IP*-recurrence and nilsystems

V. Bergelson and A. Leibman

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Abstract

By a result due to Furstenberg, a homeomorphism $T$ of a compact space is distal if and only if it possesses the property of IP*-recurrence, meaning that for any $x_0 \in X$, for any open neighborhood $U$ of $x_0$, and for any sequence $(n_i)$ in $\mathbb{Z}$, the set $R_U(x_0) = \{ n \in \mathbb{Z} : T^n x_0 \in U \}$ has a non-trivial intersection with the set of finite sums \{ $n_{i_1} + n_{i_2} + \cdots + n_{i_s} : i_1 < i_2 < \cdots < i_s$, s \in \mathbb{N} \}. We show that translations on compact nilmanifolds (which are known to be distal) are characterized by a stronger property of IP*_r-recurrence, which asserts that for any $x_0 \in X$ and any neighborhood $U$ of $x_0$ there exists $r \in \mathbb{N}$ such that for any $r$-element sequence $n_1, \ldots, n_r$ in $\mathbb{Z}$ the set $R_U(x_0)$ has a non-trivial intersection with the set \{ $n_{i_1} + n_{i_2} + \cdots + n_{i_s} : i_1 < i_2 < \cdots < i_s$, s \leq r$\}. We also show that the property of IP*_r-recurrence is equivalent to an ostensibly much stronger property of polynomial IP*_r-recurrence. (This should be juxtaposed with the fact that for general distal transformations the polynomial IP*-recurrence is strictly stronger than the IP*-recurrence.)

0. Introduction

Let $(X,T)$ be a topological dynamical system, meaning that $X$ is a compact metric space and $T$ is a self-homeomorphism of $X$. Given a point $x_0 \in X$ and an open neighborhood $U$ of $x_0$, define $R_U(x_0) = \{ n \in \mathbb{Z} : T^n x_0 \in U \}$, the set of returns of $x_0$ into $U$. Sets of returns reflect the properties of topological system, and it is of interest to characterize (and/or distinguish between) dynamical systems by arithmetic properties of these sets. An example of this kind is provided by a theorem of Furstenberg on sets of returns in distal systems. A system $(X,T)$ is said to be distal if for any distinct $x, y \in X$, $\inf_{n \in \mathbb{Z}} \text{dist}(T^n x, T^n y) > 0$. Given a sequence $n_1, n_2, \ldots$ in $\mathbb{Z}$, the set \{ $n_{i_1} + \cdots + n_{i_s} : s \in \mathbb{N}, i_1 < \cdots < i_s$\} of finite sums of distinct elements of this sequence is called an IP-set. A subset $E$ of $\mathbb{Z}$ is called an IP*-set if it intersects every IP-set. Furstenberg’s theorem says that distal systems are characterized by the IP*-recurrence property.

Theorem 0.1. ([F], Theorem 9.11) A system $(X,T)$ is distal if and only if for any $x_0 \in X$ and any open neighborhood $U$ of $x_0$ the set of returns $R_U(x_0)$ is an IP*-set.

Another relevant example involves translations on compact abelian groups. A set of differences is a set of the form \{ $n_i - n_j, j < i$\}, where $(n_i)$ is an infinite sequence in $\mathbb{Z}$; a subset $E$ of $\mathbb{Z}$ is said to be a $\Delta^*$-set if it has a nonempty intersection with every set of differences in $\mathbb{Z}$. A point $x$ in a system $(X,T)$ is said to be almost automorphic if for any

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sequence \((n_i)\) in \(\mathbb{Z}\), \(T^{n_i}x \rightarrow y\) implies \(T^{-n_i}y \rightarrow x\). It is shown in [F], Theorem 9.13, that a system has the \(\Delta^*\)-recurrence property (that is, that every set of returns in the system is a \(\Delta^*\)-set) if and only if every point in the system is almost automorphic. Next, by a theorem of Veech (see [V], Theorem 1.2; see also [AGN]) every point of a minimal\(^{(1)}\) system \((X,T)\) is almost automorphic if and only if the family \(\{T^n, n \in \mathbb{Z}\}\) of powers of \(T\) is equicontinuous. Now, it is not hard to see that for a minimal \(T\) the family \(\{T^n, n \in \mathbb{Z}\}\) is equicontinuous if and only if \((X,T)\) is isomorphic to a translation on a compact abelian group\(^{(2)}\). Thus, the recurrence property characterizing minimal group translations is that of \(\Delta^*\).

Our goal in this paper is to provide a similar characterization of nilsystems, namely, systems of the form \((X,T)\) where \(X\) is a nilmanifold (a compact homogeneous space of a nilpotent Lie group \(G\)) and \(T\) is a niltranslation (a translation on \(X\) defined by an element of \(G\)). The motivation for this study comes from the fact that nilsystems are intrinsically related to various problems arising in ergodic theory of multiple recurrence, combinatorics, and number theory, and understanding the recurrence properties of niltranslations leads to interesting applications in these areas. It is well known that nilsystems are distal (see [AGH], [Ke1], [Ke2]), and thus are IP*-recurrent; however, not every distal system is a nilsystem, and thus there must be a stronger than IP* property of recurrence that characterizes them.

For an integer \(r \in \mathbb{N}\) and an \(r\)-element sequence \(n_1, \ldots, n_r\) in \(\mathbb{Z}\), we call the set \(\{n_{i_1} + \cdots + n_{i_s} : 1 \leq s \leq r, i_1 < \cdots < i_s\}\) of sums of distinct elements of this sequence an \(IP_r\)-set. A set \(E \subseteq \mathbb{Z}\) is called an \(IP_r^*\)-set if it has a nonempty intersection with every \(IP_r\)-set in \(\mathbb{Z}\). We say that a set is an \(IP_0^*\)-set if it is an \(IP_r^*\)-set for some \(r \in \mathbb{N}\). \(IP_0^*\)-sets form a proper subfamily of the family of \(IP^*\)-sets: clearly, every \(IP_0^*\)-set is \(IP^*\), but not vice versa\(^{(3)}\). A special class of nilsystems is provided by affine skew product transformations of tori\(^{(4)}\); it follows from [B], Theorem 7.7, that every such system has the \(IP_0^*\)-recurrence property: for every \(x_0 \in \mathbb{T}^k\) and any open neighborhood \(U\) of \(x_0\) the set of returns \(R_U(x_0)\) is an \(IP_0^*\)-set. On the other hand, one can show that not every distal system is \(IP_0^*\)-recurrent (see [BL3], Section 1). It is tempting to conjecture that it is the \(IP_0^*\)-recurrence property that characterizes the nilsystems. This, however, cannot be

\(^{(1)}\) A system \((X,T)\) is minimal if it has no proper closed subsystems, or, equivalently, if the orbit of every point of \(X\) is dense in \(X\).

\(^{(2)}\) The “only if” implication follows from the fact that for any \(x_0 \in X\) one can define an additive group structure on the orbit \(\{T^n x_0, n \in \mathbb{Z}\}\) by \(T^n x_0 + T^m x_0 = T^{n+m} x_0, n,m \in \mathbb{Z}\), and then extend it, with the help of equicontinuity, to all of \(X\). This makes \(X\) a compact abelian group on which \(T\) acts as a minimal translation.

\(^{(3)}\) To see this, it is enough to exhibit an \(IP^*\)-set \(S\) which is not an \(IP\)-set. One can take, for example \(S = \bigcup_{r=1}^{\infty} S_r\), where \(S_r = \{2^r, 2 \cdot 2^r, 3 \cdot 2^r, \ldots, r \cdot 2^r\}, r \in \mathbb{N}\). Since for each \(r\), \(S_r\) is a dilation of the set \(\{1, 2, \ldots, r\}\), \(S\) contains arbitrarily large \(IP_r\)-sets, but it contains no \(IP\)-sets since the distances between consecutive elements of \(S\) form a non-decreasing sequence which tends to infinity.

\(^{(4)}\) An affine skew product transformation of the \(k\)-dimensional torus \(\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k\) is defined by the formula \(T(x_1, \ldots, x_k) = (x_1 + \alpha_1, x_2 + a_{2,1} x_1 + \alpha_2, \ldots, x_k + a_{k,k-1} x_{k-1} + \cdots + a_{k,1} x_1 + \alpha_k)\) with \(\alpha_i \in \mathbb{T}\) and \(a_{i,j} \in \mathbb{Z}\).
has natural coordinates such that under the action of a niltransla
tion $T$ coordinates of the image $X$ to prove the first statement, we use a coordinate approach. On any nilmanifold $X$ one
has natural coordinates such that under the action of a niltranslation $T$ the sequence of coordinates of the image $X^n$ of any point $x_0 \in X$ is given by generalized polynomials (see [BL2], Theorem A). We therefore need to deal with images of IP-sets under generalized polynomial mappings; these images form a subclass of generalized polynomial IP-sets. Conventional IP- and IP$_r$-sets in $\mathbb{Z}$ can be viewed as the images of mappings $\varphi: \mathcal{F}(A) \rightarrow \mathbb{Z}$ from the semigroup $\mathcal{F}(A)$ of finite subsets of $A,$ for $A = \mathbb{N}$ and, respectively, for $A = \{1, \ldots, r\},$ defined by $\varphi(\alpha) = \sum_{i \in \alpha} a_i.$ Such a mapping $\varphi$ is “linear” in the following sense:

\begin{align*}
\varphi(\alpha + \beta) &= \varphi(\alpha) + \varphi(\beta) \quad \text{whenever } \alpha, \beta \in \mathcal{F}(A) \text{ are disjoint. Let } H \text{ be an additive abelian group; one can introduce the notion of polynomial mappings } \mathcal{F}(A) \rightarrow H \text{ as follows. For a mapping } \\
\varphi: \mathcal{F}(A) \rightarrow H \text{ and a set } \beta \in \mathcal{F}(A) \text{ let the } \beta\text{-derivative } D_\beta \varphi \text{ be the mapping } \\
\mathcal{F}(A \setminus \beta) \rightarrow H \text{ defined by } D_\beta \varphi(\alpha) = \varphi(\alpha + \beta) - \varphi(\alpha). \text{ Then we say that a mapping } \\
\varphi: \mathcal{F}(\{1, \ldots, r\}) \rightarrow H \text{ is polynomial of degree } \leq d \text{ if for any disjoint } \beta_0, \beta_1, \ldots, \beta_d \in \\
\mathcal{F}(\{1, \ldots, r\}), D_{\beta_0} D_{\beta_1} \cdots D_{\beta_d} \varphi = 0. \text{ (See [BL1], Section 8.1.) Examples of quadratic (that is, of degree } \leq 2) \text{ polynomial mappings are, in increasing generality, } \varphi(\alpha) = (\sum_{i \in \alpha} a_i)^2, \\
\varphi(\alpha) = (\sum_{i \in \alpha} a_i)(\sum_{i \in \alpha} b_i) = \sum_{i,j \in \alpha} a_i b_j, \text{ and } \varphi(\alpha) = \sum_{i,j \in \alpha} c_{i,j}, \text{ where } a_i, b_j, c_{i,j} \in H. \text{ Generalized polynomial mappings are the mappings built from (conventional) polynomial mappings using the operations of addition, multiplication, and taking the integer part. (An example is } \varphi = [(\varphi_1) \varphi_2 + \varphi_3] \varphi_4 + [\varphi_5][\varphi_6] \varphi_7, \text{ which is comprised of the polynomial mappings } \varphi_1, \ldots, \varphi_7.) \text{ Let us say that a generalized polynomial mapping } \varphi \text{ has total degree } \leq D \text{ if the sum } \sum_i \deg \varphi_i \text{ of the degrees of all the “conventional” polynomial mappings } \varphi_i \text{ of which } \varphi \text{ is comprised does not exceed } D, \text{ and let us say that a generalized polynomial mapping is constant free if all the } \varphi_i \text{ vanish at } \emptyset: \varphi_i(\emptyset) = 0. \text{ Let us also say that a generalized polynomial mapping is open if it is contained in the ideal generated by the conventional constant-free polynomials of the ring of constant-free generalized polynomials. (In other words, a generalized polynomial is open if it contains no “closed” summands of the form } [\varphi_1] \cdots [\varphi_k], \text{ where } \varphi_i \text{ are generalized polynomials.) For } x \in \mathbb{R} \text{ let } \|x\| = \text{dist}(x, \mathbb{Z}). \text{ The following result of Diophantine nature, which we use to prove Theorem 0.2, is of independent interest:}

**Theorem 0.2.** Any pre-nilsystem (and so, any nilsystem) is IP$_0^*$-recurrent. Any IP$_0^*$-recurrent system is a disjoint union of pre-nilsystems.

**Remark.** In analogy with IP$_0^*$-sets, one can define $\Delta^*_\alpha$-sets as those having a nonempty intersection with every large enough finite set of differences. In contrast with IP$^*/$IP$_0^*$-recurrence, the classes of $\Delta^*$- and $\Delta^*_0$-recurrent systems coincide. (These are translations of compact abelian groups.)

The second statement of Theorem 0.2 is an easy corollary of the results from [HKM].
**Theorem 0.3.** (Cf. Theorem 1.11 below.) For any $D \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r = r(D,d,\varepsilon) \in \mathbb{N}$ such that for any open constant-free generalized polynomial mapping $\varphi: \{1, \ldots, r\} \rightarrow \mathbb{R}$ of total degree $\leq D$ there exists a nonempty $\alpha \subseteq \{1, \ldots, r\}$ for which $\|\varphi(\alpha)\| < \varepsilon$.

The VIP-sets in $\mathbb{Z}^l$ are defined as the images $\{\varphi(\alpha) : \alpha \in \mathcal{F}(\mathbb{N}), \alpha \neq \emptyset\}$ of polynomial mappings $\varphi: \mathcal{F}(\mathbb{N}) \rightarrow \mathbb{Z}^l$ with $\varphi(\emptyset) = 0$, and we say that a set $E \subseteq \mathbb{Z}^l$ is a VIP-set if $E$ has a nonempty intersection with every VIP-set in $\mathbb{Z}^l$. Similarly, for all $d,r \in \mathbb{N}$, we define VIP$_{d,r}$-sets as the images $\{\varphi(\alpha) : \alpha \subseteq \{1, \ldots, r\}, \alpha \neq \emptyset\}$ of polynomial mappings $\varphi: \mathcal{F}(\{1, \ldots, r\}) \rightarrow \mathbb{Z}^l$ of degree $\leq d$ and with $\varphi(\emptyset) = 0$, and say that a set $E \subseteq \mathbb{Z}^l$ is a VIP$_{d,r}$-set if it has a nonempty intersection with every VIP$_{d,r}$-set. We will also say that a set $E \subseteq \mathbb{Z}^l$ is a VIP$_0$-set if for any $d \in \mathbb{N}$, $E$ is a VIP$_{d,r}$-set for some $r \in \mathbb{N}$. Theorem 0.3 now implies the following result:

**Theorem 0.4.** For any $D,d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r = r(D,d,\varepsilon) \in \mathbb{N}$ such that for any $l \in \mathbb{N}$ and any open constant-free generalized polynomial mapping $\varphi: \mathbb{Z}^l \rightarrow \mathbb{R}$ of total degree $\leq D$ the set $\{n \in \mathbb{Z}^l : ||\varphi(n)|| < \varepsilon\}$ is a VIP$_{d,r}$-set.

Let $G$ be a nilpotent Lie group; an $l$-parameter polynomial sequence in $G$ is a mapping $g: \mathbb{Z}^l \rightarrow G$ of the form $T_1^{p_1(n)} \cdots T_l^{p_l(n)}$, $n \in \mathbb{Z}^l$, where $T_1 \in G$ and $p_i$ are polynomials $\mathbb{Z}^l \rightarrow \mathbb{Z}$; the naive degree of $g$ is defined as $\max_i \deg p_i$.\(^{(5)}\) Using the fact that the coordinates of a point of a nilmanifold under the action of a polynomial sequence of niltranslations are generalized polynomials, we obtain as a corollary of Theorem 0.4 the following strengthening of the first part of Theorem 0.2:

**Theorem 0.5.** (Cf. Theorem 1.12 below.) Let $X$ be a nilmanifold with metric $\rho$ (compatible with the homogeneous space structure on $X$). For any $a,d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r = r(a,d,\varepsilon) \in \mathbb{N}$ such that for any $x_0 \in X$, any $l \in \mathbb{N}$, and any $l$-parameter polynomial sequence $g$ of niltranslations on $X$ of naive degree $\leq a$ and with $g(0) = \Id_X$, the set $R_U(x_0) = \{n \in \mathbb{Z}^l : \rho(g(n)x_0, x_0) < \varepsilon\}$ is a VIP$_{d,r}$-set.

We say that a dynamical system $(X,T)$ is VIP*-recurrent if for any $x_0 \in X$ and any open neighborhood $U$ of $x_0$ the set of returns $R_U(x_0) = \{n \in \mathbb{Z} : T^n x_0 \in U\}$ is a VIP*-set, and is VIP$^*_0$-recurrent if for any $x_0 \in X$ and any open neighborhood $U$ of $x_0$ the set $R_U(x_0)$ is a VIP$^*_0$-set. The VIP*-recurrence property turns out to be strictly stronger than that of the IP*-recurrence: there exist distal but not VIP*-recurrent systems.\(^{(6)}\) As for the VIP$^*_0$-recurrence, we get, as a corollary of Theorem 0.5, that, via Theorem 0.2, VIP$^*_0$-recurrence is equivalent to IP$^*_0$-recurrence:

**Theorem 0.6.** Any pre-nilsystem is VIP$^*_0$-recurrent, and any VIP$^*_0$-recurrent system is a disjoint union of pre-nilsystems.

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\(^{(5)}\) A more fundamental notion of degree of a polynomial sequence in a nilpotent group can be defined as the number of “differentiations” which it takes in order to reduce the polynomial sequence to a constant. For our purposes, however, the “naive” degree is quite sufficient.

\(^{(6)}\) See [P], Corollary 5.1, where it is shown that for any nonlinear polynomial $p: \mathbb{Z} \rightarrow \mathbb{Z}$ there exists an affine skew product transformation $T$ such that $\liminf_{n} \dist(T^{p(n)} 0, 0) > 0$. 

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In Section 1 of the paper we prove (a more precise version of) Theorems 0.3 and 0.4 and deduce Theorem 0.5 from them. In Section 2 we obtain the second statement of Theorem 0.2.

1. Sets of visits of open bounded generalized polynomials with no constant term to a neighborhood of zero.

Let \( A \) be a set and \((H,+)\) be an abelian group. For \( r \in \mathbb{N} \) we will denote by \([1,r]\) the interval \( \{1, \ldots, r\} \) in \( \mathbb{N} \). We denote by \( \mathcal{F}(A) \) the set of finite subsets of \( A \), by \( A^{(d)} \), \( d \in \mathbb{N} \), the set of subsets of \( A \) of cardinality \( d \), and by \( A^{(\leq d)} \), \( d \in \mathbb{N} \), the set of nonempty subsets of \( A \) of cardinality \( \leq d \), \( A^{(\leq d)} = \bigcup_{d=1}^{d} A^{(d)} \).

We start with discussing polynomial mappings on \( \mathcal{F}(A) \). We say that a mapping \( \varphi: \mathcal{F}(A) \rightarrow H \) is linear if it satisfies the identity \( \varphi(\alpha \cup \beta) = \varphi(\alpha) + \varphi(\beta) \) whenever \( \alpha, \beta \in \mathcal{F}(A) \) are disjoint, and will denote the set of linear mappings \( \mathcal{L}(A) \rightarrow H \) by \( \text{Lin}(A,H) \). A mapping \( \varphi \in \text{Lin}(A,H) \) is uniquely defined by its values at singletons: for any \( \alpha \in \mathcal{F}(A) \), \( \varphi_\alpha = \sum_{a \in \alpha} \hat{\varphi}(\{a\}) \). We will call the mapping \( \hat{\varphi}: A \rightarrow H \) defined by \( \hat{\varphi}(a) = \varphi(\{a\}) \) the producing function for \( \varphi \); we then have \( \varphi(\alpha) = \sum_{a \in \alpha} \hat{\varphi}(a), \alpha \in \mathcal{F}(A) \).

For a mapping \( \varphi: \mathcal{F}(A) \rightarrow H \) and \( \beta \in \mathcal{F}(A) \) we define the \( \beta \)-derivative \( D_\beta \varphi \) of \( \varphi \) by \( D_\beta \varphi(\alpha) = \varphi(\alpha \cup \beta) - \varphi(\alpha), \alpha \in \mathcal{F}(A \setminus \beta) \). We say that a mapping \( \varphi \) is polynomial of degree \( \leq d \) if for any \( d + 1 \) pairwise disjoint sets \( \beta_0, \ldots, \beta_d \in \mathcal{F}(A) \) one has \( D_{\beta_d} \cdots D_{\beta_0} \varphi = 0 \).

We will denote by \( \text{Pol}_d(A,H) \) the group of polynomial mappings \( \mathcal{F}(A) \rightarrow H \) of degree \( \leq d \). We will mainly deal with polynomial mappings “having zero constant term”; let us denote by \( \text{Pol}_d^0(A,H) \) the subgroup \( \{ \varphi \in \text{Pol}_d(A,H): \varphi(\emptyset) = 0 \} \) of \( \text{Pol}_d(A,H) \). Notice that \( \text{Lin}(A,H) = \text{Pol}_1^0(A,H) \).

One can show (see [BL1], sections 8.3-8.5) that any polynomial mapping \( \varphi \in \text{Pol}_d^0(A,H) \) can be represented in the form \( \varphi(\alpha) = \Phi(\alpha^d), \alpha \in \mathcal{F}(A) \), for some mapping \( \Phi \in \text{Lin}(A^d,H) \), so that

\[
\varphi(\alpha) = \sum_{v \in \alpha^d} \hat{\Phi}(v), \alpha \in \mathcal{F}(A),
\]

where \( \hat{\Phi}: A^d \rightarrow H \) is the producing function for \( \Phi \). We will call \( \hat{\Phi} \) a q-producing function for \( \varphi \).

The q-producing function for a polynomial mapping \( \varphi \in \text{Pol}_d^0(A,H) \) is not canonically defined. A more natural is the t-producing function for \( \varphi \), a function \( \tilde{\Phi}: A^{(\leq d)} \rightarrow H \) such that for any \( \alpha \in \mathcal{F}(A) \),

\[
\varphi(\alpha) = \sum_{u \in \alpha^{(\leq d)}} \tilde{\Phi}(u).
\]

The t-producing function \( \tilde{\Phi} \) for \( \varphi \) is defined uniquely (and provides a natural approach to the definition of polynomial mappings in the case \( H \) is a commutative semigroup). In terms of \( \tilde{\Phi} \), \( \varphi \) is the sum of its homogeneous components, \( \varphi = \varphi_1 + \cdots + \varphi_d \), where for each \( i \), \( \varphi_i(\alpha) = \sum_{\delta \in \alpha(i)} \tilde{\Phi}(\delta) \). To obtain the t-producing function \( \tilde{\Phi} \) for \( \varphi \) from a q-producing function \( \hat{\Phi} \) one simply sums up the values of \( \hat{\Phi} \) at the elements of \( A^{(d)} \) corresponding to
the same element of \( A(\leq d) \): for any \( u \in A(\leq d) \),
\[
\tilde{\Phi}(u) = \sum_{v = (a_1, \ldots, a_d) \in \alpha^d} \tilde{\Phi}(v).
\] (1.2)

Let \( B \) be a collection of pairwise disjoint finite subsets of \( A \); we will call \( B \) a disjoint subcollection in \( A \); if \( |B| = s \) we will say that \( B \) is a disjoint \( s \)-subcollection. Given a disjoint subcollection \( B \) in \( A \), we have an injection \( \mathcal{F}(B) \to \mathcal{F}(A) \) defined by \( \gamma \mapsto \bigcup \gamma \), and we will identify \( \mathcal{F}(B) \) with its image in \( \mathcal{F}(A) \). Given a polynomial mapping \( \varphi: \mathcal{F}(A) \to H \), we call the polynomial mapping \( \varphi|_{\mathcal{F}(B)} \) a subpolynomial of \( \varphi \) corresponding to the disjoint subcollection \( B \) and denote it by \( \varphi_{\downarrow B} \). Any disjoint subcollection \( B \) of a disjoint subcollection in \( A \) induces the disjoint subcollection \( B' = \{ \bigcup C : C \in B \} \) in \( A \); abusing notation, we will denote the subpolynomial \( \varphi_{\downarrow B} \) of \( \varphi \) by \( \varphi_{\downarrow B} \).

Let \( \tilde{\Phi}: A^d \to H \) be a q-producing function for a polynomial mapping \( \varphi: \mathcal{F}(A) \to H \) of degree \( \leq d \) and let \( \Phi \in \text{Lin}(A^d, H) \) be the linear mapping produced by \( \tilde{\Phi} \). Given a disjoint \( s \)-subcollection \( B = \{ B_1, \ldots, B_s \} \) in \( A \), one finds a q-producing function for the subpolynomial \( \varphi_{\downarrow B} \) as follows. For any \( \beta \subseteq B \) we have
\[
\varphi_{\downarrow B}(\beta) = \varphi\left( \bigcup_{C \in \beta} C \right) = \Phi\left( \left( \bigcup_{C \in \beta} C \right)^d \right) = \sum_{v \in (\bigcup_{C \in \beta} C)^d} \tilde{\Phi}(v) = \sum_{v \in C_1 \times \cdots \times C_d} \tilde{\Phi}(v) = \sum_{(C_1, \ldots, C_d) \in \beta^d} \Phi(C_1, \ldots, C_d); 
\] (1.3)

thus, the mapping \( \Phi|_{B^d} \) is a q-producing function for \( \varphi_{\downarrow B} \).

The following proposition establishes the \( \text{IP}^*_r \)-recurrence property of polynomial mappings with values in the torus \( T = \mathbb{R}/\mathbb{Z} \).

**Proposition 1.1.** (Cf. [B], Theorem 7.7) For any \( k, d \in \mathbb{N} \) and \( \varepsilon > 0 \) there exists \( r = r(k, d, \varepsilon) \in \mathbb{N} \) such that for any \( \varphi_1, \ldots, \varphi_k \in \text{Pol}^0([1, r], T) \) there exists a nonempty \( \alpha \in \mathcal{F}([1, r]) \) such that \( \text{dist}(\varphi_i(\alpha), 0) < \varepsilon \) for all \( i \in \{1, \ldots, k\} \) (where \( \text{"dist" is the distance on } T \)).

**Proof.** Put \( c = [1/\varepsilon] \) and partition the torus \( T \) into \( c \) intervals of length \( \leq 1/\varepsilon \). By the Polynomial Hales-Jewett theorem (see [BL1], Theorem 0.10), there exists \( r \in \mathbb{N} \) such that for any partition of \( \mathcal{F}([1, r]^d \times [k]) \) into \( c \) subsets there exist \( \gamma \subset [1, r]^d \times [k] \) and a nonempty \( \alpha \subset [1, r] \) such that \( \gamma \cap (\alpha^d \times [k]) = \emptyset \) and the sets \( \gamma, \gamma \cup (\alpha^d \times \{1\}), \ldots, \gamma \cup (\alpha^d \times \{k\}) \) belong to the same element of the partition. Let \( \varphi_1, \ldots, \varphi_k \in \text{Pol}^0([1, r], T) \). For each \( i \) let \( \tilde{\Phi}_i: [1, r]^d \to T \) be a q-producing function for \( \varphi_i \). Define a mapping \( \tilde{\Phi}: [1, r]^d \times [k] \to T^k \) by \( \tilde{\Phi}(v, i) = \tilde{\Phi}_i(v), v \in [1, r]^d, i \in [k] \), and let \( \Phi \in \text{Lin}([1, r]^d \times [k], T) \) be the linear mapping produced by \( \tilde{\Phi} \). Then, via \( \Phi \), the partition of \( T \) defines a partition of \( \mathcal{F}([1, r]^d \times [k]) \) into \( c \) subsets. Applying the Polynomial Hales-Jewett theorem, we can find \( \gamma \subset [1, r]^d \times [k] \) and a nonempty \( \alpha \subset [1, r] \) such that \( \gamma \cap (\alpha^d \times [k]) = \emptyset \) and the sets \( \gamma, \gamma \cup (\alpha^d \times \{1\}), \ldots, \gamma \cup (\alpha^d \times \{k\}) \) belong to the same element of the partition; then for any \( i \), \( \Phi(\gamma) \) and \( \Phi(\gamma \cup (\alpha^d \times \{i\})) \)
Proposition 1.3. For any $\varphi \in \text{Pol}_d([1, r], \mathbb{T})$ there exists a disjoint $s$-subcollection $B$ in $[1, r]$ such that a $q$-producing function $\hat{\Phi}_B$ for $\varphi \downarrow_B$ satisfies $\text{dist}(\hat{\Phi}_B, 0) < \varepsilon$.

Proof. Take $r_0 = r(s^d, d, \varepsilon)$ as in Corollary 1.2, and put $A = [1, s] \times [1, r_0]$ (and $r = |A| = sr_0$). Let $\varphi \in \text{Pol}_d^0([1, r], \mathbb{T})$. Let $\hat{\Phi} : [1, r] \rightarrow \mathbb{T}$ be a $q$-producing function for $\varphi$ and let $\Phi \in \text{Lin}([1, r]^d, \mathbb{T})$ be the linear mapping produced by $\hat{\Phi}$. For each $i = (i_1, \ldots, i_d) \in [1, s]^d$ define a polynomial mapping $\varphi_I \in \text{Pol}_d^0([1, r_0], \mathbb{T})$ by $\varphi_I(\alpha) = \Phi((\{i_1\} \times \alpha) \times \cdots \times (\{i_d\} \times \alpha))$. By Corollary 1.2 there exists $\alpha \subseteq [1, r_0]$ such that $\text{dist}(\varphi_I(\alpha), 0) < \varepsilon$ for all $I \in [1, s]^d$. Take the disjoint $s$-subcollection $B = \{\{i\} \times \alpha : i \in [1, s]\}$ in $A$. By the choice of $\alpha$, for any $w \in B^d$ we have $\text{dist}(\hat{\Phi}(w), 0) < \varepsilon$. Since, by (1.3), $\Phi|_{B^d}$ is a $q$-producing function for $\varphi \downarrow_B$, we are done. \hfill \blacksquare

Replacing in Proposition 1.3 $\varepsilon$ by $\varepsilon/s^d$, we obtain:

Corollary 1.4. For any $d, s \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that for any $\varphi \in \text{Pol}_d^0([1, r], \mathbb{T})$ there exists a disjoint $s$-subcollection $B$ in $[1, r]$ such that $\text{dist}(\varphi \downarrow_B, 0) < \varepsilon$.

By formula (1.2), any value of the $t$-producing function for $\varphi \in \text{Pol}_d^0(A, \mathbb{R})$ is a sum of less than $d^d$ values of the $q$-producing function for $\varphi$. Hence, Proposition 1.3 implies the following corollary:

Proposition 1.5. For any $d, s \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that for any $\varphi \in \text{Pol}_d^0([1, r], \mathbb{T})$ there exists a disjoint $s$-subcollection $B$ in $[1, r]$ such that the $t$-producing function $\hat{\Phi}_B$ for $\varphi \downarrow_B$ satisfies $\text{dist}(\hat{\Phi}_B, 0) < \varepsilon$.

In terms of polynomial mappings with values in $\mathbb{R}$, Proposition 1.5 takes the following form:

Corollary 1.6. For any $d, s \in \mathbb{N}$ and $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that for any $\varphi \in \text{Pol}_d^0([1, r], \mathbb{R})$ there exists a disjoint $s$-subcollection $B$ in $[1, r]$ such that the $t$-producing function $\hat{\Phi}_B$ for $\varphi \downarrow_B$ satisfies $\|\hat{\Phi}_B\| < \varepsilon$. 

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Now let \( \varphi \in \text{Pol}^0_d([1,r], \mathbb{R}) \) be a polynomial mapping whose t-producing function \( \Phi \) satisfies \( \|\Phi\| < 1/r^d \). We will denote by \( [x] \) the integer and by \( \{x\} \) the fractional parts of \( x \in \mathbb{R} \). If \( x \in \mathbb{R} \) satisfies \( \|x\| < \varepsilon \), then either \( \{x\} < \varepsilon \) or \( \{x\} > 1 - \varepsilon \). If \( x_1, \ldots, x_n \in \mathbb{R} \) satisfy \( \{x_i\} < 1/n, i = 1, \ldots, n \), then \( \sum_{i=1}^n x_i = \sum_{i=1}^n [x_i] \). Thus, if \( \Phi \) satisfies \( \|\Phi\| < 1/r^d \), then for any \( \alpha \subseteq [1,r] \),

\[
[\varphi(\alpha)] = \left[ \sum_{u \in \alpha \subseteq [1,d]} \Phi(u) \right] = \sum_{u \in \alpha \subseteq [1,d]} [\Phi(u)]
\]

and so, \( \varphi \) is also a polynomial mapping, \( \varphi \in \text{Pol}^0_d([1,r], \mathbb{Z}) \), with the t-producing function \( \tilde{\Phi} \).

For any \( x \in \mathbb{R} \setminus \mathbb{Z} \), \( [x] = -[-x] - 1 \) and \( \{x\} = 1 - \{x\} \), so, if \( x_1, \ldots, x_n \in \mathbb{R} \) satisfy \( \{x_i\} > 1/2n, i = 1, \ldots, n \), then

\[
[\sum_{i=1}^n x_i] = -[-\sum_{i=1}^n x_i] - 1 = -[\sum_{i=1}^n (-x_i)] - 1 = -\sum_{i=1}^n [-x_i] - 1 = \sum_{i=1}^n (-[-x_i]) - 1.
\]

Applying this to \( \tilde{\Phi} \), we see that if \( \tilde{\Phi} \) satisfies \( \|\tilde{\Phi}\| > 1/1/r^d \), then for any \( \alpha \subseteq [1,r] \),

\[
[\varphi(\alpha)] = \left[ \sum_{u \in \alpha \subseteq [1,d]} \tilde{\Phi}(u) \right] = \sum_{u \in \alpha \subseteq [1,d]} (-[-\tilde{\Phi}(u)]) - 1.
\]

So, \( \varphi + 1 \) is a polynomial mapping, \( \varphi + 1 \in \text{Pol}^0_d([1,r], \mathbb{Z}) \), with the t-producing function \( -[-\tilde{\Phi}] \).

In the general case, when \( \|\tilde{\Phi}\| < 1/r^d \), we may have neither \( \{\tilde{\Phi}\} < 1/r^d \) nor \( \{\tilde{\Phi}\} > 1 - 1/r^d \). However, if \( \varphi \) is a homogeneous polynomial of degree \( l \leq d \) (which means that \( \varphi(\alpha) = \sum_{u \in \alpha \subseteq [1,d]} \tilde{\Phi}(u) \)), then, given \( s \in \mathbb{N} \), if \( r \) is large enough, by the classical Ramsey theorem we can choose an \( s \)-element subset \( B \) of \([1,r]\) such that either \( \{\tilde{\Phi}(u)\} < 1/r^d \) for all \( u \in B(d) \) or \( \{\tilde{\Phi}(u)\} > 1 - 1/r^d \) for all \( u \in B(d) \). Identifying \( B \) with the “singleton disjoint -subcollection” \( \{\{b\} : b \in B\} \) in \([1,r]\), we will therefore have \( \varphi_{\downarrow B} \in \text{Pol}^0_d(B, \mathbb{Z}) + e \) with \( e \in \{0,-1\} \).

For a general \( \varphi \in \text{Pol}^0_d([1,r], \mathbb{R}) \), applying this argument to all homogeneous components of \( \varphi \) and using a diagonal process, we arrive at the following lemma:

**Lemma 1.7.** For any \( d, s \in \mathbb{N} \) there exists \( r \in \mathbb{N} \) such that for any \( \varphi \in \text{Pol}^0_d([1,r], \mathbb{R}) \) whose t-producing function \( \Phi \) satisfies \( \|\Phi\| < 1/r^d \) there exists a (singleton) disjoint subcollection \( B \) in \([1,r]\) such that \( [\varphi_{\downarrow B}] \in \text{Pol}^0_d(B, \mathbb{Z}) + e \) with \( e \in \{0,-1, \ldots, -d\} \).

Combining Lemma 1.7 with Corollary 1.6 we obtain:

**Theorem 1.8.** For any \( d, s \in \mathbb{N} \) there exists \( r \in \mathbb{N} \) such that for any \( \varphi \in \text{Pol}^0_d([1,r], \mathbb{R}) \) there exists a disjoint \( s \)-subcollection \( B \) in \([1,r]\) such that \( [\varphi] \in \text{Pol}^0_d(B, \mathbb{Z}) + e \) with \( e \in \{0,-1, \ldots, -d\} \).

Using induction on \( k \), one can extend Theorem 1.8 to the case of \( k \) polynomials:

**Theorem 1.9.** For any \( k, d_1, \ldots, d_k, s \in \mathbb{N} \) there exists \( r = r(k,(d_1, \ldots, d_k),s) \in \mathbb{N} \) such that for any \( \varphi_i \in \text{Pol}^0_{d_i}([1,r], \mathbb{R}) \), \( i = 1, \ldots, k \), there exists a disjoint \( s \)-subcollection \( B \) in \([1,r]\) such that for every \( i \in \{1, \ldots, k\} \), \( [\varphi_i] \in \text{Pol}^0_{d_i}(B, \mathbb{Z}) + e_i \), with \( e_i \in \{0,-1, \ldots, -d\} \).
A generalized polynomial is a function obtained from conventional polynomials using the operations of taking the integer part, addition, and multiplication. We say that a generalized polynomial $\varphi$ is constant free if all polynomials involved in the expression of $\varphi$ have zero constant term. (More precisely, a generalized polynomial is constant free if it has a representation in which all polynomials have zero constant term. A similar convention applies to all the definitions below.) We say that a polynomial $\varphi$ is open if it is contained in the ideal, in the ring of constant free generalized polynomials, generated by the ordinary polynomials. This is equivalent to saying that $\varphi$ (or rather a representation of $\varphi$) has no summand that is a product of “closed” generalized polynomials $[\varphi_i]$. Any open constant-free generalized polynomial is representable in the form

$$\varphi = \sum_{j=1}^{m} [\varphi_{j,1}] \cdots [\varphi_{j,t_j}] \varphi_{j,0}$$  \hspace{1cm} (1.4)$$

where for every $j$, $\varphi_{j,1}, \ldots, \varphi_{j,t_j}$ are open constant-free generalized polynomials and $\varphi_{j,0}$ are conventional polynomials with zero constant term.

We now introduce the notions of height, width, and degree for (a representation of) a generalized polynomial $\varphi$:

The height $h(\varphi)$ of $\varphi$ is the maximum length of sequences of nested brackets in $\varphi$: we put $h(\varphi) = 0$ if $\varphi$ is a conventional polynomial and we say that $h(\varphi) \leq h$ if $\varphi$ has a representation (1.4) where for all $j$ and all $t \geq 1$, $h(\varphi_{j,t}) \leq h - 1$.

The width $w(\varphi)$ is the maximum number of components in $\varphi$ itself and in all its components: we put $w(\varphi) = 1$ if $\varphi$ is a conventional polynomial and we say that $w(\varphi) \leq w$ if $\varphi$ has a representation (1.4) where $w(\varphi_{j,t}) \leq w$ for all $j$ and all $t \geq 1$ and also $\sum_{j=1}^{m} (l_j + 1) \leq w$.

The degree $d(\varphi)$ of $\varphi$ is defined as usual under the assumption that $\deg[\varphi] = \deg \varphi$: we say that $d(\varphi) \leq d$ if $\varphi$ has a representation (1.4) with $\max_{j=1}^{m} (\sum_{t=0}^{l_j} \deg \varphi_{j,t}) \leq d$.

(For example, for $\varphi(x) = [x^2 + 1]x^3 + 2x]x + [x^2](x^3 + 1)x^3$ we have $h(\varphi) = 2$, $w(\varphi) = 6$, and $d(\varphi) = 7$.)

We extend the above definitions to generalized polynomial mappings with domain $\mathcal{F}(A)$, and will denote by $\text{GPol}^0_{d,h,w}(A, H)$ the set (the algebra) of open constant-free generalized polynomial mappings $\varphi: \mathcal{F}(A) \rightarrow H$, where $H = \mathbb{R}$ or $\mathbb{Z}$, with $d(\varphi) \leq d$, $h(\varphi) \leq h$, and $w(\varphi) \leq w$. Given $\varphi \in \text{GPol}^0_{d,h,w}(A, H)$ and a disjoint subcollection $B$ in $A$, we define the generalized polynomial mapping $\varphi_{\downarrow B} \in \text{GPol}^0_{d,h,w}(B, H)$ as the restriction of $\varphi$ to the set $\mathcal{F}(B)$ considered as a subset of $\mathcal{F}(A)$.

The following theorem says that generalized polynomial mappings turn into ordinary polynomial mappings after being restricted to a suitable disjoint subcollection in their domain:

**Theorem 1.10.** For any $k, d_1, \ldots, d_k, h, w, s \in \mathbb{N}$ there exists $r = r(k, (d_1, \ldots, d_k), h, w, s) \in \mathbb{N}$ such that for any $\varphi_i \in \text{GPol}^0_{d_i,h,w}([1, r], \mathbb{R})$, $i = 1, \ldots, k$, there exists a disjoint $s$-subcollection $B$ in $[1, r]$ such that $\varphi_{i \downarrow B} \in \text{Pol}^0_{d_i}(B, \mathbb{R})$, $i = 1, \ldots, k$. 

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Proof. We will use induction on \( h \); when \( h = 0 \) the statement is trivial. Take \( r_0 \) to be the maximum of the integers \( r(l, (b_1, \ldots, b_l), s) \) in Theorem 1.9 over all integers \( l \leq kw \) and all \( l \)-tuples \( (b_1, \ldots, b_l) \) of nonnegative integers with \( \sum_{j=1}^{l} b_j \leq w \sum_{i=1}^{k} d_i \). By induction on \( h \), let \( r \) be the maximum of the integers \( r(l, (d_1, \ldots, d_l), h - 1, w, r_0) \) in the assertion of Theorem 1.10 over all integers \( l \leq kw \) and all \( l \)-tuples \( (b_1, \ldots, b_l) \) of nonnegative integers with \( \sum_{j=1}^{l} b_j \leq w \sum_{i=1}^{k} d_i \). Let \( \varphi_i \in \text{GPOL}^0_{d_i, h, w}([1, r], \mathbb{R}) \), \( i = 1, \ldots, k \). For each \( i \) represent \( \varphi_i \) in the form

\[
\varphi_i = \sum_{j=1}^{m_i} [\varphi_{i,j,1}] \cdots [\varphi_{i,j,l_i,j}] \varphi_{i,j,0},
\]

where for every \( i, j \) we have \( \varphi_{i,j,0} \in \text{Pol}^0_{d_{i,j,0}}([1, r], \mathbb{R}) \) and for every \( t \geq 1 \) we have \( \varphi_{i,j,t} \in \text{GPOL}^0_{d_{i,j,t}, h-1, w}([1, r], \mathbb{R}) \) with

\[
\sum_{j=1}^{m_i} (l_{i,j} + 1) \leq w \text{ for all } i \text{ and } \sum_{t=0}^{d_{i,j}} d_{i,j,t} \leq d_i \text{ for all } i, j,
\]

so that

\[
\sum_{i=1}^{k} \sum_{j=1}^{m_i} (l_{i,j} + 1) \leq kw \text{ and } \sum_{i=1}^{k} \sum_{j=1}^{m_i} \sum_{t=0}^{d_{i,j}} d_{i,j,t} \leq w \sum_{i=1}^{k} d_i.
\]

By the choice of \( r \) there exists a disjoint \( r_0 \)-subcollection \( B_0 \subset \mathcal{F}([1, r]) \) such that \( \varphi_{i,j,t} \downarrow B_0 \in \text{GPOL}^0_{d_{i,j,t}}(B_0, \mathbb{R}) \) for all \( i, j, t \). Then by the choice of \( r_0 \) there exists a disjoint \( s \)-subcollection \( B \) in \( B_0 \) such that for all \( i, j, t \), \( [\varphi_{i,j,t} \downarrow B] \in \text{Pol}^0_{d_{i,j,t}}(B, \mathbb{Z}) \). Hence for every \( i \),

\[
\varphi_{i} \downarrow B = \sum_{j=1}^{m_i} [\varphi_{i,j,1} \downarrow B] \cdots [\varphi_{i,j,l_{i,j}} \downarrow B] \varphi_{i,j,0} \in \text{Pol}^0_{d_i}(B, \mathbb{R}).
\]

Combining Theorem 1.10 and Corollary 1.2, we obtain:

**Theorem 1.11.** For any \( k, d, h, w \in \mathbb{N} \) there exists \( r = r(k, d, h, w) \in \mathbb{N} \) such that for any \( \varphi_1, \ldots, \varphi_k \in \text{GPOL}^0_{d, h, w}([1, r], \mathbb{R}) \) there exists a nonempty \( \alpha \in \mathcal{F}([1, r]) \) such that \( \|\varphi_i(\alpha)\| < \varepsilon \), \( i = 1, \ldots, k \).

Let \( X = G/\Gamma \) be a \( k \)-dimensional compact nilmanifold; we may and will assume that \( X \) is connected. (Any nilmanifold is a subnilmanifold of a connected one.) Let \( \rho \) be a metric on \( X \) (induced by a metric on \( G \) compatible with the Lie group structure thereon). Fix a point \( x_0 \in X \), and let \( \tau = (\tau_1, \ldots, \tau_k): X \rightarrow [0, 1)^k \) be Maltsev’s coordinates on \( X \) centered at \( x_0 \). The inverse mapping \( \tau^{-1} \) is continuous, and the distance \( \rho(x, x_0) \) from \( x \in X \) to \( x_0 \) is continuous with respect to the distance from \( \tau(x) \) to the set of vertices \( \{0, 1\}^k \) of the cube \( [0, 1]^k \). (See, for example, [BL2], Section 1.5.)

Let \( g \) be an \((l\text{-parameter})\ polynomial sequence in \( G \), that is, a mapping \( g: \mathbb{Z}^l \rightarrow G \) of the form \( g(n) = T_{1,n}^{p_1} \cdots T_{b,n}^{p_b}, n \in \mathbb{Z}^l \), where \( T_1, \ldots, T_b \in G \), \( p_1, \ldots, p_b \) are polynomials \( \mathbb{Z}^l \rightarrow \mathbb{Z} \); we define \( \text{n-deg} g \), the naive degree of \( g \), as \( \max_{i=1}^{b} \text{deg} p_i \). Then for each \( i = 1, \ldots, k \), the sequences \( \psi_i(n) = \tau_i(g(n)x_0), n \in \mathbb{Z}^c \), of coordinates of \( x_0 \) under the action of \( g \) are open \([0, 1]\)-valued generalized polynomials, with parameters depending only on \( X \) and
n-deg \, g \, (\text{see [BL2], Theorem A and Theorem A**}), and if \, g(0) = 1_G, \, these \, polynomials \, can \, be \, assumed \, to \, be \, constant-free. \, For \, any \, polynomial \, mapping \, \varphi \in \text{Pol}_d([1, r], \mathbb{Z}^c), \, the \, composition \, mappings \, \psi_i \circ \varphi: F([1, r]) \rightarrow [0, 1], \, i = 1, \ldots, k, \, are \, open \, constant-free \, generalized \, polynomial \, mappings, \, with \, parameters \, only \, depending \, on \, X, \, d, \, and \, n-deg \, g. \, From \, Theorem \, 1.11 \, we \, now \, obtain \, the \, following \, result:

**Theorem 1.12.** Let \( X = G/\Gamma \) be a nilmanifold with metric \( \rho \). For any \( a, d \in \mathbb{N} \) and \( \varepsilon > 0 \) there exists \( r = r(a, d, \varepsilon) \in \mathbb{N} \) such that for any \( l \), any \( l \)-parameter polynomial sequence \( g \) in \( G \) with \( n-deg \, g \leq a \) and \( g(0) = 1_G \), any \( x_0 \in X \), and any \( \varphi \in \text{Pol}^d([1, r], \mathbb{Z}^l) \) there exists a nonempty \( \alpha \in F([1, r]) \) such that \( \rho(g(\varphi(\alpha))x_0, x_0) < \varepsilon \).

**Remark.** Theorem 1.12 easily extends to generalized polynomial sequences in nilpotent groups, that is, to sequences of the form \( g(n) = T_{p_1(n)}^r \cdots T_{p_k(n)}^r \) where \( p_i \) are generalized polynomials \( \mathbb{Z}^l \rightarrow \mathbb{Z} \).

## 2. IP\textsuperscript{*}-recurrence implies approximability by nilsystems

In this section we prove the second statement of Theorem 0.2. Let \((X, \rho)\) be a compact metric space, \( T \) be a self homeomorphism of \( X \), and assume that \((X, T)\) is IP\textsuperscript{*}-recurrent. Then, in particular, \((X, T)\) is IP\textsuperscript{*}-recurrent, so by Theorem 0.1, \((X, T)\) is distal, and thus is a disjoint union of minimal subsystems (see [F], corollary to Theorem 8.7). Hence, we may assume that \((X, T)\) is minimal.

Now, by the way of contradiction, assume that a minimal system \((X, T)\) is not a pre-nilsystem, that is, not an inverse limit of nilsystems; our goal is to show that there exists a point \( x \in X \) and \( \varepsilon > 0 \) such that for every \( r \in \mathbb{N} \) there exists a linear mapping \( \varphi \in \text{Lin}([1, r], \mathbb{Z}) \) such that \( \rho(T^{\varphi(\alpha)}x, x) > \varepsilon \) for every nonempty \( \alpha \subseteq [1, r] \).

We will use the following result ([HKM] Theorem 1.3 and Corollary 4.2): for any \( r \), the maximal \( r \)-step pro-nilfactor of \((X, T)\) is defined by a closed \( T \)-invariant equivalence relation \( \text{RP}^r \subseteq X^2 \) (called the regionally proximal relation of order \( r \)), with \((x_0, y_0) \in \text{RP}^r \) if and only if for any \( \delta > 0 \) there exists a point \( x \in X \) and a mapping \( \varphi \in \text{Lin}([1, r], \mathbb{Z}) \) such that

\[
\rho(x, x_0) < \delta \quad \text{and} \quad \rho(T^{\varphi(\alpha)}x, y_0) < \delta \quad \text{for all nonempty} \quad \alpha \subseteq [1, r].
\]  

Our assumption that \((X, T)\) is not a pre-nilsystem is equivalent to the assumption that \( \bigcap_{r=1}^{\infty} \text{RP}^r \neq \Delta \), where \( \Delta \) is the diagonal of \( X^2 \). Fix \((x_0, y_0) \in \bigcap_{r=1}^{\infty} \text{RP}^r \) with \( x_0 \neq y_0 \).

Let \( \varepsilon = \inf_{n \in \mathbb{Z}} \rho(T^n x_0, T^n y_0) \); since \((X, T)\) is distal, we have \( \varepsilon > 0 \). Since \((X, T)\) is minimal, the orbit \( \{T^n x_0\}_{n \in \mathbb{Z}} \) of \( x_0 \) is dense in \( X \). Let \( r \in \mathbb{N} \) and let \( U \subseteq X \) be an open set. Choose \( n \in \mathbb{Z} \) such that \( T^n x_0 \in U \) and choose \( \delta > 0 \) such that \( \rho(T^n x, T^n y) < \varepsilon/3 \) whenever \( \rho(x, y) < \delta \).

Find \( x \in X \) such that (2.1) holds and \( T^n x \in U \). Then \( \rho(T^n x, T^n x_0) < \varepsilon/3 \) and \( \rho(T^{\varphi(\alpha)}T^n x, T^n y_0) < \varepsilon/3 \) for all nonempty \( \alpha \subseteq [1, r] \), and since \( \rho(T^n x_0, T^n y_0) \geq \varepsilon \), we have that \( \rho(T^{\varphi(\alpha)}T^n x, T^n x) > \varepsilon/3 \) for all nonempty \( \alpha \subseteq [1, r] \). This proves that for any \( r \in \mathbb{N} \) the open set

\[
R_r = \{x \in X : \text{there exists} \, \varphi \in \text{Lin}([1, r], \mathbb{Z}) \, \text{such that} \, \rho(T^{\varphi(\alpha)}x, x) > \varepsilon/3 \}
\]

for all nonempty \( \alpha \in [1, r] \).
is dense in $X$. By Baire category theorem $\bigcap_{r=1}^{\infty} R_r$ is nonempty, which gives us what we wanted – a point $x \in X$ such that for every $r \in \mathbb{N}$ there exists a mapping $\varphi \in \text{Lin}([1, r], \mathbb{Z})$ such that $\rho(T^{\varphi(\alpha)} x, x) > \varepsilon/3$ for every nonempty $\alpha \subseteq [1, r]$.

Bibliography


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