Math 4181H **Solutions to Midterm 4 review problems**

1. (a) *If the function $f\colon [a,b] \longrightarrow \mathbb{R}$ has the property that $U(f,P) = L(f,P)$ for every partition P of $[a,b]$, what can be said about f?*

Solution. f is constant: its supremum and infimum on $[a,b]$ are equal.

(b) *If the function $f\colon [a,b] \longrightarrow \mathbb{R}$ has the property that $U(f,P) = L(f,P)$ for some partition P of $[a,b]$, what can be said about f?*

Solution. On every interval $[x_{i-1}, x_i]$ of P, the supremum and infimum of f are equal, so f is constant on $[x_{i-1}, x_i]$. Since every two adjoint intervals of the partition have nonempty intersection, this implies that f is constant on $[a,b]$. (If, in the definition of $\int$, we used sup and inf on $[x_{i-1}, x_i)$ instead of $[x_{i-1}, x_i]$, f would be only piecewise constant.)

2. *Prove that every monotone function on a closed bounded interval is integrable.*

Solution. W.l.o.g. assume that f is increasing; then f is bounded below by $f(a)$ and above by $f(b)$. Also assume that $f(a) \neq f(b)$, otherwise f is constant. Given $\varepsilon > 0$, put $\delta = \varepsilon/(f(b) - f(a))$. Now if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ with $\operatorname{mesh} P < \delta$, then

$$\Delta(f,P) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Delta x_i \leq \Big(\sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Big)\delta = (f(b) - f(a))\delta = \varepsilon.$$

3. *If $\int_a^b f = p$ and $\int_a^b g = q$, find $\int_a^b \big(\int_a^b f(x)g(y)\,dx\big)\,dy$.*

Solution. $\int_a^b \big(\int_a^b f(x)g(y)\,dx\big)\,dy = \int_a^b g(y)\big(\int_a^b f(x)\,dx\big)\,dy = \int_a^b g(y)p\,dy = p\int_a^b g(y)dy = pq.$

4. *Let f be the Riemann function ($f(x) = \frac{1}{n}$ if $x = \frac{m}{n} \in \mathbb{Q}$ in lowest terms and $f(x) = 0$ if x is irrational). Prove that f is integrable on $[0,1]$ and $\int_0^1 f = 0$.*

Solution. Let $\varepsilon > 0$. There are only finitely many points $a_1, \ldots, a_k$ in $[0,1]$ at which f is $\geq \varepsilon/2$. Choose a partition P for which the total length of the intervals containing at least one of a_i is less than $\varepsilon/2$; then the contribution of these intervals into Δf is $\leq 1 \cdot \varepsilon/2$, and of all other intervals is $< (\varepsilon/2)\cdot 1$. So, $\Delta(f,P) < \varepsilon$. Hence, f is integrable. Since $L(f,Q) = 0$ for all partitions, we have that $\int_0^1 f = L(f) = 0$.

5. *Find integrable functions whose composition is not integrable.*

Solution. Let f be the Riemann function on $[0,1]$, and let g be the indicator function of $\mathbb{R} \setminus \{0\}$, $g(x) = 1$ if $x \neq 0$ and $g(x) = 0$ if $x = 0$. Both f and g are integrable, but $g \circ f$ is Dirichlet's function, $(g \circ f)(x) = 1$ if $x \in \mathbb{Q}$ and $(g \circ f)(x) = 0$ if $x \notin \mathbb{Q}$, which is non-integrable.

6. *Let f be bounded on $[a,b]$.*

(a) *For any partition P of $[a,b]$, prove that $\Delta(|f|,P) \leq \Delta(f,P)$.*

Solution. For any interval (or any set) I and any $x,y \in I$, $|f(x)| - |f(y)| \leq |f(x) - f(y)|$; taking sup of this we get that $\operatorname{Var}_I |f| = \sup_{x,y\in I}(|f(x)| - |f(y)|) \leq \sup_{x,y\in I}|f(x) - f(y)| = |\operatorname{Var}_I f| = \operatorname{Var}_I f$. So, $\Delta(|f|,P) = \sum_i \operatorname{Var}_{[x_{i-1},x_i]} |f| \Delta x_i \leq \sum_i \operatorname{Var}_{[x_{i-1},x_i]} f \Delta x_i = \Delta(f,P)$.

(b) *If f is intergable, prove that so is $|f|$.*

Solution. Given $\varepsilon > 0$, find a partition P for which $\Delta(f,P) < \varepsilon$, then also $\Delta(|f|,P) < \varepsilon$.

Another solution. $|f|$ is a composition of f and the (uniformly) continuous function $y \mapsto |y|$.

(c) *If f and g are integrable on $[a,b]$, prove that so are $\max(f,g)$ and $\min(f,g)$.*

Solution. $\max(f,g) = \frac{1}{2}\big((f+g) + |f-g|\big)$ is integrable since $f+g$ and so $|f+g|$ are integrable. Similarly, integrable is $\min(f,g) = \frac{1}{2}\big((f+g) - |f-g|\big)$.

(d) *Prove that f is integrable iff $f^+ = \max(f,0)$ and $f^- = \min(f,0)$ are integrable.*

Solution. By (c), if f is integrable then f^+ and f^- are integrable. If f^+ and f^- are integrable, then $f = f^+ + f^-$ is integrable.

7. *Suppose that f is locally integrable on $[0,+\infty)$ and $\lim_{x\to+\infty} f(x) = a$. Prove that $\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)\,dt = a$.*
Solution. Let $\varepsilon > 0$; find N such that $|f(t) - a| < \varepsilon/2$ when $t \geq N$. Then for any $x > N$ we have

$$\Big|\frac{1}{x}\int_0^x f(t)\,dt - a\Big| = \frac{1}{x}\Big|\int_0^x f(t)\,dt - ax\Big| = \frac{1}{x}\Big|\int_0^N f(t)\,dt + \int_N^x f(t)\,dt - aN - a(x-N)\Big|$$
$$\leq \frac{1}{x}\Big|\int_0^N f(t)\,dt - aN\Big| + \frac{1}{x}\Big|\int_N^x (f(t)-a)\,dt\Big| \leq \frac{C}{x} + \frac{1}{x}\int_N^x |f(t)-a|\,dt \leq \frac{C}{x} + \frac{x-N}{x}\varepsilon/2,$$

where $C = \big|\int_0^N f(t)\,dt - aN\big|$. Since $(x-N)/x < 1$ and $C/x \to 0$ as $x \to +\infty$, there is M such that $\Big|\frac{1}{x}\int_0^x f(t)\,dt - a\Big| < \varepsilon$ for all $x > M$.

8. *If f is integrable on $[a,b]$, prove that $\int_a^b f(x)\,dx = \int_a^b f(b+a-x)\,dx$.*
Solution. Let $g(x) = g(b+a-x)$, $x \in [a,b]$. For any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$, let $P' = \{a = b+a-x_n, b+a-x_{n-1}, \ldots, b+a-x_0 = b\}$; then $P \leftrightarrow P'$ is a self-bijection of the set of all partitions of $[a,b]$. (It is easy to see that) for every partition P, $L(g,P') = L(f,P)$ and $U(g,P') = U(f,P)$, so $L(g) = L(f)$ and $U(g) = U(f)$, so g is integrable and $\int_a^b g = \int_a^b f$.

9. *Find $\int_0^1 L(x)\,dx$ where L is the Cantor ladder function.*
Solution. L is continuous, so, integrable. By construction, $L(x)$ satisfies $L(1-x) = 1 - L(x)$, $x \in [0,1]$. (This is true for $x \in [1/3, 2/3]$; then for $x \in [1/9, 2/9]$ and $x \in [7/9, 8/9]$, etc., by induction; for all other $x \in [0,1]$ it is true by continuity.) So $1 = \int_0^1 (L(x) + L(1-x))\,dx = \int_0^1 L(x)\,dx + \int_0^1 L(1-x)\,dx = 2\int_0^1 L(x)\,dx$, so $\int_0^1 L(x)\,dx = 1/2$.

10. *If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is periodic with period a and integrable on $[0,a]$, show that for any $b \in \mathbb{R}$, f is integrable on $[b, b+a]$ and $\int_b^{b+a} f = \int_0^a f$.*
Solution. First of all, for any $b, c \in \mathbb{R}$ with $b < c$, if f is integrable on an interval $[b,c]$ then, since $f(x+a) = f(x)$, f is integrable on $[b+a, c+a]$ with $\int_{b+a}^{c+a} f = \int_b^c f$. It follows (by induction) that f is integrable on $[b+ka, c+ka]$ with $\int_{b+ka}^{c+ka} f = \int_b^c f$ for all $k \in \mathbb{Z}$. In particular, f is integrable on $[-ka, ka] = \bigcup_{n=-k}^{n-1} [na, (n+1)a]$ for all $k \in \mathbb{Z}$, and so, is locally integrable on $\mathbb{R}$.

Now let $b \in \mathbb{R}$, $(k-1)a \leq b \leq ka$ with $k \in \mathbb{Z}$. Then

$$\int_b^{b+a} f = \int_b^{ka} f + \int_{ka}^{b+a} f = \int_{b-(k-1)a}^{a} f + \int_0^{b-(k-1)a} f = \int_0^a f.$$

11. *Suppose that f is integrable on $[a,b]$. Prove that there is $c \in [a,b]$ such that $\int_a^c f = \int_c^b f$. Show that it is not always possible to choose c to be in (a,b).*

Solution. Consider the function $H(x) = \int_a^x f - \int_x^b f$, $x \in [a,b]$. H is continuous (it is a difference of two integral functions), $H(a) = -\int_a^b f$, and $H(b) = \int_a^b f$. Thus, by the intermediate value theorem, there exists $c \in [a,b]$ such that $H(c) = 0$.

Take $[a,b] = [-1,1]$ and $f(x) = \operatorname{sign} x$. Then for any $c \in [-1,1]$, $\int_{-1}^c f = -c-1$ for $c \leq 0$ and $c - 1$ for $c \geq 0$, and $\int_c^1 f = c+1$ for $c \leq 0$ and $-c+1$ for $c \geq 0$. Hence, $\int_{-1}^c f = \int_c^1 f$ only if $-c-1 = c+1$ (that is, if $c = -1$) or $c - 1 = -c + 1$ (that is, if $c = 1$).

12. *Use the Fundamental Theorem of Caluclus and Darboux's theorem to give another proof of the Intermediate Value Theorem.*

Solution. If f is a continuous function on $[a,b]$ then f has a primitive: $f = F'$, where F is the integral function of f. By Darboux's theorem, f takes all values between $f(a)$ and $f(b)$.

13. *Let f be a continuous function on $[a,b]$ with the property that $\int_a^b fg = 0$ for all continuous functions g on $[a,b]$ satisfying $g(a) = g(b) = 0$. Prove that $f = 0$.*

Solution. Assume, on the way of contradiction, that $f(x_0) \neq 0$ for some $x_0 \in (a,b)$. (If $f(a) \neq 0$ or $f(b) \neq 0$, then there exists such an x_0 anyway.) W.l.o.g., let $f(x_0) > 0$; find $\delta > 0$ such that $x_0 - \delta > a$, $x_0 + \delta < b$, and $f(x) > f(x_0)/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Construct a continuous function g such that $g(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$ and $g(x) = 0$ otherwise. (Say, $g(x) = \min\{|x - y| : |y - x_0| \geq \delta\}$. Let $\tau > 0$ be such that $g(x) > g(x_0)/2$ for all $x \in [x_0 - \tau, x_0 + \tau]$. Then $\int_a^b fg = \int_{x_0-\delta}^{x_0+\delta} fg \geq \int_{x_0-\tau}^{x_0+\tau} fg > \big(f(x_0)g(x_0)/4\big)2\tau > 0$, contradiction.

14. *Let the function f be integrable and ≥ 0 on an interval $[a,b]$ and suppose there is $c \in [a,b]$ such that f is continuous at c and $f(c) > 0$. Prove that $\int_a^b f > 0$.*

Solution. Assume that $a < c < b$; if $c = a$ or b, by continuity there is $c' \in (a,b)$ such that $f(c') > 0$. Since $f(c) > 0$ and f is continuous at c, there is $\delta > 0$ such that $a \leq c - \delta$, $b \geq c + \delta$, and $f(x) > f(c)/2$ on $[c-\delta, c+\delta]$. Then $\int_{c-\delta}^{c+\delta} f \geq (f(c)/2)2\delta > 0$. Since $f \geq 0$, $\int_a^{c-\delta} f \geq 0$ and $\int_{c+\delta}^b f \geq 0$, so $\int_a^b f = \int_a^c f + \int_c^{c+\delta} f + \int_{c+\delta}^b f > 0$.

15. *Let f be integrable on $[a,b]$, let $c \in (a,b)$, and let $F(x) = \int_a^x f\,dt$, $x \in [a,b]$. Prove or disprove:*

(i) *If f is differentiable at c, then F is differentiable at c.*

Solution. This is true. If f is differentiable at c, then f is continuous at c, so by the F.T.C.1, F is differentiable at c (and $F'(c) = f(c)$).

(ii) *If f is differentiable at c, then F' is continuous at c.*

Solution. F does not have to be differentiable at all points of $[a,b]$. (But since f is integrable, it is continuous at "almost all" points of $[a,b]$, F is differentiable at these points with $F' = f$, so F' restricted to the set of such points is continuous.) Actually, F', defined on the whole set where F is differentiable, is continuous at c. (Which remains true if f is continuous, not necessarily differentiable, at c.) Indeed, for any $\varepsilon > 0$ there exists a neighborhood of c where $|f(t) - f(c)| < \varepsilon$; for any distinct x, y in this neighborhood,

$$\Big|\frac{F(y) - F(x)}{y - x} - f(c)\Big| = \Big|\frac{\int_x^y f(t)\,dt}{y - x} - f(c)\Big| < \varepsilon,$$

so, if F is differentiable at x from this neighborhood then

$$|F'(x) - f(c)| = \Big|\lim_{y \to x} \frac{F(y) - F(x)}{y - x} - f(c)\Big| \leq \varepsilon.$$

(iii) *If f' is continuous at c, then F' is continuous at c.*

Solution. Again, it is not clear whether it is assumed that f' is defined in a neighborhood of c. If it is, then f is continuous in this neighborhood, so F is differentiable and $F' = f$ in this neighborhood, so F' is continuous in this neighborhood. Otherwise, the answer is the same as in (ii): F' is continuous at c, but doesn't have to be defined in any neighborhood of c.

16. *Find $(f^{-1})'(0)$ if $f(x) = \int_0^x (1 + \sin(\sin t))\,dt$.*

Solution. Since the function $1 + \sin(\sin x)$ is continuous, we have $f'(x) = 1 + \sin(\sin x) > 0$ for all x, so f is strictly increasing, and has (a continuous, and so, differentiable) inverse. $f(0) = 0$, so $(f^{-1})'(0) = 1/f'(0) = 1$.

17. (a) *Find the derivatives of $F(x) = \int_1^x \frac{1}{t} dt$ and $G(x) = \int_b^{bx} \frac{1}{t} dt$.*

Solution. F and G are defined for all $x > 0$. Since the function $1/x$ is continuous, by the F.T.C., $F'(x) = 1/x$ and $G'(x) = (1/bx)(bx)' = (1/bx)b = 1/x$. Hence, $F = G + c$ for some $c \in \mathbb{R}$. For $x = 1$ we have $F(1) = G(1) = 0$, so $F = G$.

(b) *Use (a) to prove that for $a > 1$ and $b > 0$, $\int_1^a 1/t\,dt = \int_b^{ab} 1/t\,dt$.*

Solution. For any $a > 0$ and $b \neq 0$, $F(a) = G(a)$.

(c) *For $a, b > 1$ prove that $\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}$.*

Solution. $\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}$.

18. *Let $f(x) = \sin(1/x)$, $x \neq 0$, and $f(0) = 0$. Is the function $F(x) = \int_0^x f$ differentiable at 0?*

Solution. First of all, f is bounded and discontinuous only at 0, so it is locally integrable, and $F(x)$ is defined for every $x \in \mathbb{R}$, with $F(0) = 0$.

Yes, F is differentiable. Indeed, let $G(x) = x^2 \cos(1/x)$, $x \neq 0$, $G(0) = 0$. Then G is differentiable on $\mathbb{R}$ with $G'(x) = 2x \cos(1/x) + \sin(1/x)$ for $x \neq 0$ and $G'(0) = 0$. The function $h(x) = 2x \cos(1/x)$ for $x \neq 0$, $h(0) = 0$ is continuous on $\mathbb{R}$, so it has a primitive H with $H(0) = 0$. We then have $(G - H)'(x) = \sin(1/x) = f(x)$ for all $x \neq 0$ and also $(G - H)'(0) = G'(0) - h(0) = 0 = f(0)$. Hence, $G - H$ is a primitive of f. By the F.T.C., $F = G - H + \text{const}$, so, F is differentiable on $\mathbb{R}$ with $F' = f$.

19. *Prove that for every integer $n \geq 2$, $\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$.*

Solution. Since $\sin^2 = 1 - \cos^2$,

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^{n-2} x\, \sin^2 x\, dx = \int_0^{\pi/2} \sin^{n-2} x\, dx - \int_0^{\pi/2} \sin^{n-2} x \cos^2 x\, dx.$$

Integrating by parts we get

$$\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\, dx = \int_0^{\pi/2} \sin^{n-2} x \cos x\, d\sin x = \tfrac{1}{n-1} \int_0^{\pi/2} \cos x\, d\sin^{n-1} x$$
$$= \tfrac{1}{n-1} \cos x\, \sin^{n-1} x\Big|_0^{\pi/2} - \tfrac{1}{n-1} \int_0^{\pi/2} \sin^{n-1} x\, d\cos x = 0 + \tfrac{1}{n-1} \int_0^{\pi/2} \sin^n x\, dx.$$

Hence, $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^{n-2} x\, dx - \frac{1}{n-1} \int_0^{\pi/2} \sin^n x\, dx$, so $\left(1 + \frac{1}{n-1}\right) \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^{n-2} x\, dx$, so $\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$.

20. *Compute the improper integral $\int_0^1 \log x\, dx$.*

Solution. $\int \log x\, dx = x \log x - \int x\, d\log x = x \log x - \int dx = x \log x - x + C$, so $\int_0^1 \log x\, dx = (\log 1 - 1) - \lim_{a \to 0^+} (a \log a - a) = -1$.

21. *Prove the following version of the integration by parts for improper integrals: If f and g are locally integrable on $[a, \beta)$ (where $\beta \in \mathbb{R} \cup \{+\infty\}$) and have primitives $F = \int f$ and $G = \int g$ on $[a, \beta)$, $\lim_{x \to \beta^-} F(x)G(x)$ exists, Fg has a ptimitive, and $\int_a^\beta F(x)g(x)\, dx$ converges, then $\int_a^\beta f(x)G(x)\, dx = \lim_{x \to \beta^-} F(x)G(x) - F(a)G(a) - \int_a^\beta F(x)g(x)\, dx$.*

Solution. $\int_a^\beta f(x)G(x)\, dx = \lim_{b \to \beta^-} \int_a^b f(x)G(x)\, dx = \lim_{b \to \beta^-} \left(F(x)G(x)\big|_a^b - \int_a^b F(x)g(x)\, dx\right) = \lim_{b \to \beta^-} (F(b)G(b) - F(a)G(a)) - \lim_{b \to \beta^-} \int_a^b F(x)g(x)\, dx = \lim_{b \to \beta^-} F(b)G(b) - F(a)G(a) - \int_a^\beta F(x)g(x)\, dx$.

22. *Determine if the improper integral $\int_0^\infty \sin(x^2)\, dx$ converges.*

Solution. $\int_0^1 \sin(x^2)$ is proper, so we focus on $\int_1^{+\infty} \sin(x^2)$. We have

$$\int \sin x^2 dx = \tfrac{1}{2} \int x^{-1} \sin x^2 d(x^2) = -\tfrac{1}{2} \int x^{-1} d\cos(x^2) = -\tfrac{1}{2} x^{-1} \cos(x^2) + \tfrac{1}{2} \int \cos(x^2) d(x^{-1})$$
$$= -\tfrac{1}{2} x^{-1} \cos(x^2) - \tfrac{1}{2} \int x^{-2} \cos(x^2) dx.$$

So,

$$\int_1^{+\infty} \sin(x^2) dx = \lim_{b \to +\infty} \int_1^b \sin(x^2) dx = -\tfrac{1}{2} \lim_{b \to +\infty} x^{-1} \cos(x^2)\Big|_1^b - \tfrac{1}{2} \int_1^{+\infty} x^{-2} \cos(x^2) dx.$$

Since $\lim_{b \to +\infty} b^{-1} \cos(b^2) = 0$ and $\int_1^{+\infty} x^{-2} \cos(x^2) dx$ converges absolutely, $\int_0^{+\infty} \sin(x^2)$ converges.

23. *Using the fact that $\int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}/2$, find $(-1/2)! = \Gamma(1/2)$ and $(1/2)! = \Gamma(3/2)$.*

Solution. Using the substitution $s = \sqrt{t}$ we calculate $\Gamma(1/2) = \int_0^{+\infty} e^{-t} t^{1/2-1} dt = \int_0^{+\infty} e^{-s^2} s^{-1} d(s^2) = 2 \int_0^{+\infty} e^{-s^2} ds = \sqrt{\pi}$. And then $\Gamma(3/2) = \frac{1}{2} \Gamma(1/2) = \sqrt{\pi}/2$.

24. *Find the Taylor polynomial $P_{a,n,f}$ for*

(a) $f(x) = e^{\sin x}$, $a = 0$, $n = 3$.

Solution. $f(0) = 1$; $f'(x) = e^{\sin x}\cos x$, so $f'(0) = 1$; $f''(x) = e^{\sin x}\cos^2 x - e^{\sin x}\sin x$, so $f''(0) = 1$; $f'''(x) = e^{\sin x}\cos^3 x - e^{\sin x}\cdot 2\cos x\sin x - e^{\sin x}\sin x - e^{\sin x}\cos x$, so $f'''(0) = 0$. Hence, $P_{3,0}(x) = 1 + x + \frac{x^2}{2}$.

Another solution. The 3rd Taylor polynomials for $\sin x$ at 0 is $x - \frac{x^3}{6}$ and for e^y at 0 is $1 + y + \frac{y^2}{2} + \frac{y^3}{6}$, so $P_{3,0}$ is the degree 3 truncation of

$$1 + \big(x - \tfrac{x^3}{6}\big) + \tfrac{1}{2}\big(x - \tfrac{x^3}{6}\big)^2 + \tfrac{1}{6}\big(x - \tfrac{x^3}{6}\big)^3,$$

which is $1 + x - \frac{x^3}{6} + \frac{1}{2}x^2 + \frac{1}{6}x^3 = 1 + x + \frac{1}{2}x^2$.

(b) $f(x) = e^x$, $a = 1$.

Solution. For any n, $(e^x)^{(n)}|_{x=1} = e$, so $P_{n,1}(x) = \sum_{k=0}^{n} \frac{e}{n!}(x-1)^n$.

Another solution. For any n,

$$e^x = ee^{x-1} = e\Big(\sum_{k=0}^{n} \frac{(x-1)^n}{n!} + o((x-1)^n)\Big) = \sum_{k=0}^{n} e\frac{(x-1)^n}{n!} + o((x-1)^n),$$

so $P_{n,1}(x) = \sum_{k=0}^{n} e\frac{(x-1)^n}{n!}$.

(c) $f(x) = x^5 + x^3 + x$, $a = 0$, $n = 4$.

Solution. Since $f(x) = x + x^3 + 0(x^3)$, $P_{3,0}(x) = x + x^3$.

Another solution. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 6$, $f^{(4)}(0) = 0$, so $P_{3,0}(x) = x + \frac{6}{6}x^3 = x + x^3$.

25. (a) *For every $n \in \mathbb{N}$, find the Taylor polynomial $P_{0,4n+2,f}$ for $f(x) = \sin(x^2)$.*

Solution. $P_{0,4n+2,f}(x) = P_{0,2n+1,\sin}(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n\frac{x^{4n+2}}{(2n+1)!}$. (Since $P_{0,2n+1,\sin}(x^2)$ has degree $4n+2$ and satisfies $\sin(x^2) = P_{0,2n+1,\sin}(x^2) + o(x^{4n+2})$.)

(b) *Find $f^{(k)}(0)$ for all k.*

Solution. $f^{(k)}(0) = k!a_k$, where a_k is the coefficient before x^k in the Taylor polynomial for f at 0.

So, $f^{(k)}(0) = \begin{cases} \frac{k!}{(k/2)!} \text{ for } k = 2, 10, 18, \ldots \\ -\frac{k!}{(k/2)!} \text{ for } k = 6, 14, 22, \ldots \\ 0 \text{ otherwise.} \end{cases}$

(c) *In general, if $f(x) = g(x^m)$, find $f^{(k)}(0)$ in terms of the derivatives of g at 0.*

Solution. Let $f(x) = g(x^m)$. If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is the Taylor polynomial of degree n for g at 0, then $g(x) = P(x) + o(x^n)$. This implies that $f(x) = P(x^m) + o((x^m)^n) = P(x^m) + o(x^{nm})$. Hence, $P(x^m) = a_0 + a_1x^m + a_2x^{2m} + \ldots + a_nx^{nm}$ is the Taylor polynomial of degree nm for f at 0. We therefore have $f^{(k)}(0) = \begin{cases} k!a_{k/m} \text{ if } k \text{ is divisible by } m \\ 0 \text{ otherwise} \end{cases} = \begin{cases} \frac{k!}{(k/m)!}g^{(k/m)}(0) \text{ if } k \text{ is divisible by } m \\ 0 \text{ otherwise.} \end{cases}$

26. *Find $P_{0,5,f}$ for*

(a) $f(x) = e^x\sin x$.

Solution. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5)$ and $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$, so $e^x\sin x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} + \frac{x^5}{120} + o(x^5)$, and $P_{0,5,f}(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$.

(b) $f(x) = \tan x$.

Solution. We have $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$, so

$$\tfrac{1}{\cos x} = \tfrac{1}{1 - (\frac{x^2}{2} - \frac{x^4}{24} + o(x^5))} = 1 + \big(\tfrac{x^2}{2} - \tfrac{x^4}{24} + o(x^5)\big) + \big(\tfrac{x^2}{2} - o(x^3)\big)^2 = 1 + \tfrac{x^2}{2} + \tfrac{5x^4}{24} + o(x^5)$$

and

$$\begin{aligned} \tan x = \sin x\tfrac{1}{\cos x} = \big(x - \tfrac{x^3}{6} + \tfrac{x^5}{120} + o(x^5)\big(1 + \tfrac{x^2}{2} + \tfrac{5x^4}{24} + o(x^5)\big) &= x - \tfrac{x^3}{6} + \tfrac{x^5}{120} + \tfrac{x^3}{2} - \tfrac{x^5}{12} + \tfrac{5x^5}{24} + o(x^5) \\ &= x + \tfrac{x^3}{3} + \tfrac{2x^5}{15} + o(x^5). \end{aligned}$$

Hence, $P_{0,5,f}(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}$.

27. *Find* (a) $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x - \sin x}$.

Solution. We have $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$ and $\sin x = x - \frac{1}{6}x^3 + o(x^3)$. So,

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x - \sin x} = \lim_{x\to 0} \frac{\frac{1}{6}x^3 + o(x^3)}{\frac{1}{6}x^3 - o(x^3)} = \lim_{x\to 0} \frac{\frac{1}{6} + o(x^3)/x^3}{\frac{1}{6} - o(x^3)/x^3} = 1.$$

(b) $\lim_{x\to 0}\Big(\frac{1}{\sin^2 x} - \frac{1}{x^2}\Big)$.

Solution.

$$\lim_{x\to 0}\Big(\frac{1}{\sin^2 x} - \frac{1}{x^2}\Big) = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{x^2 - (x - \frac{1}{6}x^3 + o(x^3))^2}{x^2 (x + o(x))^2} = \lim_{x\to 0} \frac{\frac{1}{3}x^4 + o(x^4)}{x^4 + o(x^4)} = \frac{1}{3}.$$

(c) $\lim_{x\to 0}\Big(\frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}\Big)$.

Solution.

$$\lim_{x\to 0}\Big(\frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}\Big) = \lim_{x\to 0} \frac{\sin(x^2) - \sin^2 x}{\sin^2 x \sin(x^2)} = \lim_{x\to 0} \frac{(x^2 + o(x^4)) - (x - \frac{1}{6}x^3 + o(x^3))^2}{(x + o(x))^2 (x^2 + o(x^2))} = \lim_{x\to 0} \frac{\frac{1}{3}x^4 + o(x^4)}{x^4 + o(x^4)} = \frac{1}{3}.$$

28. *Use Taylor polynomials to reprove Leibniz's identity* $(fg)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(n-i)}(a) g^{(i)}(a)$.

Solution. Let f and g be n-times differentiable at a, let $P_{a,n,f}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$ and $P_{a,n,g}(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$. Then

$$\begin{aligned} P_{a,n,fg}(x) &= T_{a,n}\big(P_{a,n,f}(x) P_{a,n,g}(x)\big) \\ &= a_0 b_0 + \big(a_0 b_1 + a_1 b_0\big)(x-a) + \big(a_0 b_2 + a_1 b_1 + a_2 b_0\big)(x-a)^2 + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)(x-a)^n \\ &= \textstyle\sum_{k=0}^n \big(\sum_{i=0}^k a_i b_{k-i}\big)(x-a)^k. \end{aligned}$$

In particular,

$$(fg)^{(n)}(a) = n! \sum_{i=0}^n a_i b_{n-i} = n! \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} \cdot \frac{g^{(n-i)}(a)}{(n-i)!} = \sum_{i=0}^n \frac{n!}{i!(n-i)!} f^{(i)}(a) g^{(n-i)}(a) = \sum_{i=0}^n \tbinom{n}{i} f^{(i)}(a) g^{(n-i)}(a).$$

29. *Let* $f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$ *Taking it for granted that* f *is infnitely differentiable on* $\mathbb{R}$, *find the Taylor polynomial of degree* n *for* f *at 0, and compute* $f^{(n)}(0)$ *for all* n.

Solution. I don't see how to easily prove that f is infinitely differentiable at 0 until we study power series.

We have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n+1)!} + o(x^{n+1})$. Hence,

$$f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{(n+1)!} + o(x^n),$$

so $1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{(n+1)!}$ is the n-th Taylor polynomial of f at 0. This implies that $f'(0) = \frac{1}{2!}$, $f''(0) = 2!\frac{1}{3!} = \frac{1}{3}$, and $f^{(n)}(0) = n!\frac{1}{(n+1)!} = \frac{1}{n+1}$ for all $n \in \mathbb{N}$.

30. *Define* $f(x) = \frac{\log(1+x^2)}{x^2}$ *for* $x \neq 0$ *and* $f(0) = 1$. *Taking it for granted that* f *is infnitely differentiable on* $\mathbb{R}$, *find the derivative* $f^{(100)}(0)$.

Solution. As we know, $\log(1 + x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots + \frac{1}{51}x^{102} + o(x^{102})$, so $f(x) = \log(1 + x^2)/x^2 = 1 - \frac{1}{2}x^2 + \frac{1}{3}x^4 - \cdots + \frac{1}{51}x^{100} + o(x^{100})$. Since f is infnitely differentiable, it follows that $1 - \frac{1}{2}x^2 + \frac{1}{3}x^4 - \cdots + \frac{1}{51}x^{100}$ is 100-th Taylor polynomial of f at 0. So, $f^{(100)}(0) = 100!\frac{1}{51}$.

31. *Using the fact that* $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$, *show that* $\pi = 3.14159...$

Solution. We have $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{2n+1}(x)$, where $|R_{2n+1}(x)| < \frac{|x|^{2n+3}}{2n+3}$ (see Spivak). For $x = \frac{1}{5}$, if we want $R_{2n+1}(x)$ to be less than 10^{-7}, it suffices to take $n = 3$, and $n = 0$ works for $x = \frac{1}{239}$, so that we have $\big|4R_7\big(\frac{1}{5}\big) - R_1\big(\frac{1}{239}\big)\big| < 5 \cdot 10^{-7}$. So, with error $< 5 \cdot 10^{-7}$, we have $\pi/4 \approx 4\big(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{1}{7\cdot 5^7}\big) - \frac{1}{239} \approx 0.78539792$, and so, with error $< 2 \cdot 10^{-6}$, $\pi \approx 3.141591$.