## Topic 1

Let $p \neq 2$ be a prime integer. A group $G$ is *a $p$-group* if $g^p = \mathbf{1}_G$ for all $g \in G$.

**Lemma 1.** *Let $G$ be a nilpotent group of class $k$ with $k < p$, let $S \subseteq G$ generate $G$ and let $s^p = \mathbf{1}_G$ for every $s \in S$. Then $G$ is a $p$-group.*

**Proof.** Let $\{\mathbf{1}_G\} \subset G_1 \subset G_2 \subset \ldots \subset G_k = G$ be the lower central series of $G$. We'll use induction on $i$ to prove that $G_i$ is a $p$-group. Let $1 \leq i \leq k-1$, let $g \in G_i$, $g = \prod_{l=1}^{r} [s_l, h_l]$, $s_l \in S$, $h_l \in G_{i+1}$, $l = 1, \ldots, r$. Then, in terminology of [1], $g^n \left( \prod_{l=1}^{r} [s_l^n, h_l] \right)^{-1}$ is a polynomial sequence of degree $\leq (1, 2, \ldots)$ in $G$, lying in $G_{i-1}$. So, $g^n = \prod_{l=1}^{r} [s_l^n, h_l] \prod_{j=1}^{i-1} g_j^{\binom{n}{k-j+1}}$ for some $g_j \in G_j$, $j = 1, \ldots, i-1$ (by a modification of Hall-Petresco Theorem in [1]). Since $\binom{p}{t} \vdots p$ for $t < p$, the last product vanishes for $n = p$.

Let now $g \in G$, $g = \prod_{l=1}^{r} s_l$. Then, for the same reason as above, we have $g^n = \prod_{l=1}^{r} s_l^n \prod_{j=1}^{k-1} g_j^{\binom{n}{k-j+1}}$ for some $g_j \in G_j$, $j = 1, \ldots, k-1$, and so, $g^p = \mathbf{1}_G$. ■

**Lemma 2.** *A finitely generated nilpotent $p$-group is finite.*

**Proof.** In a Malcev basis $g_1, \ldots, g_r$ of the group, every its element is representable in the form $g_1^{d_1} g_2^{d_2} \ldots g_r^{d_r}$ with $0 \leq d_l \leq p-1$, $l = 1 \ldots, r$. ■

**Lemma 3.** *Let $H$ be a nilpotent $p$-group. Given $h_1, \ldots, h_r \in H$ and $d_1, \ldots, d_r \in \mathbb{Z}$ with $d = d_1 + \ldots + d_r \not\equiv 0 (\operatorname{mod} p)$, the mapping $\varphi\colon g \mapsto \prod_{l=1}^{r} g^{d_l} h_l$ is a one-to-one mapping of $H$ onto itself.*

**Proof.** Replace $\varphi$ by $\varphi(\mathbf{1}_G)^{-1}\varphi$. We may assume that $H$ is finitely generated, and so finite by Lemma 2. Let $H_1 \subset \ldots \subset H_k = H$ be the lower central series of $H$. The mapping $\varphi$ preserves each $H_i$, $i = 1, \ldots, k$, and the mapping $H_i/H_{i-1} \longrightarrow H_i/H_{i-1}$ induced by $\varphi$ is of the form $g \to g^d$. Since $H_i/H_{i-1}$ is a commutative $p$-group, that is a $\mathbb{Z}_p$-vector space, this mapping is surjective, and so is $\varphi$ itself. ■

For $k \in \mathbb{N}$ and a set $S$, *the free nilpotent $p$-group of class $k$ over $S$* is the group

$$\left\langle S \;\middle|\; [s_1, [s_2, \ldots [s_k, s] \ldots]] = \mathbf{1},\ s^p = \mathbf{1},\ s_1, s_2, \ldots, s_k, s \in S \right\rangle.$$

We will call free nilpotent $p$-groups simply "free".

Let $K$ be an infinite countable field of characteristic $p$, let $G$ be the group of $3 \times 3$ upper-triangular matrices with unit main diagonal: $G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},\ a, b, c \in K \right\}$. Then $G$ is a nilpotent $p$-group of class 2. We denote the commutator of $G$ by $G_1$: $G_1$ consists of matrices of the form $\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c \in K$.

**Proposition.** *Let $T$ be a unitary action of $G$ on a Hilbert space $M$ such that $T\big|_{G_1}$ is weakly mixing on $M$. Let $u \in M$, let $H$ be a finite free subgroup of $G$, let $\varepsilon > 0$. There is $g \in G \setminus H$ such that the group $H'$ generated by $g$ and $H$ is free and $\big|\langle hu, u\rangle\big| < \varepsilon$ for all $h \in H' \setminus H$.*

**Corollary (informal).** *Under the assumptions of the proposition above, $G$ contains an infinite free subgroup $F$ such that the function $\big|\langle hu, u\rangle\big|$ decreases "as fast as one wishes" on $F$.*

**Proof. 1.** Let's fix a sequence of subgroups in $G$: we enumerate the elements of $G$, and let $\Phi_m$ be the subgroup of $G$ generated by the first $m$ elements. We will measure densities of subsets in $G$ with respect to the Følner sequence $\Phi_1, \Phi_2, \ldots$, and in $G_1$ with respect to the Følner sequence $\Phi_1 \cap G_1, \Phi_2 \cap G_1, \ldots$.

**2.** If $H$ is a nilpotent $p$-group of class 2, then $H/[H,H]$ and $[H,H]$ are commutative $p$-groups and so, can be considered as vector spaces over the field $\mathbb{Z}_p$. If such $H$ is generated by a set $S$, then $H$ is free if $S$ is a basis for $H/[H,H]$ and the elements $[s_1, s_2]$ for all distinct $s_1, s_2 \in S$ form a basis for $[H,H]$. It follows that, after adding a new element $g$ to $H$ we will still have a free group if $[g,H] \cap [H,H] = \{\mathbf{1}_H\}$.

Thus the condition "the subgroup $H'$ of $G$ generated by $g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and a finite free $H \subset G$ is free" converts into finitely many inequalities of the form $at - br \neq w$, for $\begin{pmatrix} 1 & r & q \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \in H \setminus [H,H]$ and $\begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in [H,H]$. So, elements $g$ satisfying this requirement form the complement to finitely many "planes" in the "3-dimensional space" $(a,b,c)$ over $K$, that is a set of density one in $G$.

**3.** Since the subgroup $G_1$ of $G$ is weakly mixing on $M$, $G$ is weakly mixing on $M$ as well. So, the set $\Big\{f \in G : \big|[fu, v]\big| < \varepsilon\Big\}$ has density one for any $v \in M$ in both $G$ and $G_1$.

**4.** For $g \in G$, every element of the subgroup $H'$ of $G$ generated by $g$ and $H$ is of the form $g^d h_1 [g, h_2]$ with $h_1 \in H$, $h_2 \in H \setminus G_1$ and $0 \leq d \leq p-1$. Fix $h_1, h_2$ and $k$ and consider the mapping $\varphi \colon G \longrightarrow G$, $\varphi(g) = g^d h_1 [g, h_2]$. For $d \neq 0$, $\varphi$ is a self-bijection of $G$ by Lemma 3, and of any subgroup of $G$ containing both $h_1$ and $h_2$. In particular, $\varphi$ is a self-bijection of $\Phi_m$ if $m$ is big enough. So, the set of $g$ for which $\big|\langle \varphi(g)u, u\rangle\big| < \varepsilon$ is of density one in $G$ in this case.

Let $d = 0$. The mapping $g \to [g, h_2]$ is a homomorphism of $G$ *onto* $G_1$, and of $\Phi_m$ *onto* $\Phi_l \cap G_1$ if $m \gg l$. Hence, again, $g$ for which $\big|\langle \varphi(g)u, u\rangle\big| = \big|\big\langle [g, h_2]u, h_1^{-1}u\big\rangle\big| < \varepsilon$ form a subset of density one in $G$.

**5.** Thus, the set of $g \in G$ satisfying the conclusion of the proposition is the intersection of finitely many subsets of density one, and so is nonempty.

**Topic 2**

For a group $G$, we will denote by $\gamma_k G$ the $k-th$ term of the lower central series of $G$.

For a set $S$, let $F(S)$ be the free group generated by $S$. A nilpotent group $G$ is *free of class $c$ (with generating set $S$)* if $G$ is isomorphic to $F(S)/\gamma_{c+1}F(S)$. (Clearly, this group is the universal repelling object in the category of nilpotent groups of class $\leq c$ generated by the (marked) set $S$.)

Under *the rank* of a nilpotent group $G$ we will understand the rank of the abelian group $G/\gamma_2 G$. If $G$ is a free nilpotent group generated by $S$, the rank of $G$ coincides with the cardinality of $S$.

Let $G$ be a nilpotent group and $K$ be any group. The constant mapping $\varphi\colon K \longrightarrow G$, $\varphi \equiv 1$, is *polynomial*. A mapping $\varphi\colon K \longrightarrow G$ is *polynomial* if for every $a \in K$, the mapping $D_a\varphi\colon K \longrightarrow G$ defined by $D_a\varphi(b) = \varphi(ab)\varphi(b)^{-1}$ is polynomial.

**Theorem.** *Under the element-wise multiplication, polynomial mappings $K \longrightarrow G$ form a group.*

If $K$ is a finitely torsion-free nilpotent group, it has *a basis* $S_1, \ldots, S_k \in K$ with the property that every element of $K$ can be uniquely written in the form $S = S_1^{a_1} \ldots S_k^{a_k}$, $a_1, \ldots, a_k \in \mathbb{Z}$. Let both $K$ and $G$ be finitely generated torsion-free nilpotent groups, let $S_1, \ldots, S_k$ be a basis in $K$ and $T_1, \ldots, T_l$ be a basis in $G$. Let $\varphi\colon K \longrightarrow G$; we can write $\varphi(S_1^{a_1} \ldots S_k^{a_k}) = T^{b_1(a_1,\ldots,a_k)} \ldots T^{b_l(a_1,\ldots,a_k)}$.

**Theorem.** *$\varphi$ is polynomial if and only if $b_1, \ldots, b_l$ are all polynomials $\mathbb{Z} \longrightarrow \mathbb{Z}$.*

A subgroup $H$ of a group $G$ is called *closed in $G$* if for every $T \in G \setminus H$, $T^n \notin H$ for all $n \neq 0$. *The closure of $H$* is the minimal closed subgroup of $G$ containing $H$.

**Theorem.** *Let $\varphi\colon K \longrightarrow G$ be a polynomial mapping of nilpotent groups. If $H$ is a closed subgroup of $G$ and $\varphi(K) \not\subseteq H$, then $\varphi(a) \notin H$ for almost all $a \in G$ (that is, for all $a \in G$ but a set of density $0$).*

We will use the following fact:

**Theorem S.** ([2]) *Let $G$ be a finitely generated nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$, $\mathcal{H} = \bigoplus_{\alpha\in A} \mathcal{L}_\alpha$ into the direct sum of a family of pairwise orthogonal subspaces such that elements of $G$ permute the members of the family, and if $T \in G$, $T(\mathcal{L}_\alpha) = \mathcal{L}_\alpha$, then $T$ is either scalar or weakly mixing on $\mathcal{L}_\alpha$. Moreover, for every $\alpha \in A$, $G$ contains a subgroup $G'$ of finite index with the following property: for any $T \in G'$ with $T(\mathcal{L}_\alpha) \neq \mathcal{L}_\alpha$ one has $T^n(\mathcal{L}_\alpha) \neq \mathcal{L}_\alpha$ for all $n \neq 0$.*

Let $G$ be a nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$, let $K$ be a nilpotent group and let $\varphi\colon K \longrightarrow G$ be a polynomial mapping (polynomial action) with $\varphi(0) = 0$. Let $\mathcal{H} = \bigoplus_{\alpha\in A} \mathcal{L}_\alpha$ be the decomposition of $\mathcal{H}$ described in Theorem S. Fix $\alpha \in A$, let $H = \{T \in G \mid T(\mathcal{L}_\alpha) = L_\alpha\}$ and $E = \{T \in H \mid T \text{ is scalar on } \mathcal{L}_\alpha\}$. Let $\overline{H}$ and $\overline{E}$ be the closure of $H$ and of $E$ respectively.

**Proposition X.** *If $\varphi(K) \not\subseteq \overline{H}$ then $\varphi_a(\mathcal{L}_\alpha) \perp \mathcal{L}_\alpha$ for almost all $a \in G$. If $\varphi(K) \subseteq \overline{H} \setminus \overline{E}$ then $\varphi$ is weakly mixing on $\mathcal{L}_\alpha$ (that is, for any $u \in \mathcal{H}^{\mathrm{wm}}(\varphi)$ and any $\varepsilon > 0$, the set $\{T \in K \mid |\langle \varphi(T)u, u\rangle| > \varepsilon\}$ has density 0 in $G$). If $\varphi(K) \subseteq \overline{E}$ then $\varphi$ is compact on $\mathcal{L}_\alpha$ (that is, $\varphi(K)u$ is precompact for all $u \in \mathcal{H}^{\mathrm{c}}(\varphi)$).*

**Theorem.** *Let $G$ be a nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$, let $K$ be a nilpotent group and let $\varphi\colon K \longrightarrow G$ be a polynomial mapping. Then $\mathcal{H} = \mathcal{H}^{\mathrm{c}}(\varphi) \oplus \mathcal{H}^{\mathrm{wm}}(\varphi)$ so that $\varphi$ is compact on $\mathcal{H}^{\mathrm{c}}(\varphi)$ and is weakly mixing on $\mathcal{H}^{\mathrm{wm}}(\varphi)$.*

Let $w \in F(x_1, x_2, \ldots)$. Let *the weight of* $w$ be the maximal $k$ for which $x_k$ participates in $w$ (that is, $w \in F(x_1, \ldots, x_k) \setminus F(x_1, \ldots, x_{k-1})$). We say that $w$ of weight $k$ is *nondegenerate* if the total exponent of $x_k$ in $w$ is nonzero. If the weight of $w$ is $k$, we will denote by $w^0$ the element of $F(x_1, \ldots, x_{k-1})$ obtained from $w$ by erasing all appearances of $x_k$ in it. If $\tau = (T_1, T_2, \ldots)$, let $w(\tau)$ denote the word obtained by replacing each $x_i$ in $w$ by the corresponding $T_i$.

Here is our result:

**Theorem R.** *Let $G$ be a nilpotent group, of rank $d$, of unitary operators on a Hilbert space $\mathcal{H}$, and let $\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{L}_\alpha$ be the decomposition of $\mathcal{H}$ under the action of $G$ described in Theorem S. Let $W \subset F(x_1, x_2, \ldots)$ be such that for every $k \in \mathbb{N}$, $W$ contains finitely many elements of weight $k$, and for each $w \in W$ let $\varepsilon_w$ be a positive real number. Then for any $\alpha_1, \ldots, \alpha_m \in A$ and any $u_1 \in \mathcal{L}_{\alpha_1}, \ldots, u_m \in \mathcal{L}_{\alpha_m}$ there is a sequence $\tau = (T_1, T_2, \ldots) \subseteq G$ such that any $d$ elements of $\tau$ generate a subgroup of finite index in $G$, and for every $w \in W$ and $i = 1, \ldots, m$, either $|\langle w(\tau)u_i, u_i\rangle| < \varepsilon_w$, or $w(\tau)u_i = \lambda w^0(\tau)u_i$, $\lambda = \lambda(w) \in \mathbb{C}$. If the action of $G$ on $\mathcal{H}$ is weakly mixing, then for all nondegenerate $w \in W$ only the first possibility takes place.*

**Proof.** We may assume that $G$ is torsion-free, then $G/\gamma_2 G$ is isomorphic to $\mathbb{Z}^d$. Now, if we pick $T_1, T_2, \ldots \in G$ in such a way that $T_k \bmod \gamma_2 G$ is in the general position with respect to $T_1 \bmod \gamma_2 G, \ldots, T_{k-1} \bmod \gamma_2 G$, then any $d$ elements of $T_1, T_2, \ldots$ generate $G/\gamma_2 G$ and so, $G$ itself. Thus, to satisfy this condition, we may choose every $T_k$ from a set of density 1 in $G$.

Let $\alpha \in A$, $u \in \mathcal{L}_\alpha$, and assume that $T_1, \ldots, T_{k-1}$ have been already chosen. Let $w_1, \ldots, w_l$ be all elements of $W$ of weight $k$. Then for every $j$, $\varphi_j(T) = w_j(T_1, \ldots, T_{k-1}, T)$ can be considered as a mapping $G \longrightarrow G$; needless to say that $\varphi_j$ is polynomial. Consider $\varphi_1$. Let $H = \{T \in G \mid T(\mathcal{L}_\alpha) = L_\alpha\}$ and $E = \{T \in H \mid T \text{ is scalar on } \mathcal{L}_\alpha\}$. We may replace $G$ by $G'$ described in Theorem S; after this, $H$ and $E$ are closed in $G$. By Proposition X, we have 3 possibilities:

1) $\varphi_1(G) \not\subseteq H$. Then $\varphi_1(T) \notin H$ for almost all $T \in G$, and $\varphi_1(T)u \perp u$ for such $T$.

2) $\varphi_1(G)\varphi_1(0)^{-1} \subseteq H \setminus E$. Then $\varphi_1$ is weakly mixing on $L_\alpha$ and so, $|\langle \varphi_1(T)u, u\rangle| < \varepsilon_{w_1}$ for almost all $T \in G$.

3) $\varphi_1(G)\varphi_1(0)^{-1} \subseteq E$. Then $\varphi_1(T)u = \varphi_1(T)\varphi_1(0)^{-1}\varphi_1(0)u = \lambda\varphi_1(0)u$, $\lambda \in \mathbb{C}$.

In any case, the set of $T$ which can serve as $T_k$ for $w_1$ has density 1 in $G$. The same true for the other elements of $W$, $w_2, \ldots, w_l$, and, if instead of $u$ we consider several vectors $u_1, \ldots, u_m$ in $\mathcal{H}$, $u_i \in \mathcal{L}_{\alpha_i}$, then it is true for each of them, that is the set of $T_k$ which satisfy the requirements of the theorem is of density 1 in $G$.

If the action of $G$ is weakly mixing on $\mathcal{H}$, then it is easy to see that the mapping $\varphi_j$ corresponding to a nondegenerate $w_j$ is weakly mixing on $\mathcal{H}$ and so, only the case 2) takes place for such $\varphi_j$. ■

**Example.** Let $G$ be a finitely generated nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$, whose action on $\mathcal{H}$ is weakly mixing. Let $W_1$ be the set of words $w$ in alphabet $x_1, x_2, \ldots$ such that for every $x_i$, it appears in $w$ not more than $d$ times. Let $W_2$ be the set of differences of $W_1$, $W_2 = \left\{ w_1 w_2^{-1} \;\middle|\; w_1, w_2 \in W_1 \right\}$, and $W$ be the set of differences of $W_2$. $W$ has the property that for any $w \in W$, $w^0 \in W$ as well. Take any $u \in \mathcal{H}$. Let $\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{L}_\alpha$ be the decomposition of $\mathcal{H}$ described in Theorem S; replace $u$ by a close vector of the form $u_1 + \ldots + u_m$ with $u_j \in \mathcal{L}_{\alpha_j}$. Find a sequence $T_1, T_2, \ldots \in G$ as in Theorem R, corresponding to $W$ and $u_1, \ldots, u_m$. Then, for every $w \in W_1$, $w(T_1, T_2, \ldots)u$ is almost (up to an a-priory given $\varepsilon_w$) orthogonal to $u$. As for the rest of elements of $W$, for every $j = 1, \ldots, m$ the set $w(T_1, T_2, \ldots)u_j$, $w \in W$, is partitioned into classes of proportional vectors, each class almost orthogonal to $u_j$. It seems that it suffices to make all vectors $w(T_1, T_2, \ldots)u$, $w \in W_1$, to be strictly orthogonal to $u$.

## Bibliography


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