Math 4181H **Solutions to Midterm 3 review problems**

1. *Suppose that f is a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with critical points $-1, 1, 2, 3, 4$ and corresponding values $6, 1, 2, 4, 3$. Sketch the graph of f in the case n is even and in the case n is odd.*

Solution. Since f' is continuous (or by Darboux's theorem) it preserves sign on every interval not containing a critical point, so, f is strictily monotone on each of the intervals $(-\infty, -1]$, $[-1, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, +\infty)$. Whether f is strictly increasing or strictly decreasing on these intervals can be determined by comparing its values at the end points: decreasing on $[-1, 1]$, and increasing on $[1, 2]$, $[2, 3]$, and $[3, 4]$. If n is even then $f(x) \longrightarrow +\infty$ for $x \longrightarrow \pm\infty$, so f is decreasing on $(-\infty, -1)$ and increasing on $(4, +\infty)$; if n is odd then $f(x) \longrightarrow -\infty$ for $x \longrightarrow -\infty$ and $f(x) \longrightarrow +\infty$ for $x \longrightarrow +\infty$, so f is increasing on both $(-\infty, -1)$ and $(4, +\infty)$.

2. *Let f be a differentiable function on $\mathbb{R}$. If f' is odd, prove that f is even; if f' is even and $f(0) = 0$, prove that f is odd.*

Solution. Let f' be odd; put $g(x) = f(x) - f(-x)$, then $g'(x) = f'(x) + f'(-x) = 0$, so $g = c \in \mathbb{R}$, so $f(-x) = f(x) + c$ for all x. In particular, $f(0) = f(0) + c$, so $c = 0$, and $f(-x) = f(x)$ for all x.

Let f' be even; put $g(x) = f(x) + f(-x)$, then $g'(x) = f'(x) - f'(-x) = 0$, so $g = c \in \mathbb{R}$, so $f(-x) = -f(x) + c$ for all x. Since $f(0) = 0$, we have $c = 0$, so $f(-x) = -f(x)$ for all x.

3. *Let $f(x) = x/2 + x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is strictly increasing at 0 but not increasing in any neighborhood of 0.*

Solution. $f'(0) = \lim_{x\to0} f(x)/x = \lim_{x\to0}\big(1/2 + x\sin(1/x)\big) = 1/2 > 0$, so f is strictly increasing at 0.

For any $x \neq 0$, $f'(x) = 1/2 + 2x\sin(1/x) - \cos(1/x)$. When x is close to 0 (when $0 < |x| < 1/4$, to be exact), $|2x\sin(1/x)| < 1/4$. So at the points $x_n = \frac{1}{\pi/2+2n\pi}$, $n \in \mathbb{Z}$, where $\cos(1/x) = 1$, we have $f'(x_n) < 1/2 + 1/4 - 1 = -1/4$. For every n, since f' is continuous at x_n, $f' < 0$ in a neighborhood of x_n, so f is decreasing at this neghborhood. (And, every neighborhood of 0 contains x_n for some n, of course.)

4. *If f is convex on an open interval I and differentiable at $a \in I$, prove that $f(x) \geq f(a) + f'(a)(x - a)$ for all $x \in I$.*

Solution. As we know, since f is convex, $f'(a) = f'_+(a) = \inf\big\{\frac{f(x)-f(a)}{x-a} \mid x \in I,\ x > a\big\}$, so $f'(a) \leq \frac{f(x)-f(a)}{x-a}$ for all $x > a$, so (as $x - a > 0$) $f'(a)(x - a) \leq f(x) - f(a)$ for such x.

Similarly, $f'(a) = f'_-(a) = \sup\big\{\frac{f(x)-f(a)}{x-a} \mid x \in I,\ x < a\big\}$, so $f'(a) \geq \frac{f(x)-f(a)}{x-a}$ for all $x < a$, so (as $x - a < 0$) $f'(a)(x - a) \leq f(x) - f(a)$ for such x.

5. *Find* $\tan'$ *and* $\arctan'$.

Solution. $\tan'(x) = \big(\frac{\sin}{\cos}\big)'(x) = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$.

$\arctan' x = \frac{1}{\tan'(\arctan(x))} = \cos^2(\arctan x)$. Ok, if $y = \arctan x$ then $x = \tan y$. But then $\cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$, so $\arctan' x = \frac{1}{1+x^2}$.

6. *Find f' in terms of g and g' if*

(i) $f(x) = g(x + g(a))$.

Solution. Since no word is said about a, it is assumed to be a constant. Then $g(a)$ is also just a constant. So, $f'(x) = g'(x + g(a))$.

(ii) $f(x) = g(xg(a))$.

Solution. $f'(x) = g'(xg(a))g(a)$.

(iii) $f(x) = g(x + g(x))$.

Solution. $f'(x) = g'(x + g(x))(1 + g'(x))$.

(iv) $f(x) = g(xg(x))$.

Solution. $f'(x) = g'(xg(x))(g(x) + xg'(x))$.

7. *Let $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$. Find $f'(0)$ and $f''(0)$.*

Solution. $f'(0) = \lim_{x\to 0} \frac{(\sin x)/x - 1}{x} = \lim_{x\to 0} \frac{\sin x - x}{x^2}$. Since $\lim_{x\to 0}(\sin x - x) = \lim_{x\to 0} x^2 = 0$ and $(x^2)' = 2x \neq 0$ if $x \neq 0$, by L'Hospital's rule $\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}$ if this last limit exist. But $\lim_{x\to 0} \frac{\cos x - 1}{x} = \cos' 0 = -\sin 0 = 0$, so $f'(0) = 0$.

We also have $f'(x) = (\cos x)/x - (\sin x)/x^2$ for all $x \neq 0$. Hence, $f''(0) = \lim_{x\to 0} \frac{(\cos x)/x - (\sin x)/x^2}{x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$. L'Hospital's rule is, again, applicable, and we get

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{3x} = \frac{-1}{3}\sin' 0 = \frac{-1}{3}.$$

8. *Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable at 0 and $f(0) = g(0) = 0$.*

Solution. We have $(fg)'(0) = f'(0)g(0) + f(0)g'(0) = 0$, whereas $x'|_{x=0} = 1$.

9. *Let $f(x) = x^3 - 3x + a$ for some $a \in \mathbb{R}$. Prove that f cannot have more than one root in $[-1, 1]$.*

Solution. If f had two roots $a, b \in [-1.1]$, then by Rolle's theorem there would be $x \in (a, b) \subseteq (-1, 1)$ such that $f'(x) = 0$; but $f'(x) = 3x^2 - 3 < 0$ for all $x \in (-1, 1)$.

10. *If f is twice differentiable with $f(0) = 0$, $f(1) = 1$, and $f'(0) = f'(1) = 0$, then $|f''(x)| \geq 4$ for some $x \in (0, 1)$.*

Solution. Assume that $|f''(x)| < 4$ for all $x \in (0, 1)$. Then $f'(x) < f'(0) + 4x = 4x$ for all $x \in \left[0, \frac{1}{2}\right]$, so $f(x) < f(0) + 2x^2 = 2x^2$ for all such x, so $f(1/2) < 1/2$. Also, $f'(x) < f'(1) + 4(1 - x) = 4(1 - x)$ for all $x \in \left[\frac{1}{2}, 1\right]$, so $f(x) > f(1) - 2(1 - x)^2 = 1 - 2(1 - x)^2$ for all such x, so $f(1/2) > 1/2$, contradiction.

11. *If f is invertible, differentiable, satisfies $f' = f^2$ on an interval I, and $f(x) \neq 0$ for all $x \in I$, find $(f^{-1})'$, f^{-1}, and f.*

Solution. For any $y \in f(I)$, $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{y^2}$, so $f^{-1}(y) = \frac{-1}{y} + c$ for some $c \in \mathbb{R}$, so $f(x) = \frac{-1}{x-c}$.

12. *Prove that the function $f(x) = 1/x$ is convex on $(0, +\infty)$ and use this to prove that for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n > 0$, $\frac{x_1 + \cdots + x_n}{n} \geq \left(\frac{x_1^{-1} + \cdots + x_n^{-1}}{n}\right)^{-1}$.*

Solution. f is strictly convex on $(0, +\infty)$ since $f''(x) = 2/x^3$ is positive on $(0, +\infty)$. By Jensen's inequality for any $x_1, \ldots, x_n > 0$, $f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$, that is, $\left(\frac{x_1 + \cdots + x_n}{n}\right)^{-1} \leq \frac{x_1^{-1} + \cdots + x_n^{-1}}{n}$. Since f is decreasing, this implies that $f\left(\left(\frac{x_1 + \cdots + x_n}{n}\right)^{-1}\right) \geq f\left(\frac{x_1^{-1} + \cdots + x_n^{-1}}{n}\right)$, that is, $\frac{x_1 + \cdots + x_n}{n} \geq \left(\frac{x_1^{-1} + \cdots + x_n^{-1}}{n}\right)^{-1}$.

13. *Prove that of all rectangles with given perimeter, the square has the greatest area.*

Solution. Let p be the given perimeter. If a rectangle with sides x and y has perimeter p then $y = p/2 - x$, and the area of such a rectangle is $S(x) = xy = x(p/2 - x)$. The function $S(x)$ is defined on the interval $[0, p/2]$, is positive on $(0, p/2)$, and satisfies $S(0) = S(p/2) = 0$. Thus, it must have a global maximum at some point of $(0, p/2)$. We have $S'(x) = p/2 - 2x$, so $S'(x) = 0$ iff $x = p/4$, hence the maximum of S is reached at this point. Thus, the area of the rectangle is maximal when, and only when, $x = p/4 = y$, that is, when the rectangle is a square.

14. *Suppose that $f\colon [0, 1] \longrightarrow [0, 1]$ is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $f'(x) \neq 1$ for all $x \in [0, 1]$. Show that there is exactly one $x \in [0, 1]$ such that $f(x) = x$.*

Solution. First, there is at least one such x by the I.V.T., since for the continuous function $g(x) - x$ we have $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. And if there are two points $x_1 < x_2$ for which $f(x_1) = x_1$ and $f(x_2) = x_2$, then by the M.V.T., $1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$ for some $c \in (x_1, x_2)$, which contradicts the assumption.

15. *Suppose function f is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, $f(0) = 0$, and f' is increasing on $(0, +\infty)$. Prove that the function $g(x) = f(x)/x$ is increasing on $(0, +\infty)$.*

Solution. Since f is differentiable on $(0, +\infty)$ g is also differentiable on $(0, +\infty)$ and $g'(x) = \frac{f'(x)x - f(x)}{x^2}$, $x > 0$. To prove that g is increasing it suffices to show that $g'(x) \geq 0$ for all $x > 0$. For any $x > 0$ by the M.V.T. we have $f(x) = f(x) - f(0) = f'(t)x$ for some $t \in (0, x)$, and since f' is increasing, $f'(t) \leq f'(x)$. So, $f'(x)x - f(x) = f'(x)x - f'(t)x \geq 0$, thus $g'(x) \geq 0$.

Another solution. Since f' is increasing, f is convex on $(0,+\infty)$. So, the function $g(x) = \frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0}$ is increasing. (Well, the interval $[0,+\infty)$ is not open, so we cannot apply the theory of convex functions on open intervals as such; it has to be adapted a little.)

16. Let f be continuous on $[a,b]$ and differentiable on (a,b).

(a) *Prove that if $f'(x) \geq M$ for all $x \in (a,b)$, then $f(b) \geq f(a) + M(b-a)$.*

Solution. By the M.V.T., $f(b) - f(a) = f'(c)(b-a) \geq M(b-a)$ for some $c \in (a,b)$.

(b) *Prove that if $f'(x) \leq M$ for all $x \in (a,b)$, then $f(b) \leq f(a) + M(b-a)$.*

Solution. By the M.V.T., $f(b) - f(a) = f'(c)(b-a) \leq M(b-a)$ for some $c \in (a,b)$.

17. *Suppose that the functions f and g are differentiable on an interval I, $f'(x) \geq g'(x)$ for all $x \in I$ and $f(a) = g(a)$ for some $a \in I$. Show that $f(x) \geq g(x)$ for all $x \in I$ with $x > a$ and $f(x) \leq g(x)$ for all $x \in I$ with $x < a$.*

Solution. Consider the function $h = f - g$. Since $h'(x) = f'(x) - g'(x) > 0$ for all $x \in I$, h is strictly increasing on I. Hence, $f(x) - g(x) = h(x) < h(a) = 0$ for $x < a$ and $f(x) - g(x) = h(x) > h(a) = 0$ for $x > a$.

18. *Suppose f is continuous on $[0,1]$ and differentiable on $(0,1)$ with $f'(x) \geq M > 0$ for all $x \in [0,1]$. Show that there is an interval of length $\frac{1}{4}$ where $|f| \geq M/4$.*

Solution. If $f\big(\frac{1}{2}\big) \geq 0$, then since $f' \geq M$,

$$f(x) \geq f\big(\tfrac{1}{2}\big) + M\big(x - \tfrac{1}{2}\big) \geq M\big(x - \tfrac{1}{2}\big) = Mx - M/2$$

for all $x \geq \frac{1}{2}$, so $f(x) \geq M/4$ for $x \in \big[\frac{3}{4}, 1\big]$. If $f\big(\frac{1}{2}\big) \leq 0$, then since $f' \geq M$,

$$f(x) \leq f\big(\tfrac{1}{2}\big) + M\big(x - \tfrac{1}{2}\big) \leq M\big(x - \tfrac{1}{2}\big) = Mx - M/2$$

for all $x \leq \frac{1}{2}$, so $f(x) \leq -M/4$ for $x \in \big[0, \frac{1}{4}\big]$.

Another solution. For any two points $x, y \in [0,1]$ such that $|f(x)|, |f(y)| < M/4$ we have $|f(x) - f(y)| < M/2$; but since $|f(x) - f(y)| \geq M|x-y|$, we obtain that $|x-y| < 1/2$. So, all points $x \in [0,1]$ with $|f(x)| < M/4$ are contained in a subinterval of $[0,1]$ of length $\leq 1/2$. Hence, the set of points where $|f(x)| \geq M/4$ contains at least one interval of length $\geq 1/4$.

19. *If a function f is differentiable on an interval I and has a unique critical point $a \in I$, prove that either f is monotone on I or attains the (global on I) maximum or minimum at a.*

Solution. Let $I = (\alpha, \beta)$m where α and β can be infinite. Since $f' \neq 0$ on (α, a) by Darboux's theorem f' preserves sign on (α, a), so f is either increasing or decreasing on (α, a); for the same reason f is either decreasing or increasing on (a, β). We therefore have four cases:
if f is increasing on both (α, a) and on (a, β), then f is increasing on I;
if f is increasing on (α, a) and is decreasing on (a, β), then f has maximum at a;
if f is decreasing on (α, a) and is increasing on (a, β), then f has minimum at a;
if f is decreasing on both (α, a) and on (a, β), then f is decreasing on I.

20. *Let $c \neq 0$ and let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentible dunction satisfying $f'(x) = cf(x)$. Prove that $f(x) = ae^{cx}$ for some $a \in \mathbb{R}$.*

Solution. Put $g(x) = f(x/c)$; then $g'(x) = f'(x/c)/c = cf(x/c)/c = f(x/c) = g(x)$. Hence, $g(x) = ae^x$ and $f(x) = g(cx) = ae^{cx}$ for some $a \in \mathbb{R}$.

Another solution. Let $h(x) = \frac{f(x)}{e^{cx}}$, then $h'(x) = \frac{f'(x)e^{cx} - f(x)ce^{cx}}{e^{2cx}} = \frac{cf(x)e^{cx} - f(x)ce^{cx}}{e^{2cx}} = 0$, so $h = \text{const}$.

21. *Suppose that f is a differentiable function on $(0,+\infty)$ such that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. Prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$ (and so, $f(x) = \log_a x$ for some $a > 0$).*

Solution. Fix any $y > 0$ and consider the function $g(x) = f(xy)$, $x > 0$. For any $x > 0$ we have $g'(x) = f'(xy)y = \frac{1}{xy}y = \frac{1}{x} = f'(x)$. It follows that $g = f + c$ for some $c \in \mathbb{R}$; for $x = 1$ we have $g(1) = f(y)$ and $g(1) = f(1) + c = c$, so $c = f(y)$. So, $f(xy) = g(x) = f(x) + f(y)$ for all $x > 0$.

22. *Let f be a function differentiable on $(0,+\infty)$ such that $\lim_{x\to+\infty} f'(x) = c > 0$.*

(a) *Prove that $\lim_{x\to+\infty} f(x)/x = c$.*

Solution. Both f and $g(x) = x$ are defined and differentiable on $(0,+\infty)$. Since $\lim_{x\to+\infty} f'(x) > 0$, there is K such that $f'(x) > 1$ for all $x \geq K$, so $f(x) > f(K) + (x-K)$ for all $x \geq K$, so $\lim_{x\to+\infty} f(x) = +\infty$; also $\lim_{x\to+\infty} g(x) = +\infty$. For all x, $g'(x) = 1 \neq 0$, and $\lim_{x\to+\infty} f'(x)/g'(x) = \lim_{x\to+\infty} f'(x)$ exists. Hence, L'Hospital's rule is applicable and gives $\lim_{x\to+\infty} f(x)/g(x) = \lim_{x\to+\infty} f'(x)/g'(x) = c$.

Another solution. Let $0 < \varepsilon < c$. Find $K > 0$ such that $c - \varepsilon/2 < f'(x) < c + \varepsilon/2$ for all $x \geq K$. Then for every $x > K$, $f(K) + (c-\varepsilon/2)x < f(x) < f(K) + (c+\varepsilon/2)x$, so $\frac{f(K)}{x} + c - \varepsilon/2 < f(x)/x < \frac{f(K)}{x} + c + \varepsilon/2$ for all $x > K$. Find $M > K$ such that $|f(x)/x| < \varepsilon/2$ for all $x \geq M$. Then for all $x \geq M$, $c - \varepsilon < f(x)/x < c + \varepsilon$, so $|f(x) - c| < \varepsilon$. Hence, $\lim_{x\to+\infty} f(x)/x = c$.

(b) *Does this imply that $f(x) - cx$ is a bounded function?*

Solution. No, the function $f(x) = cx + \log(1+x)$ is a counterexample: since $\log(1+x) \longrightarrow +\infty$ and $(\log(1+x))/x \longrightarrow 0$ as $x \longrightarrow +\infty$, $f(x)/x \longrightarrow c$ and $f(x) - cx \longrightarrow +\infty$.

23. (a) *Give an example of a function f differentiable on $(0,+\infty)$ for which a finite $\lim_{x\to+\infty} f(x)$ exists, but $\lim_{x\to+\infty} f'(x)$ does not exist.*

Solution. As f we can take a function that tends, say, at zero but oscilates faster and faster, so that the derivative doesn't stabilize.

For example, we can take $f(x) = (\sin(x^2))/x$. We have $\lim_{x\to\infty} f(x) = 0$ (since $\sin(x^2)$ is bounded and $1/x \longrightarrow 0$ as $x \longrightarrow \infty$), and $f'(x) = \frac{2x^2\cos(x^2) - \sin(x^2)}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}$ has no limit as $x \longrightarrow +\infty$.

(b) *Prove that if $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f'(x)$ both exist and are finite, then $\lim_{x\to+\infty} f'(x) = 0$.*

Solution. If $\lim_{x\to\infty} f'(x) = b \neq 0$, say $b > 0$, then for some $N \in \mathbb{R}$, $f'(x) > b/2$ for all $x > N$, so $f(x) > (b/2)(x-N) + f(N)$ for $x > N$, so $\lim_{x\to+\infty} f(x) = +\infty$.

(c) *Prove that if $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f''(x)$ both exist and are finite, then $\lim_{x\to+\infty} f''(x) = 0$.*

Solution. If $\lim_{x\to+\infty} f''(x) = b \neq 0$, say $b > 0$, then for some $N \in \mathbb{R}$, $f''(x) > b/2$ for all $x > N$, so $f'(x) > (b/2)(x-N) + f'(N)$ for $x > N$, so $\lim_{x\to++\infty} f'(x) = +\infty$, so there is K such that $f'(x) > 1$ for all $x > K$, so $f(x) > x - N + f(K)$ for $x > K$, so $\lim_{x\to+\infty} f(x) = +\infty$.

24. (a) *Prove that for any $x, y \in \mathbb{R}$ such that $\cos x, \cos y, \cos(x+y) \neq 0$, $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.*

Solution. For any $x, y \in \mathbb{R}$ for which $\cos x, \cos y, \cos(x+y) \neq 0$, we have $\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

(b) $\arctan$ *is the inverse of* $\tan|_{(-\frac{\pi}{2},\frac{\pi}{2})}$. *Prove that $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$, indicating all the necessary restrictions on x and y.*

Solution. Since tan is strictly increasing on the interval $(-\pi/2, \pi/2)$ with $\lim_{x\to(-\frac{\pi}{2})^+} \tan x = -\infty$ and $\lim_{x\to(\frac{\pi}{2})^-} \tan x = +\infty$, arctan is defined on $(-\infty,+\infty) = \mathbb{R}$ and $\mathrm{Rng}(\arctan) = (-\pi/2, \pi/2)$. Let $x, y \in \mathbb{R}$; put $u = \arctan x$ and $v = \arctan y$ (then $u, v \in (-\pi/2, \pi/2)$). Then $\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{x+y}{1-xy}$. This, indeed, implies that $u + v = \arctan\left(\frac{x+y}{1-xy}\right)$, but if and only if $u + v \in (-\pi/2, \pi/2)$.

25. *Prove that* $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

Solution. First of all, $0 < \arctan\frac{1}{2} + \arctan\frac{1}{3} < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. So $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\frac{1/2+1/3}{1-(1/2)(1/3)} = \arctan 1 = \pi/4$.

26. *If a function f is twice differentiable on $\mathbb{R}$ and satisfies $f'' = f$, prove that $f(x) = ae^x + be^{-x}$ for some $a, b \in \mathbb{R}$.*

Solution. Put $g = f'$, then $g' = f$. Hence, $(f+g)' = f + g$ and $(f-g)' = -(f-g)$, so $f(x) + g(x) = ce^x$ and $f(x) - g(x) = de^{-x}$ for some $c, d \in \mathbb{R}$, so $f(x) = \frac{1}{2}(ce^x + de^{-x})$.

27. *Find $(f^{-1})''(f(a))$ in terms of the derivatives of f at a.*

Solution. For any x in a neighborhood of $b = f(a)$, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. So, by the chain rule,

$$(f^{-1})''(b) = -\frac{1}{f'(f^{-1}(b))^2} f''(f^{-1}(b))(f^{-1})'(b) = -\frac{1}{f'(a)^2} f''(a) \frac{1}{f'(a)} = \frac{-f''(a)}{f'(a)^3}.$$

28. *Suppose that f is continuous on $[a,b]$, n-times differentiable on (a,b), and that $f(x) = 0$ for $n+1$ distinct points x in $[a,b]$. Prove that $f^{(n)}(x) = 0$ for some x in (a,b).*

Solution. Let $x_0 < x_1 < \ldots < x_n$ be the points where $f = 0$. By Rolle's theorem, for each $i = 1, \ldots, n$, there exists a point $y_i \in (x_{i-1}, x_i)$ such that $f'(y_i) = 0$. The function f' is $(n-1)$-times differentiable on (a,b) and vanishes at n different points of this interval; thus we may use induction and claim that $f^{(n)} = (f')^{(n-1)}$ vanishes at at least one point.

29. *Let $n \in \mathbb{N}$, and let $f(x) = x^n$ for $x > 0$ and $f(x) = 0$ for all $x \leq 0$. Prove that f is $(n-1)$-times differentiable but not n-times differentiable at 0.*

Solution. For any $k \leq n$, $f^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$ for all $x > 0$ and $= 0$ for all $x < 0$. If, by induction on k, we know that $f^{(k)}(0) = 0$, and $k < n-1$, then $f_+^{(k+1)}(0) = \lim_{x\to 0^+} f^{(k)}(x)/x = \lim_{x\to 0^+} \frac{n!}{(n-k)!}x^{n-k-1} = 0$. Also, clearly, $f_-^{(k+1)}(0) = 0$, so f is $(k+1)$-times differentiable at 0 with $f^{(k+1)}(0) = 0$. For $k = n-1$ we have $f_+^{(n)}(0) = f_+^{(k+1)}(0) = \lim_{x\to 0^+} = n!$ and $f_-^{(n)}(0) = f_-^{(k+1)}(0) = 0$, so $f^{(n)}(0)$ doesn't exist.

30. (a) *Let $n \in \mathbb{N}$ and $f(x) = x^{2n}\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is n times differentiable on $\mathbb{R}$ and that $f^{(n)}$ is discontinuous at 0.*

Solution. For all $x \neq 0$, $f'(x) = 2nx^{2n-1}\sin(1/x) + x^{2n}\cos(1/x)(-1/x^2) = -x^{2n-2}\cos(1/x) + 2nx^{2n-1}\sin(1/x)$. By induction on k, for any $k \leq n$, for any $x \neq 0$, $f^{(k)}(x) = \sum_{m=2n-2k}^{2n} a_m f_{k,m}$ where $a_{2n-2k} \neq 0$, for every m, $a_m \in \mathbb{R}$, and $f_{k,m}(x) = x^m \sin(1/x)$ or $x^m\cos(1/x)$. By induction on k, for all $k \leq n$, $f^{(k)}(0) = 0$: indeed, if $f^{(k-1)}(0) = 0$, then $f^{(k)}(0) = \lim_{x\to 0} f^{(k-1)}(x)/x$, and for every $m \geq (2n - 2(k-1)) \geq 2$ we have $\lim_{x\to 0} f_{k,m}(x)/x = 0$, so $\lim_{x\to 0} f^{(k-1)}(x)/x = 0$.

$\lim_{x\to 0} f^{(n)}(x)$ doesn't exist: for all $m > 2n - 2n = 0$, $\lim_{x\to 0} f_{n,m}(x) = 0$, but for $m = 0$ this limit doesn't exist.

(b) *Let $n \in \mathbb{N}$ and $f(x) = x^{2n+1}\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is n times differentiable on $\mathbb{R}$, and that $f^{(n)}$ is Lipschitz but not differentiable at 0.*

Solution. As in (a), for any $k \leq n$, $f^{(k)}(x) = \sum_{m=2n+1-2k}^{2n+1} a_m f_{k,m}$, and $f^{(k)}(0) = 0$. $f^{(n)}$ is Lipschitz at 0 since all the functions $f_{n,m}$ with $m \geq 2n+1-2n = 1$ are Lipschitz at 0. But $f^{(n+1)}(0) = \lim_{x\to 0} f^{(n)}(x)/x$ doesn't exist: for all $m > 1$, $\lim_{x\to 0} f_{n,m}(x)/x = 0$, but for $m = 1$ this limit doesn't exist.

31. *If f and g are n times differentiable at a, prove by induction that $(fg)^{(n)}(a) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(a)g^{(n-k)}(a)$ (Leibniz's formula).*

Solution. It would be much easier to get this formula using Taylor polynomials. But ok, induction on n: For $n = 0$, $(fg)^{(0)} = fg = f^{(0)}g^{(0)}$ works. Assume that, for some n, if f and g are n times differentiable at a, then $(fg)^{(n)}(a) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(a)g^{(n-k)}(a)$. Let f and g be $(n+1)$-times differentiable at a, then f and g are n-times differentiable in a neighborhood of a and so, by assumption, $(fg)^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)$ for all x in this neighborhood. Then

$$\begin{aligned}
(fg)^{(n+1)}(a) &= \Big(\textstyle\sum_{k=0}^{n}\binom{n}{k} f^{(k)}g^{(n-k)}\Big)'(a) = \sum_{k=0}^{n}\binom{n}{k}\big(f^{(k+1)}(a)g^{(n-k)}(a) + f^{(k)}(a)g^{(n-k+1)}(a)\big)\\
&= \textstyle\sum_{k=0}^{n}\binom{n}{k} f^{(k+1)}(a)g^{(n-k)}(a) + \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a)\\
&= \textstyle\sum_{k=0}^{n-1}\binom{n}{k} f^{(k+1)}(a)g^{(n-k)}(a) + f^{(n+1)}(a)g^{(0)}(a) + f^{0}(a)g^{(n+1)}(a) + \sum_{k=1}^{n}\binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a)\\
&= \textstyle\sum_{k=0}^{n-1}\binom{n}{k} f^{(k+1)}(a)g^{(n-k)}(a) + f^{(n+1)}(a)g^{(0)}(a) + f^{0}(a)g^{(n+1)}(a) + \sum_{k=0}^{n-1}\binom{n}{k+1} f^{(k+1)}(a)g^{(n-k)}(a)\\
&= \textstyle\sum_{k=0}^{n-1}\Big(\binom{n}{k} + \binom{n}{k+1}\Big) f^{(k+1)}(a)g^{(n-k)}(a) + f^{(n+1)}(a)g^{(0)}(a) + f^{0}(a)g^{(n+1)}(a)\\
&= \textstyle\sum_{k=0}^{n-1}\binom{n+1}{k+1} f^{(k+1)}(a)g^{(n-k)}(a) + f^{(n+1)}(a)g^{(0)}(a) + f^{0}(a)g^{(n+1)}(a)\\
&= \textstyle\sum_{k=1}^{n}\binom{n+1}{k} f^{(k)}(a)g^{(n-k+1)}(a) + f^{(n+1)}(a)g^{(0)}(a) + f^{0}(a)g^{(n+1)}(a) = \sum_{k=0}^{n+1}\binom{n+1}{k} f^{(k)}(a)g^{(n+1-k)}(a).
\end{aligned}$$

32. *Find* $\lim_{x\to 0} \dfrac{\sin(x^2) - x^2}{(\cos x - 1)^3}$.

Solution. I'll use L'Hospital's rule three times (each time checking its applicability :) and the fact that $\lim_{x\to 0} \frac{\sin x}{x} = \sin' 0 = \cos 0 = 1$ and so, also $\lim_{x\to 0} \frac{x}{\sin x} = 1$. First,

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2}{(\cos x - 1)^3} = \lim_{x\to 0} \frac{\cos(x^2)2x - 2x}{3(\cos x - 1)^2(-\sin x)} = \frac{-2}{3} \lim_{x\to 0} \frac{\cos(x^2) - 1}{(\cos x - 1)^2} \lim_{x\to 0} \frac{x}{\sin x} = \frac{-2}{3} \lim_{x\to 0} \frac{\cos(x^2) - 1}{(\cos x - 1)^2}.$$

Further,

$$\lim_{x\to 0} \frac{\cos(x^2) - 1}{(\cos x - 1)^2} = \lim_{x\to 0} \frac{-\sin(x^2)2x}{2(\cos x - 1)(-\sin x)} = \lim_{x\to 0} \frac{-\sin(x^2)2x}{2(\cos x - 1)(-\sin x)} = \lim_{x\to 0} \frac{\sin(x^2)}{\cos x - 1} \lim_{x\to 0} \frac{x}{\sin x}$$
$$= \lim_{x\to 0} \frac{\sin(x^2)}{\cos x - 1}.$$

Finally,

$$\lim_{x\to 0} \frac{\sin(x^2)}{\cos x - 1} = \lim_{x\to 0} \frac{\cos(x^2)2x}{-\sin x} = \lim_{x\to 0} \cos(x^2) \lim_{x\to 0} \frac{2x}{-\sin x} = -2.$$

So, $\lim_{x\to 0} \frac{\sin(x^2) - x^2}{(\cos x - 1)^3} = 4/3$.

33. *Given* $c \in \mathbb{R}$, *find* $\lim_{x\to 0} (1 + cx)^{1/x}$.

Solution.

$$\lim_{x\to 0} (1 + cx)^{1/x} = \lim_{x\to 0} \big((1 + cx)^{1/cx}\big)^c = \big(\lim_{x\to 0} (1 + cx)^{1/cx}\big)^c$$

since the function $z \mapsto z^c$ is continuous. The function $(1 + cx)^{1/cx}$ is the compsition of $x \mapsto cx$ and $y \mapsto (1 + y)^{1/y}$; since (as we know) $\lim_{y\to 0}(1 + y)^{1/y} = e$, we obtain that $\lim_{x\to 0}(1 + cx)^{1/cx} = e$ and $\lim_{x\to 0}(1 + cx)^{1/x} = e^c$.

Another solution.

$$\lim_{x\to 0} \log(1 + cx)^{1/x} = \lim_{x\to 0} \frac{\log(1 + cx)}{x} = \log(1 + cx)'|_{x=0} = c \log' 1 = c.$$

Since exp is continuous,

$$\lim_{x\to 0} (1 + cx)^{1/x} = \lim_{x\to 0} \exp\big(\log(1 + cx)^{1/x}\big) = \exp\big(\lim_{x\to 0} \log(1 + cx)^{1/x}\big) = \exp(c) = e^c.$$