Theorem (Hermite, 1873). *e is transcendental.*

Proof. By the way of contradiction, assume that e is algebraic: let $a_n e^n + \ldots + a_1 e + a_0 = 0$ where $a_i \in \mathbb{Z}$ and $a_0 \neq 0$. For every prime integer p define

$$f_p(x) = \frac{1}{(p-1)!} x^{p-1}(x-1)^p(x-2)^p \ldots (x-n)^p.$$

Then $f_p \in \mathbb{Q}[x]$ (that is, is a polynomial with rational coefficients), $\deg f_p = np + p - 1$, and so, $f_p^{(np+p)} = 0$. Let us find

$$I_p = \sum_{k=0}^{n} \Big(a_k e^k \int_0^k e^{-x} f_p(x) dx \Big).$$

This could be done by integration by parts, but we will compute I_p in a different way. Define $F_p = f_p + f_p' + f_p^{(2)} + \ldots + f_p^{(np+p-1)}$ (the sum of all the nonzero derivatives of f_p). Then $F_p' = F_p - f_p$ and so,

$$\big(e^{-x} F_p(x)\big)' = e^{-x} F_p'(x) - e^{-x} F_p(x) = -e^{-x} f_p(x).$$

For every $k \in \mathbb{N}$ we therefore have

$$e^k \int_0^k e^{-x} f_p(x) dx = -e^{k-x} F_p(x) \Big|_0^k = e^k F_p(0) - F_p(k),$$

and so,

$$I_p = \Big(\sum_{k=0}^{n} a_k e^k \Big) F_p(0) - \sum_{k=0}^{n} a_k F_p(k) = - \sum_{k=0}^{n} a_k F_p(k).$$

I now claim that for all $k = 0, 1, \ldots, n$ we have $F_p(k) \in \mathbb{Z}$, $F_p(0) \equiv (-1)^{np}(n!)^p \bmod p$, and $F_p(k) \equiv 0 \bmod p$ for all $k = 1, \ldots, n$. Indeed, for any $k = 1, \ldots, n$, recalling Leibniz's rule, we see that a nonzero term in the polynomial $f_p^{(l)}(k)$, $l = 1, \ldots, n$, only appears if the factor $(x-k)^p$ is differentiated exactly p-times, which produces the coefficient $p!/(p-1)! = p$. As for $F_p(0)$, a nonzero term in $f_p^{(l)}(0)$ arises only when the factor x^{p-1} is differentiated $p-1$ times, which produces the coefficient $(p-1)!/(p-1)! = 1$, and if we additionally differentiate anything else, the coefficient becomes divisible by p. So, the only term of $F_p(0)$ that can be non-divisible by p is $f_p^{(p-1)}(0) = (-1)^p \ldots (-n)^p$.

It follows that $I_p = -\sum_{k=0}^{n} a_k F_p(k)$ is an integer with $I_p \equiv -a_0(-1)^{np}(n!)^p \bmod p$, and so, $\neq 0$ for if $p > \max\{|a_0|, n\}$. On the other hand, on the interval $x \in [0, n]$ we have $|f_p(x)| \leq \frac{n^{np+p-1}}{(p-1)!}$, so

$$\Big| \int_0^k e^{-x} f_p(x) dx \Big| \leq k \frac{n^{np+p-1}}{(p-1)!} \leq \frac{n^{np+p}}{(p-1)!}$$

and so, $I_p \longrightarrow 0$ as $p \longrightarrow \infty$; contradiction. ■

Theorem (Lindemann, 1882). *π is transcendental.*

The proof of this theorem is harder, it uses some complex analysis and some algebra. We will prove that the number $\alpha = i\pi$ (where $i = \sqrt{-1}$) is transcendental based on the fact that $e^{\alpha} = -1$; this implies (via the theory of fields) that π is also transcendental.

Proof. Assume that α satisfies $g(\alpha) = 0$ for some $g \in \mathbb{Q}[z]$; we may assume that g is monic and irreducible. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the roots of g in $\mathbb{C}$, then by the fundamental theorem of algebra, $g(z) = (z - \alpha_1)\ldots(z - \alpha_n)$. The coefficients of g are rational numbers and are $\pm$ elementary symmetric polynomials in $\alpha_1, \ldots, \alpha_n$.

Let $\beta_1, \ldots, \beta_k$ be all sums of distinct α_i-s: $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1+\alpha_2, \alpha_1+\alpha_3, \ldots, \alpha_1+\alpha_2+\alpha_3, \ldots, \alpha_1+\alpha_2+\ldots+\alpha_n$. Since $e^{\alpha_1} = -1$, $0 = \prod_{i=1}^{n}(e^{\alpha_i}+1) = \sum_{j=1}^{k} e^{\beta_j}+1$, so $\sum_{j=1}^{k} e^{\beta_j} = -1$. Define $G(z) = (z - \beta_1)\ldots(z - \beta_k)$; the coefficients of G are symmetric polynomials in α_i-s and so, by the fundamental theorem of symmetric polynomials, are integer polynomials in the coefficients of g. Hence, $G \in \mathbb{Q}[z]$; let $G(z) = z^k + \frac{d_{k-1}}{d} z^{k-1} + \ldots + \frac{d_0}{d}$ with $d \in \mathbb{N}$ and $d_0, \ldots, d_{k-1} \in \mathbb{Z}$.

For every prime integer p define $g_p(z) = z^{p-1}G(z)^p$, $z \in \mathbb{C}$. For every $l = 0, \ldots, kp+p-1$ the sum $\sum_{j=1}^{k} g_p^{(l)}(\beta_j)$ is a symmetric polynomial in β_j-s of degree $< (k+1)p$, so it is an integer polynomial in the coefficients of G of degree $< (k+1)p$; hence, $d^{(k+1)p}\sum_{j=1}^{k} g_p^{(l)}(\beta_j) \in \mathbb{Z}$ (without even taking into account the additional multipliers appearing as the result of differentiation of the factors of g). As in the proof of Hermite's theorem, for every l and j, for $g_p^{(l)}(\beta_j)$ to be nonzero the factor $(z - \beta_j)^p$ has to be differentiated p times, which produces the additional multiplier $p!$. Put $f_p = \frac{d^{(k+1)p}}{(p-1)!} g_p$ and $F_p = f_p + f_p' + \ldots + f_p^{(kp+p-1)}$, then F_p is a polynomial with integer coefficients such that for every j, $\sum_{j=1}^{k} F_p(\beta_j)$ is an integer divisible by p. And as in the proof of Hermite's theorem, $F_p(0) \in \mathbb{Z}$ is not divisible by p if p is large enough.

As $\big(e^{-z}F_p(z)\big)' = -e^{-z}f_p(z)$, for any $z \in \mathbb{C}$ we have

$$e^{-z}F_p(z) - F_p(0) = -\int_0^z e^{-t} f_p(t)dt = -z\int_0^1 e^{-z\tau} f_p(z\tau)d\tau,$$

and so,

$$F_p(z) - e^z F_p(0) = -z\int_0^1 e^{z(1-\tau)} f_p(z\tau)d\tau.$$

Hence,

$$\sum_{j=1}^{k} F_p(\beta_j) - F_p(0)\sum_{j=1}^{k} e^{\beta_j} = -\sum_{j=1}^{k} \beta_j \int_0^1 e^{\beta_j(1-\tau)} f_p(\beta_j\tau)d\tau.$$

Put $I_p = \sum_{j=1}^{k} F_p(\beta_j) - F_p(0)\sum_{j=1}^{k} e^{\beta_j} = \sum_{j=1}^{k} F_p(\beta_j) + F_p(0)$ (as $\sum_{j=1}^{k} e^{\beta_j} = -1$), then $I_p \in \mathbb{Z}$. Since for large p, $\sum_{j=1}^{k} F_p(\beta_j)$ is divisible by p and $F_p(0)$ is not, $I_p \neq 0$ for large p. On the other hand, as $p \longrightarrow \infty$, $f_p \longrightarrow 0$ locally uniformly, so $\int_0^1 e^{\beta_j(1-\tau)} f_p(\beta_j\tau)d\tau \longrightarrow 0$ for all j, thus $I_p \longrightarrow 0$. This is a contradiction. ■