\sqrt[n]{n!} > M$ for all $n \ge k$.

8. *Let* $a \in \mathbb{R}$, *let* $x_1 = a$ *and* $x_{n+1} = x_n^2 - x_n + 1$ *for all* $n \in \mathbb{N}$. *Prove that the sequence* (x_n) *converges if* $0 \le a \le 1$ *and diverges to* $+\infty$ *otherwise.*

Solution. For every n we have $x_{n+1} - x_n = x_n^2 - 2x_n + 1 = (x_n - 1)^2 \ge 0$, so the sequence (x_n) is increasing and therefore has a limit. If this limit is $b \in \mathbb{R}$, then b satisfies $b = b^2 - b + 1$, so $(b-1)^2 = 0$, so $b = 1$; otherwise, the limit is $+\infty$. If $0 \le x_1 = a \le 1$, then $x_1^2 \le x_1$, so $0 \le x_1 \le x_2 = x_1^2 - x_1 + 1 \le 1$, and by induction $0 \le x_n \le 1$ for all n. Hence, (x_n) is bounded, and $\lim x_n = 1$. If $a < 0$ or $a > 1$, then $x_2 > 1$, and since (x_n) increases, (x_n) cannot converge to 1. Hence, $\lim x_n = +\infty$ in this case.

9. (a) *Let* (x_n) *and* (y_n) *be two sequences such that* $|y_n - y_m| \le |x_n - x_m|$ *for all* $n, m \in \mathbb{N}$. *Prove that if* (x_n) *converges then* (y_n) *also converges.*

Solution. If (x_n) converges it is Cauchy. Let $\varepsilon > 0$, find k such that $|x_n - x_m| < \varepsilon$ for all $n, m \ge k$. Then also $|y_n - y_m| < \varepsilon$ for all $n, m \ge k$. This shows that (y_n) is also Cauchy and thus converges.

(b) *Give an example of two sequences* (x_n) *and* (y_n) *such that* $|y_{n+1} - y_n| \le |x_{n+1} - x_n|$ *for all* $n \in \mathbb{N}$, (x_n) *converges but* (y_n) *diverges.*

Solution. Let $(a_n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{8}, \frac{1}{16}, \ldots\right)$. Define $x_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$ and $y_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$, $n \in \mathbb{N}$. Then for every n, $|x_{n+1} - x_n| = |y_{n+1} - y_n| = a_{n+1}$, (x_n) converges and $y_n \longrightarrow +\infty$.

10. *Consider the sequence* $(x_n) = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \ldots\right)$. *For which numbers* α *is there a subsequence of* (x_n) *converging to* α? *Also find* $\limsup x_n$ *and* $\liminf x_n$.

Solution. (x_n) runs over all rational numbers in the interval $(0,1)$. Since rational numbers are dense in $\mathbb{R}$, every point $a \in [0,1]$ is a limit point of (x_n), thus for every $a \in [0,1]$ there exists a subsequence of (x_n) converging to a. Every point $b \notin [0,1]$ is not a limit point of (x_n), therefore there is no subsequence of (x_n) converging to b. Since the maximal and the minimal limit points of (x_n) are 1 and -1 respectively, we have $\limsup x_n = 1$ and $\liminf x_n = -1$.

11. *For* $d \in \mathbb{N}$, *construct a sequence that has exactly* d *limit points.*

Solution. $(1, 2, 3, \ldots, d, 1, 2, 3, \ldots, d, 1, 2, 3, \ldots, d, \ldots)$.

12. (a) *If* $\lim x_n = a$ *prove that every subsequence of* (x_n) *converges to* a.

Solution. Let $x_n \longrightarrow a$ and let (x_{n_i}) be a subsequence of (x_n). Let $\varepsilon > 0$. Find k such that $|x_n - a| < \varepsilon$ for all $n \ge k$. Find l such that $n_l \ge k$. Then for any $i \ge l$ we have $n_i \ge n_l \ge k$, so $|x_{n_i} - a| < \varepsilon$. Hence, $x_{n_i} \longrightarrow a$.

(b) *Suppose a sequence* (x_n) *and* $a \in \mathbb{R}$ *are such that every subsequence of* (x_n) *has a subsequence that converges to* a. *Prove that* $\lim x_n = a$.

Solution. I'll prove the contrapositive: Assume that (x_n) doesn't converge to a; then there exists $\varepsilon > 0$ such that there are infinitely many n such that $|x_n - a| > \varepsilon$. Let $n_1 < n_2 < \cdots$ be such that $|x_{n_i} - a| > \varepsilon$ for all i; then the subsequence (x_{n_i}) of (x_n) has no subsequence that converges to a.

13. *Find* $\lim\bigl(1 + 1/(2n)\bigr)^n$.

Solution. For all n, $\bigl(1 + 1/(2n)\bigr)^n = \sqrt{\bigl(1 + 1/(2n)\bigr)^{2n}}$. The sequence $\left(\bigl(1 + 1/(2n)\bigr)^{2n}\right)$ is a subsequence of the sequence $\left(\bigl(1 + 1/n\bigr)^n\right)$ that converges to Euler's number e, so $\bigl(1 + 1/(2n)\bigr)^{2n} \longrightarrow e$ as well. Since the function $x \mapsto \sqrt{x}$ is continuous, $\lim \sqrt{\bigl(1 + 1/(2n)\bigr)^{2n}} = \sqrt{\lim\bigl(1 + 1/(2n)\bigr)^{2n}} = \sqrt{e}$.

14. *If the sequence* (x_n) *satisfies* $\limsup |x_n| = 0$*, prove that it converges to* 0.

Solution. Since $|x_n| \ge 0$ for all n, $\liminf |x_n| \ge 0$, so $\liminf |x_n| = \limsup |x_n| = 0$, so $\lim |x_n| = 0$, so $x_n \longrightarrow 0$.

15. *Let* (x_n) *be a sequence of positive numbers. Prove that* $\limsup(x_{n+1}/x_n) \ge \limsup \sqrt[n]{x_n}$ *and* $\liminf(x_{n+1}/x_n) \le \liminf \sqrt[n]{x_n}$*. If* $\lim(x_{n+1}/x_n)$ *exists, prove that* $\lim \sqrt[n]{x_n}$ *also exists.*

Solution. This is simply a reformulation of Theorem 2.7.6 from Lecture notes, with x_n instead of $x_1 \cdots x_n$. But ok, let's reprove it. Let $a = \limsup(x_{n+1}/x_n)$, let $b > a$. Find k such that $x_{n+1}/x_n < b$ for all $n \ge k$. Then for any $n > k$,

$$x_n = x_k \frac{x_{k+1}}{x_k} \cdots \frac{x_n}{x_{n-1}} < x_k b^{n-k} = (x_k/b^k)b^n,$$

so $\sqrt[n]{x_n} < b\sqrt[n]{x_k/b^k}$. Hence,

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} b\sqrt[n]{x_k/b^k} = b \lim_{n\to\infty} \sqrt[n]{x_k/b^k} = b.$$

Since this is true for every $b > a$, $\limsup \sqrt[n]{x_n} \le a$.

Similarly, $\liminf \sqrt[n]{x_n} \ge \liminf(x_{n+1}/x_n)$. So,

$$\liminf(x_{n+1}/x_n) \le \liminf \sqrt[n]{x_n} \le \limsup \sqrt[n]{x_n} \le \limsup(x_{n+1}/x_n).$$

If (finite or infinite) $\lim(x_{n+1}/x_n)$ exists, then $\liminf(x_{n+1}/x_n) = \limsup(x_{n+1}/x_n)$, so $\liminf \sqrt[n]{x_n} = \limsup \sqrt[n]{x_n}$, so $\lim \sqrt[n]{x_n}$ exists (and equals $\lim(x_{n+1}/x_n)$).

16. *Let* $f\colon A \longrightarrow \mathbb{R}$ *be a function, let* a *be a limit point of* A*. Prove that* $\lim_{x\to a} f(x) = b$ *iff for every monotone sequence* (x_n) *in* $A \setminus \{a\}$ *with* $x_n \longrightarrow a$ *one has* $f(x_n) \longrightarrow b$.

Solution. If $\lim_{x\to a} f(x) = b$ then for every (not only monotone) sequence (x_n) in $A \setminus \{a\}$ with $x_n \longrightarrow a$ we have $f(x_n) \longrightarrow b$. Now suppose that $f(x) \not\longrightarrow b$ as $x \longrightarrow a$; then there are $\varepsilon > 0$ and a sequence (x_n) in $A \setminus \{a\}$ with $x_n \longrightarrow a$ such that $|f(x_n) - b| \ge \varepsilon$ for all n. And the sequence (x_n) has a monotone subsequence (x_{n_i}) which also converges to a and satisfies $|f(x_{n_i}) - b| \ge \varepsilon$ for all i, so that $f(x_{n_i}) \not\longrightarrow b$.

17. (a) *Suppose that* g *and* h *are continuous at* a *and that* $g(a) = h(a)$*. Define* $f(x) = \begin{cases} g(x), & x \ge a \\ h(x), & x \le a. \end{cases}$ *Prove that* f *is continuous at* a.

Solution. We have $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = g(a) = f(a)$ and $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} h(x) = h(a) = f(a)$. (If these limits make sense, that is, if a is a limit point of corresponding sets.) Hence, $\lim_{x\to a} f(x) = f(a)$.

18. *A function* $f\colon A \longrightarrow \mathbb{R}$ *is said to be Lipschitz at a point* $a \in A$ *if there is* $C > 0$ *such that* $|f(x) - f(a)| \le C|x - a|$ *for all* $x \in A$ *in a neighborhood of* a*. Prove that if* f *is Lipschitz at* a *then* f *is continuous at* a.

Solution. Let $\delta > 0$ be such that $|f(x) - f(a)| \le C|x - a|$ for all $x \in A$ with $|x - a| < \delta$. Given $\varepsilon > 0$, if $x \in A$ is such that $|x - a| < \min\{\delta, \varepsilon/C\}$, then $|f(x) - f(a)| \le C|x - a| < C(\varepsilon/C) = \varepsilon$.

19. (a) *Prove that if* f *is continuous at* a *then so is* $|f|$.

Solution. The function $|f(x)|$ is the composition of f and of the continuous function $y \mapsto |y|$.

Another solution. Given $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ when $x \in \mathrm{Dom}(f)$, $|x - a| < \delta$, then for any such x, $\big||f(x)| - |f(a)|\big| \le |f(x) - f(a)| < \varepsilon$.

(b) *Prove that if* f *and* g *are continuous at* a *then so are* $\max\{f, g\}$ *and* $\min\{f, g\}$.

Solution. We have $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$, so both are continuous by (a).

(c) *Prove that every continuous* f *can be written* $f = g - h$*, where* g *and* h *are nonnegative and continuous.*

Solution. Put $g = \max\{f, 0\}$ and $h = -\min\{f, 0\}$, then $f = g - h$ and both $g, h \ge 0$ and are continuous (at all points) by (b).

20. *Find* $\lim 2^{(1+1/n)^n}$.

Solution. $\lim(1+1/n)^n = e$, the function $x \mapsto 2^x$ is continuous, thus $2^{(1+1/n)^n} \longrightarrow 2^e$.

21. *Suppose A_n, $n \in \mathbb{N}$, are pairwise disjoint subsets of $\mathbb{R}$ with no limit points. (That is, for any n and any $a \in \mathbb{R}$ there is $\delta > 0$ such that $(a-\delta, a+\delta) \cap A_n \setminus \{a\} = \emptyset$, and $A_n \cap A_m = \emptyset$ for any distinct n and m). Define f by $f(x) = \begin{cases} 1/n, & x \in A_n \\ 0, & x \notin A_n \text{ for all } n. \end{cases}$ Prove that $\lim_{x \to a} f(x) = 0$ for all $a \in [0,1]$. Deduce that f is discontinuous at every point of $\bigcup_{n=1}^{\infty} A_n$ and continuous at all other points of $\mathbb{R}$.*

Solution. Let $a \in \mathbb{R}$. Let $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Let $\delta > 0$ be such that $(a-\delta, a+\delta) \cap (A_1 \cup \cdots \cup A_n) \setminus \{a\} = \emptyset$. Then for any $x \in (a-\delta, a+\delta) \setminus \{a\}$ we have $x \notin A_1 \cup \cdots \cup A_n$, so either $f(x) = 0$ or $f(x) = 1/k$ with $k > n$ and so, $|f(x)| < 1/n < \varepsilon$.

22. *Let f be a monotone function that takes all rational values (that is, $\mathrm{Rng}(f) \supseteq \mathbb{Q}$). Prove that f is continuous.*

Solution. Since $\mathrm{Rng}(g) \supseteq \mathbb{Q}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, the range of f has no "gaps" (intervals between $\inf \mathrm{Rng}(f)$ and $\sup \mathrm{Rng}(f)$ containing no points of $\mathrm{Rng}(f)$). Hence, f is continuous by Theorem 4.2.1 from Lecture notes. (For every limit point a of $\mathrm{Dom}(f)$, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ (if both limits make sense) and $= f(a)$ (if $a \in \mathrm{Dom}(f)$); thus, f is continuous at a.)

23. *A function $g \colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be even if $g(-x) = g(x)$ for all $x \in \mathbb{R}$ and odd if $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Prove that every function f continuous on $\mathbb{R}$ can be written as $f = E + O$, where E is a continuous even function and O is a continuous odd function.*

Solution. Put $E(x) = \frac{1}{2}(f(x) + f(-x))$ and $O(x) = \frac{1}{2}(f(x) - f(-x))$, $x \in \mathbb{R}$. The functions f and $f(-x)$ are continuous ($f(-x)$ is the composition of two continuous functions, $x \mapsto -x$ and f), so E and O are continuous; E is even and O is odd.

24. *Let $A \subseteq \mathbb{R}$, let f be a function $A \longrightarrow \mathbb{R}$, let $a = \inf A$. Define the function $\widetilde{f} \colon (a, +\infty) \longrightarrow \mathbb{R}$ by $\widetilde{f}(x) = \sup\{f(z) : z \in A,\ z \leq x\}$. Prove that $\widetilde{f}$ is an increasing function. If f is an increasing function, prove that $\widetilde{f}$ is an extension of f.*

Solution. For any $x, y \geq a$ we have

$$\widetilde{f}(x) = \sup\{f(z) : z \in A,\ z \leq x\} \leq \sup\{f(z) : z \in A,\ z \leq y\} = \widetilde{f}(y),$$

so $\widetilde{f}$ is increasing. If f is increasing, for every $x \in A$ we have $f(z) \leq f(x)$ for all $z \in A$ with $z \leq x$, so $\widetilde{f}(x) = \sup\{f(z) : z \in A,\ z \leq x\} = f(x)$.

25. *Find an integer n such that $f(x) = x^3 - x + 3$ has a root in $[n, n+1]$.*

Solution. We have $f(-2) = -8 + 2 + 3 < 0$, and $f(-1) = -1 + 1 + 3 > 0$, so by the I.V.T. (the intermediate value theorem) there exists $x \in [-2, 1]$ such that $f(x) = 0$.

26. *Suppose that f is continuous on $[a,b]$ and that $f(x) \in \mathbb{Q}$ for all $x \in [a,b]$. What can be said about f?*

Solution. It can be said that f is constant. Indeed, let $y, z \in [a,b]$, $y < z$, and assume that $f(y) \neq f(z)$. Since irrational numbers are dense in $\mathbb{R}$, there exists an irrational α between $f(y)$ and $f(z)$. By the intermediate value theorem there exists $x \in (y, z)$ such that $f(x) = \alpha$; but this contradicts the assumption that $f(x) \in \mathbb{Q}$ for all $x \in [a,b]$.

27. *Suppose f is continuous on $[0,1]$ and $\mathrm{Rng}(f) \subseteq [0,1]$. Prove that $f(x_0) = x_0$ for some $x_0 \in [0,1]$.*

Solution. Let $g(x) = x$, $x \in [0,1]$. We have $f(0) \geq 0 = g(0)$ and $f(1) \leq 1 = g(1)$. If $f(0) = g(0)$ or $f(1) = g(1)$ we are done, so let's assume that $f(0) > g(0)$ and $f(1) < g(1)$; then there exists $x_0 \in (0,1)$ such that $f(x_0) = g(x_0) = x_0$.

28. *Let f be any polynomial function. Prove that there is $x_0 \in \mathbb{R}$ such that $|f(x_0)| \leq |f(x)|$ for all $x \in \mathbb{R}$.*

Solution. If f is constant, then any point x_0 works. If not, then $|f| \longrightarrow +\infty$ as $x \longrightarrow \pm\infty$, so such x_0 exists (as was proved in class).

29. (a) *Suppose that f is continuous on an interval $[a, b]$ and let c be any number. Prove that there is a point on the graph of f which is closest to $(c, 0)$.*

Solution. Consider the function

$$\varphi(x) = \big[\text{the distance between } (x, f(x)) \text{ and } (c, 0)\big] = \sqrt{(x-c)^2 + f(x)^2},$$

on the interval $[a, b]$. This function is continuous on $[a, b]$ (by the theorems about sums/products/compositions of continuous functions), so it attains its minimum at a point $x_0 \in [a, b]$.

(b) *Show that this assertion is not necessarily true if $[a, b]$ is replaced by (a, b).*

Solution. I'll give a very simple example: let $f(x) = 0$ on (a, b) and $c = b$. Then for any $x \in (a, b)$ "the distance between the points $(x, f(x))$ and $(c, 0)$" equals $b - x$, and there is no point x on (a, b) that minimizes this distance.

(c) *Show that this assertion is true if $[a, b]$ is replaced by $\mathbb{R}$.*

Solution. In this case the continuous function $\varphi(x)$, introduced in (a), tends to $+\infty$ as $x \to \pm\infty$ (since $f(x) \geq |x - c|$ for all x), so it attains its minimal value on $\mathbb{R}$.

30. *Suppose that f is continuous on an interval (a, b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$, which may be finite or $\pm\infty$. Prove that f has a maximum on all of (a, b) or a minimum on all of (a, b).*

Solution. The problem has an easy solution if $c = \lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ is finite ($\neq \infty$): Define $f(a) = f(b) = c$, then f becomes continuous on $[a, b]$, so attains its maximal value M and its minimal value m. If both values are taken at the endpoints, then $M = m = c$, so f is constant. Otherwise, f takes at least one of the values M, m at a point $x_0 \in (a, b)$.

If $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$ is infinite, this argument doesn't work. But then, if the limits are equal to $+\infty$, we know that f attains a minimal value, and if both are equal to $-\infty$, then $-f$ attains a minimal value, so f attains a maximal value at that point.

31. *Suppose that function f is continuous but not uniformly continuous on an interval $[a, b)$. Prove that $\lim_{x\to b^-} f(x)$ does not exist or is infinite.*

Solution. If $\lim_{x\to b^-} f(x) = c$ exists we may extend f to b by $f(b) = c$, then f becomes continuous on $[a, b]$, and so, uniformly continuous on $[a, b]$. Since f is not uniformly continuous the limit doesn't exist.

32. *If a function f is uniformly continuous on an interval $[a, b]$ and on the interval $[b, c]$, prove that f is uniformly continuous on $[a, c]$.*

Solution. Let $\varepsilon > 0$. Find $\delta_1, \delta_2 > 0$ such that for any $x, y \in [a, b]$ with $|x-y| < \delta_1$ we have $|f(x)-f(y)| < \varepsilon/2$ and for any $x, y \in [b, c]$ with $|x - y| < \delta_2$ we have $|f(x) - f(y)| < \varepsilon/2$. Put $\delta = \min\{\delta_1, \delta_2\}$. Let $x, y \in [a, c]$, $0 < y - x < \delta$. If $x, y \in [a, b]$ or $\in [b, c]$, then $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$. If $x \leq b \leq y$, then $|b - x| < \delta$ and $|y - b| < \delta$, so $|f(x) - f(b)| < \varepsilon/2$ and $|f(y) - f(b)| < \varepsilon/2$, thus $|f(x) - f(y)| < \varepsilon$.

33. *A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be periodic with period a, where a is a positive real number, if $f(x + a) = f(x)$ for all $x \in \mathbb{R}$.*

(b) *Prove that every continuous periodic function is bounded and attains its maximal and minimal values.*

Solution. Let f be continuous and periodic with period a. f is continuous on the interval $[0, a]$ thus $f|_{[0,a]}$ is bounded and attains its maximal and minimal values M and m. For any $x \in \mathbb{R}$ there is $n \in \mathbb{Z}$ such that $x - na \in [0, a)$ (namely, $n = [x/a]$), so $f(x) = f(x - na)$, and therefore $m \leq f(x) \leq M$.

(a) *Prove that every continuous periodic function is uniformly continuous.*

Solution. Let f be continuous and periodic with period a. The function $f|_{[0,2a]}$ is continuous and so uniformly continuous. Given $\varepsilon > 0$ let $0 < \delta < a$ be such that for any $x, y \in [0, 2a]$ with $|x - y| < \delta$ one has $|f(x) - f(y)| < \varepsilon$. Now let $x, y \in \mathbb{R}$, $0 < y - x < \delta$. Find $n \in \mathbb{Z}$ such that $0 \leq x - na < a$, then $0 < y - na < 2a$, so for $x' = x - na$ and $y' = y - na$ we have $x', y' \in [0, 2a]$ and $|x' - y'| < \delta$, so $|f(x') - f(y')| < \varepsilon$. Since f is periodic with period a, (it is easy to prove by induction that) $f(x) = f(x')$ and $f(y) = f(y')$, thus $|f(x) - f(y)| < \varepsilon$.

34. *If a function* $f\colon \mathbb{R} \longrightarrow [0, +\infty)$ *satisfies* $f(x+y) = f(x)f(y)$ *for all* x *and* y *and is continuous at* 0 *prove that* f *is continuous (at all points).*

Solution. Let $a \in \mathbb{R}$. Write $f(x) = f(x - a + a) = f(x - a)f(a)$. Since f is continuous at 0, $f(x - a)$ is continuous at a, thus $f(x) = f(x - a)f(a)$ is continuous at a.