Lower bounds for ergodic averages

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Abstract

We compute the exact lower bounds for some averages arising in ergodic theory. In particular, we prove that for any measure preserving system $(X,\mathcal{B},\mu,T)$ with $\mu(X)<\infty$, any $A\in\mathcal{B}$ and any $N\in\mathbb{N}$, $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A)\geq\sqrt{\mu(A)^2+(\mu(X)-\mu(A))^2}+\mu(A)-\mu(X)$.

1. Lower bound for the averages $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A)$

1.1. Let T be a measure preserving transformation of a probability measure space $(X,\mathcal{B},\mu)$. Let $0<a\leq 1$; it follows from the mean ergodic theorem that if A is a subset of X with $\mu(A)\geq a$, then the limit of the averages

$$\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A) \tag{1.1}$$

exists and satisfies $\lim\limits_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A)\geq a^2$ ([Kh]). This does not apriori guarantee that there is a uniform positive lower bound of the averages (1.1) for all A with $\mu(A)\geq a$, that is, that there is $c=c(a)>0$ such that for any X, T and A with $\mu(A)\geq a$ one has $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A)\geq b$ for all $N\in\mathbb{N}$. Indeed, for the more general expressions $\frac{1}{N-M}\sum_{n=M}^{N-1}\mu(A\cap T^{-n}A)$ one still has $\lim\limits_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}\mu(A\cap T^{-n}A)\geq a^2$ ([Kh]), while, if $a<\frac{1}{2}$, for arbitrarily large $N-M$ one may have $\frac{1}{N-M}\sum_{n=M}^{N-1}\mu(A\cap T^{-n}A)=0$ for appropriately chosen T, A and M. (For example, take $X=[0,1]$, $A=[0,a]$ with $a<\frac{1}{2}$ and $T(x)=(x+\alpha)\bmod 1$ with $\alpha\ll 1-2a$; then there are large intervals of n for which $\mu(A\cap T^{-n}A)=\emptyset$.)

The existence of positive lower bound for averages of the form $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T_1^{-n}A\cap\ldots\cap T_k^{-n}A)$, where $T_1,\ldots,T_k$ are pairwise commuting measure preserving transformations of X, is proven in [BHMP]. We compute the exact lower bound of the averages (1.1):

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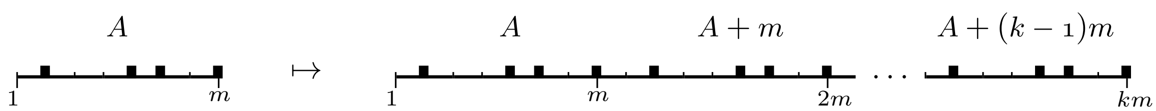
1.2. Theorem. *Let* $0 \le a \le 1$*.*
(a) *For any probability measure preserving system* $(X, \mathcal{B}, \mu, T)$ *and any* $A \in \mathcal{B}$ *with* $\mu(A) \ge a$ *one has* $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A \cap T^{-n}A) \ge \sqrt{a^2 + (1-a)^2} + a - 1$ *for all* $N \in \mathbb{N}$*.*
(b) *For any* $\delta > 0$ *there exist a measure preserving system* $(X, \mathcal{B}, \mu, T)$*,* $A \in \mathcal{B}$ *with* $\mu(A) = a$ *and* $N \in \mathbb{N}$ *such that* $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A \cap T^{-n}A) < \sqrt{a^2 + (1-a)^2} + a - 1 + \delta$*.*

Proof. Passing, if needed, to the natural extension of $(X, \mathcal{B}, \mu, T)$ ([R]), we may assume that T is invertible. We may also assume that X is finite with $\mu(B) = |B|/|X|$, $B \in \mathcal{B}$. Indeed, given $A \in \mathcal{B}$, $\mu(A) = a$, for any $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite set $\hat{X}$, a permutation $\hat{T}$ of $\hat{X}$ and a set $\hat{A} \subseteq \hat{X}$ such that $\left|\frac{|\hat{A}|}{|\hat{X}|} - a\right| < \varepsilon$ and $\left|\frac{|\hat{A} \cap \hat{T}^{-n}\hat{A}|}{|\hat{X}|} - \mu(A \cap T^{-n}A)\right| < \varepsilon$ for all $n \le N$. (One can deduce this fact from the Rohlin lemma, or prove it directly.) Thus, we arrive at the following problem: given a permutation T of a finite set X, a subset A of X with $|A| = a|X|$ and $N \in \mathbb{N}$, we have to estimate $\frac{1}{N|X|}\sum_{n=0}^{N-1}|A \cap T^{-n}A|$.

First, let us assume that T is a cyclic permutation: $X = \{1, \dots, m\}$ and $Tx = (x \bmod m) + 1$. Let $A \subseteq \{1, \dots, m\}$ with $|A| = b = ma$. For any $k \in \mathbb{N}$, if we replace X by $\{1, \dots, km\}$ and A by $A \cup (A + m) \cup \ldots \cup (A + (k-1)m)$:

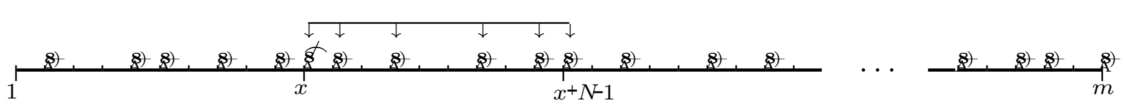


then the quantities $|A|/|X|$ and $|A \cap T^{-n}A|/|X|$, $n \in \mathbb{Z}$, do not change. Hence, we may assume that m is arbitrarily large. Fix $\varepsilon > 0$ and assume that $N/m < \varepsilon$. Under this assumption, we will estimate from below the sum

$$S = \sum_{n=0}^{N-1}\left|A \cap (A-n)\right| = \sum_{x \in A}\left|A \cap [x, x+N-1]\right|,$$

which does not exceed $\sum_{n=0}^{N-1}|A \cap T^{-n}A|$.

To make the argument more transparent, let us reformulate the problem in combinatorial language. Assume that b *archers* are positioned at the points $1, 2, \dots, m$ of the real line, no more than one archer at a point: there is an archer at x iff $x \in A$. Every archer *threatens* himself and all other archers positioned at his right at the distance $< N$. (That is, the archer located at a point x threatens the archers located in the interval $[x, x+N-1]$.)

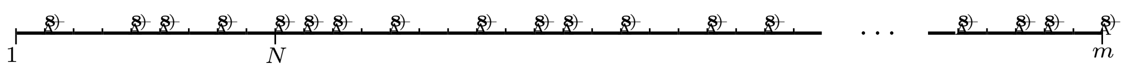


The question is: how should one position the archers in order to minimize "the total number of threats"

$$S = \sum_{R \text{ is an archer}} \text{the number of archers threatened by } R,$$

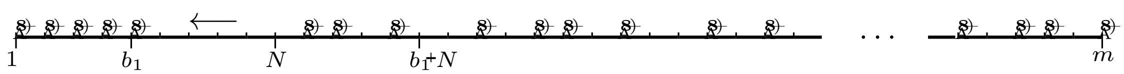
and what is the minimal value of S?

We start with an arbitrary positioning of archers

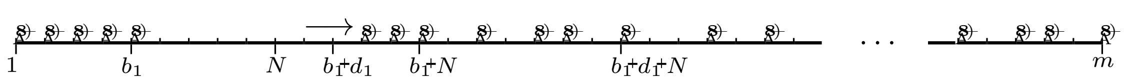


and will "improve" it by moving the archers in such a way that S will not increase.

Step 1. Assume that b_1 archers are located at the points of the interval $[1, N]$. If $b_1 > 0$, we move these archers to the left end of the interval $[1, N]$; clearly, this does not increase S. As a result, all (integer) points in the interval $[1, b_1]$ become occupied (we will say that $[1, b_1]$ *is full*), while all points in the interval $[b_1 + 1, N]$ become free (we will say that $[b_1 + 1, N]$ *is empty*):

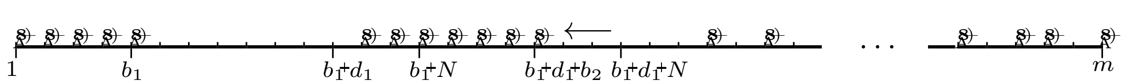


Step 2. Now, if an archer R is located at a point $x \in [N, b_1 + N - 1]$ and the point $x + 1$ is not occupied, then R can be moved to $x + 1$. Indeed, after this relocation R is no longer threatened by the archer located at $x - N + 1 \in [1, b_1]$ and so, the number of archers threatening R decreases by 1. On the other hand, the number of archers threatened by R increases by at most 1 and, hence, the total number of threats S does not increase. This allows us to move all archers located in $[N + 1, b_1 + N]$ to the right end of this interval:



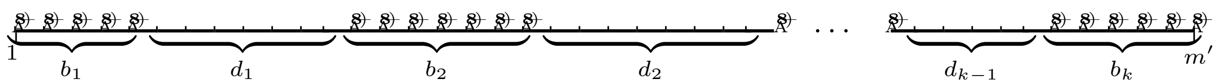
Assume that there are c_1 archers in $[N + 1, b_1 + N]$ (possibly, $c_1 = 0$) and put $d_1 = N - c_1$; then after this rearrangement the interval $[N + 1, b_1 + d_1]$ becomes empty and the interval $[b_1 + d_1 + 1, b_1 + N]$ becomes full. Note that $c_1 \le b_1$ and so, $b_1 + d_1 \ge N$.

Step 3. We shift the archers located in $[b_1 + N + 1, b_1 + N + d_1]$ to the left end of this interval; we can do this since, at any position, these archers are threatened by all archers from the interval $[b_1 + d_1 + 1, b_1 + N]$ and are not threatened by the archers from $[1, b_1]$:



Assume that the interval $[b_1 + N + 1, b_1 + N + d_1]$ contains e_1 archers and put $b_2 = c_1 + e_1$. Then, after this rearrangement, the interval $[b_1 + d_1 + 1, b_1 + d_1 + b_2]$ becomes full and the interval $[b_1 + d_1 + b_2 + 1, b_1 + d_1 + N]$ becomes empty. Note that $b_2 \ge c_1$ and so, $d_1 + b_2 \ge d_1 + c_1 = N$.

We repeat Steps 2 and 3 starting at the point b_1+d_1+1 instead of 1, and obtain an empty interval $[b_1+d_1+b_2+1, b_1+d_1+b_2+d_2]$ and a full interval $[b_1+d_1+b_2+d_2+1, b_1+d_1+b_2+d_2+b_3]$. And so on, until we reach the last archer. In the process of the last application of Step 2 some archers will possibly be forced to cross the boundary of the interval $[1,m]$ and move to the interval $[m+1,m']$ with $m' \le m+N$. The resulting configuration will represent an alternating sequence of full/empty intervals of lengths, respectively, $b_1, d_1, \dots, b_{k-1}, d_{k-1}, b_k$, where b_i, d_i satisfy $0 \le b_i \le N$ for $i = 1, \dots, k$; $0 \le d_i \le N$, $b_i + d_i \ge N$ and $d_i + b_{i+1} \ge N$ for $i = 1, \dots, k-1$; $b_1 + \ldots + b_k = b$ and $d_1 + \ldots + d_{k-1} = m' - b$.



In this situation, the first (from the left) archer of the i-th group of archers threatens all b_i members of this group, the next one threatens $b_i - 1$ archers, and so on. In addition, the last archer of the i-th group threatens $N - d_i - 1$ members of the $(i+1)$-st group, the next-to-last archer threatens $N - d_i - 2$ archers of the $(i+1)$-st group, and so on. Hence, the number of threats coming from the members of the i-th group is

$$\big(b_i + (b_i - 1) + \ldots + 1\big) + \big((N-d_i-1) + (N-d_i-2) + \ldots + 1\big) = \tfrac{b_i(b_i+1)}{2} + \tfrac{(N-d_i)(N-d_i-1)}{2}.$$

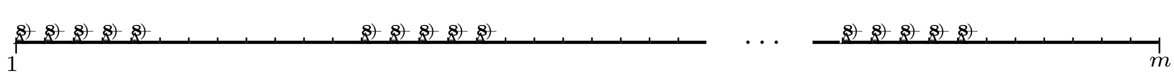
The total number of threats S is, therefore,

$$\begin{aligned} S &= \sum_{i=1}^{k} \frac{b_i(b_i+1)}{2} + \sum_{i=1}^{k-1} \frac{(N-d_i)(N-d_i-1)}{2} \\ &= \frac{1}{2}\sum_{i=1}^{k} b_i^2 + \frac{1}{2}\sum_{i=1}^{k-1}(N-d_i)^2 + \frac{1}{2}\sum_{i=1}^{k} b_i - \frac{1}{2}\sum_{i=1}^{k-1}(N-d_i) \\ &\ge \frac{1}{2k}\Big(\sum_{i=1}^{k} b_i\Big)^2 + \frac{1}{2(k-1)}\Big(\sum_{i=1}^{k-1}(N-d_i)\Big)^2 + \frac{1}{2}\sum_{i=1}^{k-1}(b_i + d_i - N) \\ &\ge \frac{1}{2k}b^2 + \frac{1}{2k}\big((k-1)N - m' + b\big)^2 = \frac{1}{2k}\big(b^2 + (kN - m'' + b)^2\big), \end{aligned} \tag{1.2}$$

where $m'' = M' + N$. Considering the right hand part of (1.2) as a function of k, one finds that its minimum is reached when $k = \frac{\sqrt{b^2+(m''-b)^2}}{N}$ and equals

$$N\sqrt{b^2 + (m''-b)^2} - N(m''-b) = mN\Big(\sqrt{a^2 + (\tfrac{m''}{m} - a)^2} + a - \tfrac{m''}{m}\Big).$$

Since $1 < \frac{m''}{m} \le \frac{m+2N}{m} < 1 + 2\varepsilon$ and ε can be taken arbitrarily small, we have $S \ge mN\big(\sqrt{a^2 + (1-a)^2} + a - 1\big)$. (Returning to the archers, we see that, if we ignore the fact that k, b/k and m/k must be integers, the "safest" configuration is the following one: the b archers form $k = \frac{\sqrt{b^2+(m-b)^2}}{N}$ equal groups with equal distances between the groups:



For this configuration $S = N\sqrt{b^2+(m-b)^2} - N(m-b)$.)

We obtain, therefore, that in the case T is a cyclic permutation,

$$\frac{1}{N}\sum_{n=0}^{N-1}|A\cap T^{-n}A| \geq \frac{1}{N}\sum_{n=0}^{N-1}|A\cap(A-n)| = \tfrac{1}{N}S \geq m\big(\sqrt{a^2+(1-a)^2}+a-1\big).$$

Now let T be an arbitrary permutation of an m-element set X. Let $X = X_1\cup\ldots\cup X_l$ be the partition of X into the union of disjoint cycles of T and let $m_j = |X_j|$, $j = 1,\ldots,l$. Let $A\subseteq X$, $|A| = b$, $A_j = A\cap X_j$ and $a_j = |A_j|/|X_j|$, $j = 1,\ldots,l$. Then for any $N\in\mathbb{N}$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1}|A\cap T^{-n}A| = \frac{1}{N}\sum_{j=1}^{l}\sum_{n=0}^{N-1}|A_j\cap T^{-n}A_j| \geq \sum_{j=1}^{l}m_j\big(\sqrt{a_j^2+(1-a_j)^2}+a_j-1\big).$$

Since the function $\varphi(a) = \sqrt{a^2+(1-a)^2}+a-1$ is convex, the conditions $m_1+\ldots+m_l = m$ and $\frac{1}{m}(a_1m_1+\ldots+a_lm_l) = a$ imply $\sum_{j=1}^{l}m_j\varphi(a_j) \geq m\varphi(a)$. Hence,

$$\frac{1}{N}\sum_{n=0}^{N-1}|A\cap T^{-n}A| \geq m\big(\sqrt{a^2+(1-a)^2}+a-1\big)$$

and

$$\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A) = \frac{1}{mN}\sum_{n=0}^{N-1}|A\cap T^{-n}A| \geq \sqrt{a^2+(1-a)^2}+a-1.$$

To prove part (b) of the theorem, we take $X = [0,1]$, $A = [0,a]$ and $T(x) = (x+\frac{1}{m}) \bmod 1$ with m to be specified later. We may assume that a is rational and, moreover, that $a = \frac{b}{m}$, $b\in\mathbb{N}$. Then for $m-b\leq N\leq m$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A) &= \frac{a}{mN}\Big(\frac{b(b+1)}{2}+\frac{(N-m+b)(N-m+b-1)}{2}\Big)\\ &= \frac{1}{2y}\big(a(a+\tfrac{1}{m})+(y+a-1)(y+a-1-\tfrac{1}{m})\big),\end{aligned}\tag{1.3}$$

where we put $y = N/m$. By taking m large enough we may make (1.3) to be less then $\frac{1}{2y}\big(a^2+(y+a-1)^2\big)+\frac{\delta}{2}$ for all $y\in[0,1]$. For $y = \sqrt{a^2+(1-a)^2}$ one has $\frac{1}{2y}\big(a^2+(y+a-1)^2\big) = \sqrt{a^2+(1-a)^2}+a-1$. Therefore, choosing N and m so that $y = \frac{N}{m}$ is sufficiently close to $\sqrt{a^2+(1-a)^2}$, we get $\frac{1}{2y}\big(a^2+(y+a-1)^2\big) < \sqrt{a^2+(1-a)^2}+a-1+\frac{\delta}{2}$ and so, $\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A) < \sqrt{a^2+(1-a)^2}+a-1+\delta$. ∎

1.3. Given $a>0$, a positive lower bound also exists for the averages $\frac{1}{N}\sum_{n=0}^{N-1}\int fT^nf\,d\mu$ where f is a nonnegative function with $\int f\,d\mu=a$:

Theorem. *Let $a>0$.*
(a) *For any probability measure preserving system $(X,\mathcal{B},\mu,T)$ and any nonnegative integrable function f on X with $\int f\,d\mu\geq a$ one has $\frac{1}{N}\sum_{n=0}^{N-1}\int fT^nf\,d\mu\geq\frac{a^2}{2}$ for all $N\in\mathbb{N}$.*
(b) *For any $\delta>0$ there exist a measure preserving system $(X,\mathcal{B},\mu,T)$, a measurable function f on X with $\int f\,d\mu=a$ and $N\in\mathbb{N}$ such that $\frac{1}{N}\sum_{n=0}^{N-1}\int fT^nf\,d\mu<\frac{a^2}{2}+\delta$.*

Proof. Fix $N\in\mathbb{N}$. Again, we may replace our system by a finite one and assume that T is a permutation of a finite set X, $|X|=m$, and that f takes on only integer values. We have to estimate the sum $\sum_{n=0}^{N-1}\sum_{x\in X}f(x)f(T^nx)$, where $f(x)$, $x\in X$, are nonnegative integers satisfying $\sum_{x\in X}f(x)=am$.

First, let T be a cyclic permutation: $X=\{1,\ldots,m\}$, $Tx=(x\bmod m)+1$. Then the problem is equivalent to the following one: $b=am$ archers are positioned at the points $1,\ldots,m$, $f(x)$ archers at a point x. An archer located at x threatens the archers located in the interval $[x,x+N-1]$, totally $\sum_{n=0}^{N-1}f(x+n)$ archers. We have to estimate

$$S=\sum_{n=0}^{N-1}\sum_{x=1}^{m}f(x)f(x+n)=\sum_{x=1}^{m}f(x)\sum_{n=0}^{N-1}f(x+n)=\sum_{x=1}^{m}\sum_{r=1}^{f(x)}\Bigl(\sum_{n=0}^{N-1}f(x+n)\Bigr)$$
$$=\sum_{R\text{ is an archer}}(\text{the number of archers threatened by }R).$$

Having replaced $X=\{1,\ldots,m\}$ by $\{1,\ldots,Nm\}$ and extended f to $\{1,\ldots,Nm\}$ by $f(x)=f(x-m)$ for $x>m$, we may assume that m is divisible by N. Let us subdivide $\{1,\ldots,m\}$ into $\frac{m}{N}$ intervals of length N. Let b_i, $i=1,\ldots,\frac{m}{N}$, be the number of archers located in the i-th interval. Fix i and enumerate the archers of the i-th interval in succession from the left to the right. Then the first archer threatens all b_i archers in the interval, the second archer threatens at least b_i-1 archers, etc. The total number of threats coming from the archers located in the i-th interval (to the archers in the same interval) is $\geq\frac{b_i(b_i+1)}{2}\geq\frac{b_i^2}{2}$. Hence, the total number of threats S satisfies

$$S\geq\sum_{i=1}^{m/N}\frac{b_i^2}{2}\geq\frac{N}{2m}\Bigl(\sum_{i=1}^{m/N}b_i\Bigr)^2=\frac{Nb^2}{2m}.$$

We therefore have $\frac{1}{N}\sum_{n=0}^{N-1}\sum_{x\in X}f(x)f(T^nx)\geq\frac{1}{N}S\geq\frac{b^2}{2m}$.

Now let T be an arbitrary permutation of an m-element set X. Let $X=X_1\cup\ldots\cup X_l$ be the partition of X into the union of disjoint cycles of T, let $m_j=|X_j|$ and $b_j=\sum_{x\in X_j}f(x)$, $j=1,\ldots,l$. We have

$$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{x\in X}f(x)f(T^nx)=\frac{1}{N}\sum_{j=1}^{l}\sum_{n=0}^{N-1}\sum_{x\in X_j}f(x)f(T^nx)\geq\sum_{j=1}^{l}\frac{b_j^2}{2m_j}.\qquad(1.4)$$

Under the conditions $m_1 + \ldots + m_l = m$ and $b_1 + \ldots + b_l = ma$, the minimal value of the right hand side of (1.4) is reached when $\frac{b_1}{m_1} = \ldots = \frac{b_l}{m_l} = a$ and equals $\frac{1}{2}ma^2$. Hence,

$$\frac{1}{N}\sum_{n=0}^{N-1}\int fT^n f\,d\mu = \frac{1}{Nm}\sum_{n=0}^{N-1}\sum_{x\in X} f(x)f(T^n x) \geq \frac{a^2}{2}.$$

To prove part (b) of the theorem, take f to be $\frac{a}{c}1_A$, where A is a set of measure $c > 0$ in X. By Theorem 1.2, for appropriately chosen X, A, T and N we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}A) < \sqrt{c^2+(1-c)^2} + c - 1 + \frac{\delta c^2}{2a^2},$$

and so,

$$\frac{1}{N}\sum_{n=0}^{N-1}\int fT^n f\,d\mu = \frac{1}{N}\sum_{n=0}^{N-1}\Big(\frac{a}{c}\Big)^2\mu(A\cap T^{-n}A) < \Big(\frac{a}{c}\Big)^2\big(\sqrt{c^2+(1-c)^2} + c - 1\big) + \frac{\delta}{2}.$$

Since $\lim\limits_{c\to 0}\frac{a^2}{c^2}\big(\sqrt{c^2+(1-c)^2}+c-1\big) = \frac{a^2}{2}$, we have $\frac{1}{N}\sum_{n=0}^{N-1}\int fT^n f\,d\mu < \frac{a^2}{2} + \delta$ when c is small enough. ■

2. Lower bounds for some non-conventional ergodic averages

2.1. Let $T_1, \ldots, T_k$ be pairwise commuting measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$ and let A be a set of positive measure in X. Let us consider the averages

$$\begin{aligned}\frac{1}{N_1\ldots N_k}&\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_k=0}^{N_k-1}\mu\Big(\bigcap_{S\subseteq\{1,\ldots,k\}}\big(\prod_{i\in S}T_i^{-n_i}A\big)\Big)\\ &= \frac{1}{N_1\ldots N_k}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_k=0}^{N_k-1}\mu\big(A\cap T_1^{-n_1}A\cap T_2^{-n_2}A\cap\ldots\cap T_1^{-n_1}\cdots T_k^{-n_k}A\big).\end{aligned} \tag{2.1}$$

The convergence of (2.1) as $N_1, \ldots, N_k \to \infty$ is known only in the case $T_1 = \ldots = T_k$ for $k = 2$ (due to V. Bergelson) and $k = 3$ (B. Host and B. Kra).

2.2. If $T_1, \ldots, T_k$ do not commute the limit of the averages (2.1) may not exist:

Example. Let a measure preserving transformation P of a probability measure space $(Y,\mathcal{D},\nu)$ and a set $B\in\mathcal{D}$ with $\nu(B)=a$, $a\neq 0,1$, be such that $\nu\bigl(B\cap P^{-n}(B)\bigr)=a^2$ for all $n>0$. Let $S\subseteq\mathbb{N}$ with $1\notin S$; define $P_n=P$ if $n\in S$ and $P_n=\mathrm{Id}_Y$ otherwise. Take $(X,\mathcal{B},\mu)=(Y,\mathcal{D},\nu)^{\mathbb{N}}$, $A=B\times Y\times Y\times\ldots$ and define $T_1,T_2\colon X\longrightarrow X$ by $T_1(y_1,y_2,\ldots)=(P_1y_1,P_2y_2,\ldots)$ and $T_2(y_1,y_2,\ldots)=(y_2,y_3,\ldots)$. Then for any $n_1,n_2\geq 1$ one has $\mu\bigl(A\cap T_1^{-n_1}A\cap T_2^{-n_2}A\cap T_1^{-n_1}T_2^{-n_2}A\bigr)=a^3$ if $n_2\in S$ and $=a^2$ if $n_2\notin S$. Therefore, if S is such that the density $d(S)=\lim\limits_{N\to\infty}\frac{1}{N}\bigl|S\cap[1,N]\bigr|$ is not defined, then $\lim\limits_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\mu\bigl(A\cap T_1^{-n_1}A\cap T_2^{-n_2}A\cap T_1^{-n_1}T_2^{-n_2}A\bigr)$ does not exist.

2.3. Nevertheless, a positive lower bound of the averages (2.1) exists even for noncommuting $T_1,\ldots,T_k$. Put $\varphi(a)=\sqrt{a^2+(1-a)^2}+a-1$, $\varphi_1=\varphi$ and $\varphi_k(a)=\varphi\bigl(\varphi_{k-1}(a)\bigr)$, $k=2,3,\ldots$.

Theorem. *Let $T_1,\ldots,T_k$ be measure preserving transformations of a probability measure space $(X,\mathcal{B},\mu)$ and let $A\in\mathcal{B}$, $\mu(A)=a$. Then for any $N_1,\ldots,N_k\in\mathbb{N}$*

$$\frac{1}{N_1\ldots N_k}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_k=0}^{N_k-1}\mu\Bigl(\bigcap_{S\subseteq\{1,\ldots,k\}}\bigl(\prod_{i\in S}T_i^{-n_i}A\bigr)\Bigr)\geq\varphi_k(a). \tag{2.2}$$

Proof. We use induction on k; the case $k=1$ is Theorem 1.2. For all $n_1,\ldots,n_{k-1}\in\mathbb{Z}_+$ define $A_{n_1,\ldots,n_{k-1}}=\bigcap\limits_{S\subseteq\{1,\ldots,k-1\}}\bigl(\prod_{i\in S}T_i^{-n_i}A\bigr)$ and $a_{n_1,\ldots,n_{k-1}}=\mu(A_{n_1,\ldots,n_{k-1}})$.

Fix $N_1,\ldots,N_k$. By induction hypothesis we have

$$\frac{1}{N_1\ldots N_{k-1}}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_{k-1}=0}^{N_{k-1}-1}a_{n_1,\ldots,n_{k-1}}\geq\varphi_{k-1}(a). \tag{2.3}$$

The left hand part of (2.2) equals

$$\frac{1}{N_1\ldots N_{k-1}}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_{k-1}=0}^{N_{k-1}-1}\Bigl(\frac{1}{N_k}\sum_{n_k=0}^{N_k-1}\mu(A_{n_1,\ldots,n_{k-1}}\cap T^{-n_k}A_{n_1,\ldots,n_{k-1}})\Bigr)$$

By Theorem 1.2, for any $n_1,\ldots,n_{k-1}$ one has $\frac{1}{N_k}\sum_{n_k=0}^{N_k-1}\mu(A_{n_1,\ldots,n_{k-1}}\cap T^{-n_k}A_{n_1,\ldots,n_{k-1}})\geq\varphi(a_{n_1,\ldots,n_{k-1}})$. Since φ is a convex increasing function on $[0,1]$, taking into account (2.3) we get

$$\frac{1}{N_1\ldots N_{k-1}}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_{k-1}=0}^{N_{k-1}-1}\varphi(a_{n_1,\ldots,n_{k-1}})\geq\varphi\Bigl(\frac{1}{N_1\ldots N_{k-1}}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_{k-1}=0}^{N_{k-1}-1}a_{n_1,\ldots,n_{k-1}}\Bigr)$$
$$\geq\varphi\bigl(\varphi_{k-1}(a)\bigr)=\varphi_k(a).$$

■

2.4. We now pass to the averages

$$\frac{1}{N_1 \dots N_k} \sum_{n_1=0}^{N_1-1} \dots \sum_{n_k=0}^{N_k-1} \mu(A \cap T_1^{-n_1} A \cap \dots \cap T_k^{-n_k} A) \tag{2.4}$$

Theorem. *Let $T_1, \dots, T_k$ be (not necessarily commuting) measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$. For any $A \in \mathcal{B}$, $\mu(A) = a$,*

$$\lim_{N_1,\dots,N_k \to \infty} \frac{1}{N_1 \dots N_k} \sum_{n_1=0}^{N_1-1} \dots \sum_{n_k=0}^{N_k-1} \mu(A \cap T_1^{-n_1} A \cap \dots \cap T_k^{-n_k} A)$$

exists and is not less than a^{k+1}.

Proof. We have

$$\begin{aligned} &\lim_{N_1,\dots,N_k \to \infty} \frac{1}{N_1 \dots N_k} \sum_{n_1=0}^{N_1-1} \dots \sum_{n_k=0}^{N_k-1} \mu(A \cap T_1^{-n_1} A \cap \dots \cap T_k^{-n_k} A) \\ &\quad = \lim_{N_1,\dots,N_k \to \infty} \frac{1}{N_1 \dots N_k} \sum_{n_1=0}^{N_1-1} \dots \sum_{n_k=0}^{N_k-1} \int_X 1_A \cdot T_1^{n_1}(1_A) \cdot \ldots \cdot T_k^{n_k}(1_A)\, d\mu \\ &\quad = \int_X 1_A \cdot \big(\lim_{N_1 \to \infty} \tfrac{1}{N_1} \textstyle\sum_{n=0}^{N_1-1} T_1^n(1_A) \big) \cdot \ldots \cdot \big(\lim_{N_k \to \infty} \tfrac{1}{N_k} \textstyle\sum_{n=0}^{N_k-1} T_k^n(1_A) \big)\, d\mu = \int_A f_1 \dots f_k \, d\mu, \end{aligned}$$

where $f_i = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_i^n(1_A)$, $i = 1, \dots, k$.

2.5. Lemma. *Let T be a measure preserving transformation of a probability measure space $(X, \mathcal{B}, \mu)$, let $A \in \mathcal{B}$, $\mu(A) > 0$, and let $f = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n(1_A)$. Then $0 \le f \le 1$, $f(x) \ne 0$ for almost all $x \in A$ and $\int_A \frac{d\mu}{f} \le 1$.*

Proof. Without loss of generality we may assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue space. Let $\pi\colon X \longrightarrow Y$, $\mu = \int_Y \mu_y \, d\nu$ be the ergodic decomposition of μ and let $B = \big\{ y \in Y \bigm| \mu_y(A) > 0 \big\}$. For almost every $y \in Y$ we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n(1_A) = \mu_y(A)$ in $L^1(X, \mu_y)$ and so, $f|_{\pi^{-1}(y)} = \mu_y(A)$. Therefore,

$$\mu\big(\{x \in A \bigm| f(x) = 0\}\big) = \mu\big(A \setminus \pi^{-1}(B)\big) \le \int_{Y \setminus B} \mu_y(A)\, d\nu = 0$$

and

$$\int_A \frac{d\mu}{f} = \int_B \Big(\int_A \frac{d\mu_y}{f} \Big) d\nu = \int_B \frac{\mu_y(A)}{\mu_y(A)}\, d\nu = \nu(B) \le 1.$$

■

2.6. We have, therefore, to determine the minimum of $F = \int_A f_1 \dots f_k \, d\mu$ under the conditions $f_i|_A > 0$ and $\int_A \frac{d\mu}{f_i} = 1$, $i = 1, \dots, k$. We pass to a finite model: $A = \{1, \dots, m\}$ and $f_i(j) = x_{i,j} > 0$, $j = 1, \dots, m$, with $\sum_{j=1}^m \frac{1}{x_{i,j}} = 1$, $i = 1, \dots, k$. We have to minimize the function $F(x_{1,1}, \dots, x_{k,m}) = \sum_{j=1}^m x_{1,j} \dots x_{k,j}$. At a point of extremum of F it must be $\operatorname{grad} F \in \operatorname{Span}\big\{\operatorname{grad}\big(\sum_{j=1}^m \frac{1}{x_{1,j}}\big), \dots, \operatorname{grad}\big(\sum_{j=1}^m \frac{1}{x_{k,j}}\big)\big\}$, that is, for some $c_1, \dots, c_k \in \mathbb{R}$, $\frac{x_{1,j} \dots x_{k,j}}{x_{i,j}} = \frac{c_i}{x_{i,j}^2}$ for $i = 1, \dots, k$, $j = 1, \dots, m$. This implies $x_{i,1} = \dots = x_{i,m}$, $i = 1, \dots, k$, that is, $f_1, \dots, f_k$ are constant on A. Hence, the minimum of F is attained when $f_1|_A = \dots = f_k|_A = a$ and equals $a^k \mu(A) = a^{k+1}$. ■

2.7. The same proof works for the uniform version of Theorem 2.4:

Theorem. *For any measure preserving transformations $T_1, \dots, T_k$ of a probability measure space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$, $\mu(A) = a$,*

$$\lim_{N_1 - M_1, \dots, N_k - M_k \to \infty} \frac{1}{(N_1 - M_1) \dots (N_k - M_k)} \sum_{n_1 = M_1}^{N_1 - 1} \dots \sum_{n_k = M_k}^{N_k - 1} \mu(A \cap T_1^{-n_1} A \cap \dots \cap T_k^{-n_k} A)$$

exists and is not less than a^{k+1}.

2.8. A lower bound for the averages (2.4) (which is not exact, of course) can be taken from Theorem 2.3:

Corollary of Theorem 2.3. *Let $T_1, \dots, T_k$ be measure preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ and let $A \in \mathcal{B}$, $\mu(A) = a$. Then for any $N_1, \dots, N_k \in \mathbb{N}$*

$$\frac{1}{N_1 \dots N_k} \sum_{n_1 = 0}^{N_1 - 1} \dots \sum_{n_k = 0}^{N_k - 1} \mu\big(A \cap T_1^{-n_1} A \cap \dots \cap T_k^{-n_k} A\big) \geq \varphi_k(a).$$

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