Geometric And Functional Analysis 1016-443X/94/0600648-12$1.50+0.20/0
Vol. 4, No. 6 (1994)

MULTIPLE RECURRENCE THEOREM FOR NILPOTENT GROUP ACTIONS

A. LEIBMAN

Abstract

We prove that if X is a compact space and $T_1, \dots, T_t$ are homeomorphisms of X generating a nilpotent group, then there exist $x \in X$ and $n_1, n_2, \dots \in \mathbf{N}$ such that $T_j^{n_m} x \underset{m\to\infty}{\longrightarrow} x$ for each $j = 1, \dots, t$.

0. Introduction

0.1. In 1978 H. Furstenberg and B. Weiss ([FW]) published a topological theorem generalizing Birkhoff's recurrence theorem and having interesting combinatorial corollaries (in particular, van der Waerden's theorem about arithmetic progressions). Here is one of its formulations (Birkhoff's theorem corresponds to the case $t = 1$):

THEOREM. *Let X be a compact metric space and let Γ be a commutative group of homeomorphisms of X; let $T_1, \dots, T_t \in \Gamma$. Then there exist $x \in X$ and $n_1, n_2, \dots \in \mathbf{N}$ such that $T_j^{n_m} x \underset{m\to\infty}{\longrightarrow} x$ for each $j = 1, \dots, t$.*

A simple example due to Furstenberg shows that the statement is not, generally speaking, valid when the assumption that Γ is commutative is omitted (see, for example, [F, p. 40]). In Furstenberg's example, the group Γ is solvable and, moreover, metabelian (its commutator is abelian).

0.2. In the dissertation of D. Hendrick ([H]), the following conjecture, due to S. Yuzvinsky, was formulated: the multiple recurrence theorem, Theorem 0.1, holds true for Γ nilpotent. The work of Hendrick demonstrates that this is so in some special cases.

Supported by British Technion Society.

0.3. In this paper we confirm the conjecture of Yuzvinsky. Our proof follows in general that of Furstenberg and Weiss, or its modification in [BlPT]. The main difference is that we use the so-called PET-induction process instead of the ordinary one (on the number t of homeomorphisms under consideration).

This induction was introduced by V. Bergelson in [B]. The author investigated there sequences of transformations (of probability spaces) of the form $T^{p(n)}$, where T is a (measure preserving weakly mixing) transformation and p is a polynomial with rational coefficients taking on integer values on the integers. One can see from the equality

$$(T_1T_2)^n = T_1^nT_2^n[T_1,T_2]^{\frac{n(n-1)}{2}} \text{ in } \Gamma/\big[[\Gamma,\Gamma],\Gamma\big], \quad T_1,T_2 \in \Gamma ,$$

how such polynomial exponents arise in a natural way when a group Γ is noncommutative.

0.4. As a matter of fact, we prove the following "polynomial" generalization of Yuzvinsky's conjecture:

THEOREM. *Let (X,ρ) be a compact metric space, let Γ be a nilpotent group of its homeomorphisms, let $T_1,\dots,T_t \in \Gamma$, let $k \in \mathbb{N}$ and let $p_{i,j}$, $i=1,\dots,k$, $j=1,\dots,t$, be polynomials with rational coefficients taking on integer values at the integers and zero at zero. Then there exist $x \in X$ and $n_1,n_2,\dots \in \mathbb{N}$ such that $T_1^{p_{i,1}(n_m)}\dots T_t^{p_{i,t}(n_m)}x \underset{m\to\infty}{\longrightarrow} x$ for each $i=1,\dots,k$.*

The special case of this theorem corresponding to commutative Γ is proved in [BL].

0.5. In the first section we give some technical definitions concerning nilpotent groups. In the second section we describe the PET-induction used in section 3, where we prove our main result, Theorem 2.2. Theorem 0.4 is a simple corollary of this theorem, this is shown in 4.1. In section 4 we give, in addition, some natural corollaries of this theorem and, in particular, its combinatorial equivalent, Corollary 4.3.

Acknowledgement. I am very grateful to Prof. V. Bergelson for his guidance and help during my work on this paper. I thank also Prof. H. Furstenberg, Prof. V.Ya. Lin and Prof. D. Hendrick for reading this manuscript and giving helpful advice.

1. Γ-polynomials

Γ will always denote a finitely generated nilpotent group without torsion (since every nilpotent group is a factor of a nilpotent group without torsion, we may assume this without loss of generality).

1.1. All the information about nilpotent groups which we need is contained in the following proposition (see, for example, [KM]):

THEOREM. *Let Γ be a finitely generated nilpotent group without torsion. Then there exists a set of elements $\{S_1,\dots,S_s\}$ of Γ (the so called, "Malcev basis") such that:*

1. *for any $1 \le i < j \le s$, $[S_i,S_j]$ belongs to the subgroup of Γ generated by $S_1,\dots,S_{i-1}$;*
2. *every element T of Γ can be uniquely represented in the form*

$$T = \prod_{j=1}^{s} S_j^{r_j(T)}\ , \qquad r_j(T) \in \mathbf{Z},\ j = 1,\dots,s\ ;$$

the mapping $r\colon \Gamma \to \mathbf{Z}^s$, $r(T) = (r_1(T),\dots,r_s(T))$, being polynomial in the following sense: there exist polynomial mappings $R\colon \mathbf{Z}^{2s} \to \mathbf{Z}^s$, $R'\colon \mathbf{Z}^{s+1} \to \mathbf{Z}^s$ such that, for any $T, T' \in \Gamma$ and any $n \in \mathbf{N}$,

$$r(TT') = R\big(r(T), r(T')\big)\ ,\quad r(T^n) = R'\big(r(T), n\big)\ .$$

1.2. Let us introduce some technical terms. We fix from now on a Malcev basis $\{S_1,\dots,S_s\}$ of Γ.

An *integral polynomial* is a polynomial taking on integer values at the integers.

The *group* $\mathrm{P}\Gamma$ is the minimal subgroup of the group $\Gamma^{\mathbf{Z}}$ of the mappings $\mathbf{Z} \to \Gamma$ which contains the constant mappings and is closed with respect to raising to integral polynomials powers: if $g, h \in \mathrm{P}\Gamma$ and p is an integral polynomial, then $gh \in \mathrm{P}\Gamma$, where $gh(n) = g(n)h(n)$, and $g^p \in \mathrm{P}\Gamma$, where $g^p(n) = g(n)^{p(n)}$. The elements of $\mathrm{P}\Gamma$ are called *Γ-polynomials*. Γ itself is a subgroup of $\mathrm{P}\Gamma$ and is presented by the *constant Γ-polynomials*.

Γ-polynomials taking on the value 1_Γ at zero form a subgroup of $\mathrm{P}\Gamma$; we denote it by $\mathrm{P}\Gamma_0$: $\mathrm{P}\Gamma_0 = \{g \in \mathrm{P}\Gamma : g(0) = 1_\Gamma\}$. Note that the Γ-polynomials $T_1^{p_{i,1}(n)}\dots T_t^{p_{i,t}(n)}$, $i = 1,\dots,k$, arising in the formulation of Theorem 0.4 belong to $\mathrm{P}\Gamma_0$.

1.3 LEMMA. *Every Γ-polynomial g can be uniquely represented in the form*

$$g(n) = \prod_{j=1}^{s} S_j^{p_j(n)}\ ,$$

where p_j are integral polynomials.

The proof is evident from the inductive definition of Γ-polynomials: if $g, h \in \mathrm{P}\Gamma$ can be represented in such a form and p is an integral polynomial, Theorem 1.1 shows that gh and g^p also can.

1.4 Remark: Define the *differentiation with step* $a \in \mathbf{Z}$ as the mapping $D_a : \Gamma^{\mathbf{Z}} \to \Gamma^{\mathbf{Z}}$ acting by the rule $D_a(g)(n) = g(n)^{-1}g(n+a)$. A *primitive* of $g \in \Gamma^{\mathbf{Z}}$ with step $a \in \mathbf{Z}$ is $h \in \Gamma^{\mathbf{Z}}$ such that $D_a(h) = g$. It is readily checked that the group of Γ-polynomials $P\Gamma$ is the minimal subgroup of $\Gamma^{\mathbf{Z}}$ closed with respect to taking primitives (recall that Γ is assumed nilpotent). We shall not use this fact in the sequel.

1.5. The *weight*, $w(g)$, of a Γ-polynomial g, $g(n) = \prod_{j=1}^{s} S_j^{p_j(n)}$, is the pair (l,d), $l \in \{0,\dots,s\}$, $d \in \mathbf{Z}_+$, for which $p_j = 0$ for any $j > l$ and, if $l \neq 0$, then $p_l \neq 0$ and $\deg(p_l) = d$. A weight (l,d) *is greater* than a weight (k,c) if $l > k$ or $l = k$, $d > c$.

EXAMPLE: The Γ-polynomial $S_1^{n^5+2n^2+3}S_2^{\frac{1}{2}(n^7+n)+1}$ has weight $(2,7)$, the Γ-polynomial $S_1^{n^2+7}S_4^{n+15}$ has weight $(4,1)$ which is greater than $(2,7)$.

1.6. The ordering described in 1.5 defines a well ordered structure on the set W of all weights; the set Φ of functions $W \to \mathbf{Z}_+$ having finite support also gets an ordering:

$$\text{for } \varphi_1,\varphi_2 \in \Phi\ ,\quad \varphi_1 \succ \varphi_2 \ \text{ if there exists } w \in W \text{ such that}$$
$$\varphi_1(w) > \varphi_2(w) \text{ and } \varphi_1(w') = \varphi_2(w')\ \forall w' > w$$

and is well ordered with respect to it.

Let us write down functions $\varphi\colon W \to \mathbf{Z}_+$ with finite supports in the form of a list $(\varphi(w_1)w_1,\dots,\varphi(w_p)w_p)$, where $w_1 > \dots > w_p$ and $\varphi(w) = 0$ for $w \notin \{w_1,\dots,w_p\}$. Then, for $a_q \neq b_q$,

$$(a_1w_1,\dots,a_{q-1}w_{q-1},a_qw_q,\dots,a_pw_p)\succ(a_1w_1,\dots,a_{q-1}w_{q-1},b_qw_q,\dots,b_pw_p)$$

if and only if $a_q > b_q$.

1.7. Let us now define an equivalence relation on $P\Gamma$: g, $g(n)=\prod_{j=1}^{s} S_j^{p_j(n)}$, is equivalent to h, $h(n) = \prod_{j=1}^{s} S_j^{q_j(n)}$, if $w(g) = w(h)$ and, if it is (l,d), the leading coefficients of the polynomials p_l and q_l coincide; we write then $g \sim h$. The *weight* of an equivalence class is the weight of any of its elements.

EXAMPLE: $S_1^nS_3^{5n^{12}+n+3}$ is equivalent to $S_1^{n^{100}+2n^{13}+4}S_2^{n+3}S_3^{5n^{12}+7n^{11}+8n^{10}-9}$.

2. Systems and the PET-induction

2.1. A *system* is a finite subset of $\mathrm{P}\Gamma$.

For every system A we define its *weight vector* $\varphi(A) \in \Phi$:

$$\varphi(A)(w) = \left\{ \begin{array}{l} \text{the number of the equivalence classes} \\ \text{of weight } w \text{ which contain elements of } A. \end{array} \right.$$

(in the notation of 1.5 and 1.7).

A system A' *precedes* a system A if $\varphi(A) \succ \varphi(A')$.

EXAMPLE: The weight vector of the system $\{S_1^{13n+1} S_1^{6n^2+2}, S_1^{7n^2+7n}, S_1^{7n^2+4}, S_1^{4n^4-9} S_2^{n^2+2}, S_1^{n^3+12} S_2^{n^2+2n}, S_1^{25n^{25}+3n^3} S_2^{2n^3+3n-8}, S_1^{8n^{10}} S_2^{n^3+4n^2+4n+1}, S_2^{n^3+2n+2}, S_1^{n^5+5} S_2^{n^3+2n^2+3}\}$ is $(1(1,1), 2(1,2), 1(2,2), 2(2,3))$.

2.2. A system A is called *a system of recurrence* if for any compact topological space X provided with a continuous action of Γ and minimal with respect to this action, and any open $U \subseteq X$ there exist $n \in \mathbf{N}$ such that

$$U \cap \Big(\bigcap_{g \in A} g(n)^{-1} U \Big) \neq \emptyset\,.$$

We shall prove the following topological version of Theorem 0.4.

THEOREM. *Every system contained in* $\mathrm{P}\Gamma_0$ *is a system of recurrence.*

2.3. The *PET-induction* we use in the proving of Theorem 2.2 is an induction along the well ordered set Φ of the weight vectors. That is, we shall show that a system $A \subset \mathrm{P}\Gamma_0$ is a system of recurrence assuming that all the systems preceding A and contained in $\mathrm{P}\Gamma_0$ are systems of recurrence.

The beginning of the induction process is clear: the system of minimal (zero) weight contained in $\mathrm{P}\Gamma_0$ consists only of the identity, and the statement is trivial for this system. The following lemma (or, more exactly, its corollary) is the main tool used in the PET-induction.

2.4 LEMMA. *Let g be a Γ-polynomial.*

(i) *If h is a $\Gamma-$polynomial and $g' = h^{-1}gh$, then $g' \sim g$.*

(ii) *If $m \in \mathbf{N}$ and g' is defined by $g'(n) = g^{-1}(m)g(n+m)$, then $g' \sim g$.*

(iii) a)*If g', h are Γ-polynomials such that $g' \sim g$, $h \not\sim g$ and $w(h) \leq w(g)$, then $g'h^{-1} \sim gh^{-1}$ and $w(gh^{-1}) = w(g)$.*

b) *If $h \neq 1_\Gamma$ is a Γ-polynomial such that $h \sim g$, then $w(gh^{-1}) < w(g)$.*

Proof: For every $i = 0, \ldots, s$, denote by Γ_i the subgroup of Γ generated by $S_1, \ldots, S_i$.

Let $g(n) = \tilde{g}(n)S_i^{p(n)}$, $\tilde{g}(n) \in \Gamma_{i-1}$ $\forall n \in \mathbb{Z}$, and let $p \neq 0$ be an integral polynomial; then $w(g) = (i, \deg(p))$.

(i) Here

$$g'(n) = h(n)^{-1}g(n)h(n) = \big([h(n), g(n)^{-1}]\tilde{g}(n)\big)S_i^{p(n)} .$$

Since, by item 1 of Theorem 1.1, $[\Gamma, \Gamma_i] \subseteq \Gamma_{i-1}$, the expression in the large parentheses belongs to Γ_{i-1} for any $n \in \mathbb{Z}$; hence, $g' \sim g$.

(ii) Here

$$\begin{aligned} g'(n) &= g^{-1}(m)g(n+m) = S_i^{-p(m)}\tilde{g}(m)^{-1}\tilde{g}(n+m)S_i^{p(n+m)} \\ &= \big([S_i^{p(m)}, \tilde{g}(n+m)^{-1}\tilde{g}(m)]\tilde{g}^{-1}(m)\tilde{g}(n+m)\big)S_i^{p(n+m)-p(m)} \quad \forall n \in \mathbb{Z} . \end{aligned}$$

Since the expression in the large parentheses belongs to Γ_{i-1} for any $n \in \mathbb{Z}$ and the polynomial $\tilde{p}(n) = p(n+m) - p(m)$ has the same degree and the same leading coefficient as $p(n)$, we have $g' \sim g$.

(iii) Let $h(n) = \tilde{h}(n)S_i^{q(n)}$, $g'(n) = \tilde{g}'(n)S_i^{p'(n)}$, where $\tilde{h}(n), \tilde{g}'(n) \in \Gamma_{i-1}$. Then, similar to that in (i) and (ii),

$$g(n)h(n)^{-1} = f(n)S_i^{p(n)-q(n)} , \quad g'(n)h(n)^{-1} = f'(n)S_i^{p'(n)-q(n)} ,$$

where $f(n), f'(n) \in \Gamma_{i-1}$ $\forall n \in \mathbb{Z}$.

a) When $h \not\sim g$, one has either $\deg(q) < \deg(p)$ or the leading coefficients of p and q are different; then $\deg(p-q) = \deg(p)$ and, so, $w(gh^{-1}) = (i, \deg(p))$. Since $g' \sim g$, the degrees and the leading coefficients of p and p' coincide; the same holds true for the polynomials $p - q$ and $p' - q$ and, hence, $g'h^{-1} \sim gh^{-1}$.

b) When $h \sim g$, the degrees and the leading coefficients of p and q coincide; hence, $\deg(p-q) < \deg(p)$ and, so, $w(gh^{-1}) < (i, \deg(p))$. □

2.5 Corollary. *Let A be a system.*

(i) *If A' is a system consisting of Γ-polynomials of the form $g' = h^{-1}gh$ for $g \in A$ and h being a Γ-polynomial, then $\varphi(A') \preceq \varphi(A)$.*

(ii) *If A' is a system consisting of Γ-polynomials g' satisfying the equality $g'(n) = g^{-1}(m)g(n+m)$ for some $g \in A$ and some $m \in \mathbb{N}$, then $\varphi(A') \preceq \varphi(A)$.*

(iii) *Let $h \in A$, $h \neq 1_\Gamma$, be a Γ-polynomial of weight minimal in A: $w(h) \leq w(g)$ for any $g \in A$. If A' is a system consisting of Γ-polynomials of the form $g' = gh^{-1}$, $g \in A$, then $\varphi(A') \prec \varphi(A)$.*

Proof: In both (i) and (ii), all the elements of A' are equivalent to some elements of A.

In (iii), the equivalence classes whose elements are members of A change when we pass to A', but the equivalence of elements is preserved and their weights remain the same. The only exception is the equivalence class containing h; it is replaced by equivalence classes, having smaller weights. □

3. Proof of Theorem 2.2

From now on we fix a compact topological space X and a continuous action of Γ on X, assuming that X is minimal with respect to this action. All the Γ-polynomials we shall deal with will belong to $P\Gamma_0$, and we shall not mention this specifically.

After all these preparations, the proof of Theorem 2.2 is not difficult.

3.1. Fix an open $U \subseteq X$. We have to find $x \in U$ and $n \in \mathbf{N}$ such that $g(n)x \in U$ for every $g \in A$.

As $X$is assumed minimal, there exist $R_1, \dots, R_c \in \Gamma$ for which

$$\bigcup_{k=1}^{c} R_r^{-1}(U) = X .$$

Put $U_r = R_r^{-1}(U)$, $r = 1, \dots, c$. Define a system A' by

$$A' = \{R_r^{-1} g R_r \ , \ \ g \in A \ , \ r = 1, \dots, c\} .$$

By Corollary 2.5, $\varphi(A') = \varphi(A)$.

Let $h \in A$ be an element whose weight is minimal in A'; we may assume that A and, hence, A' do not contain constant Γ-polynomials and, therefore, $h \neq \mathbf{1}_\Gamma$.

3.2. Put $r_0 = 1$ and

$$A_0 = \{f : f(n) = g(n)h(n)^{-1} \ , \ g \in A\} .$$

Due to Corollary 2.5, A_0 precedes A' and, so, precedes A. Hence, there exist $x_0 \in U_{r_0}$ and $n_1 \in \mathbf{N}$ such that $f(n_1)x_0 \in U_{r_0}$ for every $f \in A_0$. We put $x_1 = h(n_1)^{-1}x_0$; then, for any $g \in A'$,

$$g(n_1)x_1 \in U_{r_0} .$$

3.3. Let $1 \leq r_1 \leq c$ be such that $x_1 \in U_{r_1}$. We put now

$$V_1 = U_{r_1} \cap \left(\bigcap_{g \in A'} g(n_1)^{-1}(U_{r_0}) \right) ;$$

it is open and nonempty since $x_1 \in V_1$.

We define a new system:

$$A_1 = \left\{ f: \ f(n) = \begin{cases} g(n)h(n)^{-1} \\ g(n_1)^{-1}g(n+n_1)h(n)^{-1} \end{cases} , \quad g \in A' \right\} .$$

By Corollary 2.5, A_1 precedes A. Hence, there exist $y_1 \in V_1$ and $n_2 \in \mathbf{N}$ such that $f(n_2)y_1 \in V_1$ for every $f \in A_1$. We put $x_2 = h(n_2)^{-1}y_1$; then, for any $g \in A'$,

$$g(n_2)x_2 \in V_1 \subseteq U_{r_1}$$

$$g(n_1)^{-1}g(n_2+n_1)x_2 \in V_1 \implies g(n_2+n_1)x_2 \in U_{r_0} .$$

3.4. We continue this process: assume that points $x_0, \ldots, x_i$ and numbers $n_1, \ldots, n_i$ and $r_0, \ldots, r_{i-1}$ have been already chosen and are such that, for any $0 \leq l < m \leq i$ and any $g \in A'$,

$$x_m \in U_{r_m} , \quad g(n_m + \ldots + n_{l+1})x_m \in U_{r_l} . \tag{3.1}$$

Let $1 \leq r_i \leq c$ be such that $x_i \in U_{r_i}$.

We put now

$$V_i = U_{r_i} \cap \left(\bigcap_{j=0}^{i-1} \bigcap_{g \in A} g(n_i + \ldots + n_{j+1})^{-1}(U_{r_j}) \right) ;$$

since V_i contains x_i, it is nonempty. We next define a new system A_i by

$$A_i = \left\{ f : f(n) = \begin{cases} g(n)h(n)^{-1} \\ g(n_i)^{-1}g(n+n_i)h(n)^{-1} \\ \vdots \\ g(n_i + \ldots + n_1)^{-1}g(n + n_i + \ldots + n_1)h(n)^{-1} \end{cases} , g \in A' \right\};$$

By Corollary 2.5, A_i precedes A; so, there exist $y_i \in V_i$ and $n_{i+1} \in \mathbf{N}$ such that $f(n_{i+1})y_i \in V_i$ for every $f \in A_i$. We put $x_{i+1} = h(n_{i+1})^{-1}y_i$; then, for any $g \in A'$,

$$g(n_{i+1})x_{i+1} \in V_i \subseteq U_{r_i}$$

$$g(n_i)^{-1}g(n_{i+1}+n_i)x_{i+1} \in V_i \implies g(n_{i+1}+n_i)x_{i+1} \in U_{r_{i-1}} .$$

$$\vdots$$

$$g(n_i + \ldots + n_1)^{-1}g(n_{i+1} + \ldots + n_1)x_{i+1} \in V_i \implies$$
$$g(n_{i+1} + \ldots + n_1)x_{i+1} \in U_{r_0} .$$

3.5. We continue this process up to $i = c$; then, since $r_i \leq c$ for every $i = 0, \dots, c$, we have $r_l = r_m$ for some $0 \leq l < m \leq c$. We put

$$x = R_{r_l} x_m \in U \ , \quad n = n_m + \dots + n_{l+1} \ .$$

Then, since for any $g \in A$ one has $R_{r_l}^{-1} g R_{r_l} \in A'$, by (3.1) we have

$$R_{r_l}^{-1} g(n) x = R_{r_l}^{-1} g(n) R_{r_m} x_m = R_{r_l}^{-1} g(n) R_{r_l} x_m \in U_{r_l} = R_{r_l}^{-1}(U)$$

for every $g \in A$. Hence, for any $g \in A$, we obtain

$$g(n) x \in U \ . \qquad \square$$

4. Corollaries and Generalizations

4.1. Since any compact topological space contains a minimal closed invariant subset with respect to any set of its homeomorphisms, it suffices to prove Theorem 0.4 for such a minimal subspace only. In this case we have a stronger statement:

COROLLARY. *Let (X, ρ) be a compact matrix space, let Γ be a nilpotent group of its homeomorphisms, such that (X, Γ) is minimal with respect to the action of Γ, let $T_1, \dots, T_t \in \Gamma$, let $k \in \mathbf{N}$ and let $p_{i,j}$, $i = 1, \dots, k$, $j = 1, \dots, t$, be polynomials with rational coefficients taking on integer values at the integers and zero at zero. Then there exists a residual set $Y \subseteq X$ such that for any $x \in Y$ there exists a sequence $n_1, n_2, \dots \in \mathbf{N}$ such that $T_1^{p_{i,1}(n_m)} \dots T_t^{p_{i,t}(n_m)} x \underset{m \to \infty}{\longrightarrow} x$, for each $i = 1, \dots, k$.*

Proof: Define Γ-polynomials $g_i(n) = T_1^{p_{i,1}(n)} \dots T_t^{p_{i,t}(n)}$, $i = 1, \dots, k$, and a system $A = \{g_1, \dots, g_k\} \subset \mathrm{P}\Gamma_0$. Define a function δ on X by

$$\delta(x) = \inf_{n \in \mathbf{N}} \left(\max_{g \in A} \rho(g(n)x, x) \right) \ .$$

Theorem 2.2 gives that, for any $\varepsilon > 0$, the points x for which $\delta(x) < \varepsilon$ are dense in X. As an upper-semicontinuous function, δ has a residual set of points of continuity and δ vanishes at every such point. $\square$

4.2. The IP-set generated by a sequence $\{s_i\}_{i\in\mathbf{N}}$ is the set

$$\left\{\sum_{i\in F} s_i \ , \ F \in \mathbf{N} \ , \#F < \infty\right\} .$$

A subset of $\mathbf{N}$ is called an *IP^*-set* if it has nonempty intersection with every IP-set.

COROLLARY (of the proof of Theorem 2.2). *Let Γ be a nilpotent group acting on a compact topological space X and let X be minimal with respect to this action. Let $A \subset \mathrm{P}\Gamma_0$ be a system of Γ-polynomials without constant parts: $g(0) = \mathbf{1}_\Gamma$ for $g \in A$. Then, for any open $U \subseteq X$, the set $P = \{n : g(n)U \cap \left(\bigcap_{g\in A} g(n)^{-1}U\right) \neq \emptyset\}$ is an IP^*-set.*

Indeed, it is easy to see that, for any IP-set S, the numbers n_i in the proof 3.2–3.4 of Theorem 2.2 can be chosen from S so that the resulting integer n will be in S itself. This shows that $P \bigcap S \neq \emptyset$.

4.3. As with to Theorem 0.1 of Furstenberg and Weiss, Theorem 0.4 has an equivalent combinatorial formulation:

COROLLARY. *Let Γ be a nilpotent group and let $A \in \mathrm{P}\Gamma_0$ be a system of Γ-polynomials without constant parts. Let $k \in \mathbf{N}$, and let $\eta : \Gamma \to \{1, \dots, k\}$ be a mapping. Then there exist $T \in \Gamma$ and $n \in \mathbf{N}$ such that η is constant on the set $\{Tg(n), g \in A\}$.*

Proof: We may assume that Γ is finitely generated. Let $B_1 \subseteq B_2 \subseteq \dots$ be a chain of finite subsets of Γ such that $\bigcup_{m=1}^{\infty} B_i = \Gamma$. Define on the set $\mathbf{K} = \{1, \dots, k\}^\Gamma$ of all mappings of Γ to $\{1, \dots, k\}$ a metric ρ: for $\chi_1, \chi_2 \in \mathbf{K}$ we put

$$\rho(\chi_1, \chi_2) = \left(\max\{m\colon \ \chi_1|_{B_m} = \chi_2|_{B_m}\}\right)^{-1} .$$

Then $(\mathbf{K}, \rho)$ is a metric compact, and Γ acts on $\mathbf{K}$ by

$$(T\chi)(S) = \chi(ST) \ , \quad \chi \in \mathbf{K} \ , \quad T, S \in \Gamma \ .$$

Let X denote the closure of the orbit of η in $\mathbf{K}$: $X = \overline{\{T\eta, \ T \in \Gamma\}}$. Then, by Theorem 0.4 , there exist $\chi \in X$ and $n \in \mathbf{N}$ such that $\rho(\chi, g(n)\chi) < 1$ for every $g \in A$. This means that $\chi(\mathbf{1}_\Gamma) = \chi(g(n))$ for every $g \in A$.

Let $m \in \mathbf{N}$ be such that $\{\mathbf{1}_\Gamma, g(n) : g \in A\} \subseteq B_m$. Find $T \in \Gamma$ for which $\rho(T\eta, \chi) < m^{-1}$; then, for every $g \in A$,

$$\eta\big(g(n)T\big) = T\eta\big(g(n)\big) = \chi\big(g(n)\big) = \chi(\mathbf{1}_\Gamma) = T\eta(\mathbf{1}_\Gamma) = \eta(T) \ . \qquad \square$$

4.4 EXAMPLE: Fix $t \in \mathbf{N}$ and consider the following set of affine transformations of $\mathbf{Z}^t$:

$$R_{i,j}(x_1,\dots,x_j,\dots,x_i,\dots,x_t) = (x_1,\dots,x_j,\dots,x_i+x_j,\dots,x_t) \qquad 1 \le j < i \le t\,,$$

$$S_i(x_1,\dots,x_i,\dots,x_t) = (x_1,\dots,x_i+1,\dots,x_t)\,, \quad 1 \le i \le t\,.$$

Then

$$R_{i,1}^{d_1}\dots R_{i,i-1}^{d_{i-1}}S_i^d(x_1,\dots,x_i,\dots,x_t) = (x_1,\dots,x_i+d_1x_1+\dots+d_{i-1}x_{i-1}+d,\dots,x_t)$$

for any $d_1,\dots,d_t$, $d \in \mathbf{Z}$. The transformations $R_{i,j}$, $1 \le j < i \le t$, S_i, $1 \le i \le t$, generate a nilpotent group; denote it by Γ.

Let $\mathbf{Z}^t = C_1 \cup \dots \cup C_k$ be a partition of $\mathbf{Z}^t$; define a mapping $\eta : \Gamma \to \{1,\dots,k\}$ by

$$\eta(T) = i \Rightarrow T(0,\dots,0) \in C_i\,,$$

Let $p_{i,j}$, $1 \le j < i \le t$, p_i, $1 \le i \le t$, be polynomials with rational coefficients taking on integer values at the integers and zero at zero. Now, Corollary 4.3, applying to the described η and the system

$$A = \left\{ g_i = \left(\prod_{j=1}^{i-1} R_{i,j}^{p_{i,j}} \right) S_i^{p_i}\,, \quad i = 1,\dots,r \right\}$$

gives the following van der Wearden type theorem:

With the assumptions above, there exist $x = (x_1,\dots,x_t) \in \mathbf{Z}^t$ and $n \in \mathbf{N}$ such that x and all points

$$\left(x_1,\dots,x_i + \sum_{j=1}^{i-1} p_{i,j}(n)x_j + p_i(n),\dots,x^t \right) \quad \text{for} \quad i = 1,\dots,t$$

belong to the same set C_m.

4.5. Since in the formulation and the proof of Theorem 0.4 we dealt with only finite numbers of elements of Γ, we could assume that Γ is *locally nilpotent* (that is, only its finitely generated subgroups are nilpotent) instead of requiring Γ to be nilpotent.

4.6. It was noted by V. Bergelson that the conclusion of Theorem 0.4 also holds true for the groups containing nilpotent subgroups of finite index, that is, for the groups of *polynomial growth*.

4.7. Let us define a *nilpotent semigroup* in the following way: a semigroup Γ is nilpotent if there exists a finite chain of its subsets $\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_s = \Gamma$ such that, for any $1 \leq i,j \leq s$ and any $T \in \Gamma_i$, $S \in \Gamma_j$, one has

$$TS = STR\,, \quad \text{where} \quad R \in \Gamma_{\min(i,j)-1}\,.$$

Let Γ be a nilpotent semigroup of continuous mappings of a compact topological space. Then the statement of Theorem 0.4 is valid for such Γ as well; this can be ascertained by a method similar to that used in the commutative situation (see, for example, Theorem 2.6 in [F]).

References

[B] V. Bergelson, Weakly mixing PET, Ergod. Th. and Dynam. Sys. 7 (1987), 337-349.

[BL] V. Bergelson, A. Leibman, Polynomial extensions of van der Waerden's and Szemerèdi's theorems, submitted.

[BlPT] A. Blaszczyk, S. Plewik, S. Turek, Topological multidimensional van der Waerden theorem, Comment. Math. Univ. Carolinae 30:4 (1989), 783-787.

[F] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, 1981.

[FW] H. Furstenberg, B. Weiss, Topological dynamics and combinatorial number theory, J. d'Analyse Math. 34 (1978), 61-85.

[H] D. Hendrick, Ph.D. Thesis, University of Oregon (1989).

[KM] M. Kargapolov, Ju. Merzljakov, Fundamentals of the Theory of Groups, Springer Verlag, New York, Heidelberg, Berlin, 1979.

A. Leibman
Department of Mathematics
Technion
Haifa 32000, Israel

Submitted: May 1994
Final version: August 1994