Pointwise convergence of ergodic averages for polynomial actions of $\mathbb{Z}^d$ by translations on a nilmanifold

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Abstract

Generalizing the one-parameter case, we prove that the orbit of a point on a compact nilmanifold X under a polynomial action of $\mathbb{Z}^d$ by translations on X is uniformly distributed on the union of several sub-nilmanifolds of X. As a corollary we obtain the pointwise ergodic theorem for polynomial actions of $\mathbb{Z}^d$ by translations on a nilmanifold.

1. Formulations

1.1. Let G be a nilpotent Lie group, Γ be a closed uniform subgroup of G and X be the compact *nilmanifold* G/Γ. G acts on X by left translations: for $a \in G$ and $x = b\Gamma \in X$ one defines $ax = ab\Gamma$.

We will say that a mapping $g\colon \mathbb{Z}^d \longrightarrow G$ is *polynomial* if g can be written in the form $g(n) = a_1^{p_1(n)} \cdot\ldots\cdot a_m^{p_m(n)}$, where $a_1, \ldots, a_m \in G$ and $p_1, \ldots, p_m$ are polynomial mappings $\mathbb{Z}^d \longrightarrow \mathbb{Z}$. Such a mapping will also be called a *polynomial action* of $\mathbb{Z}^d$ on X by translations, in contrast with a homomorphism $\mathbb{Z}^d \longrightarrow G$, which will be referred to as a *linear action*. We are going to show the following:

1.2. Theorem A. *Let g be a polynomial mapping $\mathbb{Z}^d \longrightarrow G$. For any $x \in X$, $f \in C(X)$ and Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}^d$, $\lim\limits_{N\to\infty} \frac{1}{|\Phi_N|} \sum\limits_{n\in\Phi_N} f\big(g(n)x\big)$ exists.*

An analogous result for polynomial actions of $\mathbb{R}^d$, in a much more general situation, was obtained in [Sh1]. The one-parameter case $d = 1$ of Theorem A was proved in [L].

1.3. Let $\varphi\colon A \longrightarrow X$ be a mapping from a countable amenable group A and let Y be a *sub-nilmanifold of* X, that is, a closed subset of the form $Y = Hy$ where H is a closed subgroup of G and $y \in Y$. Let B be a subset of A; we will say that $\{\varphi(a)\}_{a\in B}$ is *well distributed in* Y if $\varphi(B) \subseteq Y$ and for any $f \in C(Y)$ and any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in A one has $\lim\limits_{N\to\infty} \frac{1}{|\Phi_N\cap B|} \sum\limits_{a\in\Phi_N\cap B} f\big(\varphi(a)\big) = \int_Y f\,d\mu_Y$, where μ_Y is the H-invariant probability measure on Y. In particular, this implies $\overline{\varphi(B)} = Y$.

1.4. In order to prove Theorem A we will show that the closure $Y = \overline{\mathrm{Orb}(x)}$ of the orbit $\mathrm{Orb}(x) = \{g(n)x\}_{n\in\mathbb{Z}^d}$ of $x \in X$ is a disjoint finite union of sub-nilmanifolds of X and that $\{g(n)x\}_{n\in\mathbb{Z}^d}$ is well distributed in the connected components of Y. This fact is known for linear actions by translations:

Theorem. *Let A be a finitely generated amenable group and let $\varphi\colon A \longrightarrow G$ be a homomorphism. For any $x \in X$ there exists a closed subgroup $E \subseteq G$ such that $\varphi(A) \subseteq E$, $\overline{\varphi(A)x} = Ex$ and $\{\varphi(a)x\}_{a\in A}$ is well distributed in Ex.*

For a simple proof of this theorem see [L]. A more general theorem can be found in [Sh2].

1.5. In the case of polynomial actions the situation is a little bit more complicated; we prove the following:

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Theorem B. *Let $g\colon \mathbb{Z}^d \longrightarrow G$ be a polynomial mapping and let $x \in X$. There exist a connected closed subgroup H of G and points $x_1, x_2, \ldots, x_k \in X$ such that $\overline{\{g(n)x\}_{n\in\mathbb{Z}^d}} = \bigcup_{j=1}^k Hx_j$ and for each $j = 1, \ldots, k$, $\{g(n)x\}_{n:g(n)x\in Hx_j}$ is well distributed in Hx_j. In particular, if $Y = \overline{\{g(n)x\}_{n\in\mathbb{Z}^d}}$ is connected then $\{g(n)x\}_{n\in\mathbb{Z}^d}$ is well distributed in Y.*

1.6. A more detailed information about the behavior of $g(n)x$ is given by the following theorem:

Theorem B*. *Let $g\colon \mathbb{Z}^d \longrightarrow G$ be a polynomial mapping and let $x \in X$. There exist a connected closed subgroup H of G, a homomorphism $\omega\colon \mathbb{Z}^d \longrightarrow W$ onto a finite group W and a set $\{x_w,\ w \in W\} \subseteq X$ such that the sets $Y_w = Hx_w$, $w \in W$, are closed in X and $\{g(n)x\}_{n\in\omega^{-1}(w)}$ is well distributed in Y_w for every $w \in W$.*

Notice that the sets Y_w are not assumed to be all distinct.

1.7. Corollary. *For any $f \in C(X)$ and any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$,*

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\big(g(n)x\big) = \frac{1}{|W|} \sum_{w\in W} \int_{Y_w} f\, d\mu_{Y_w}.$$

In particular, Theorem A follows.

1.8. Let $A_1, \ldots, A_l$ be finitely generated subgroups of G. Theorem 1.4 says that, for every i, the orbit of any $x \in X$ under the action of A_i is well distributed in a sub-nilmanifold of X. It now follows from Theorem B that the orbit $A_1 \ldots A_l x$ of x under the product $A_1 \ldots A_l$ is also well distributed in the union of several disjoint submanifolds of X.

Corollary. *For any $x \in X$ there exist a connected closed subgroup H of G and points $x_1, x_2, \ldots, x_k \in X$ such that $\overline{A_1 \ldots A_l x} = \bigcup_{j=1}^k Hx_j$, and for each $j = 1, \ldots, k$, $\{ax\}_{a\in A_1\ldots A_l : ax\in Hx_j}$ is well distributed in Hx_j (in the sense clear from the proof).*

Proof. For each $i = 1, \ldots, l$, the finitely generated nilpotent group A_i possesses a finite *basis*, that is, $a_{i,1}, \ldots, a_{i,r_i} \in A_i$ such that every element of A_i is representable in the form $a_{i,1}^{n_1} \ldots a_{i,r_i}^{n_{r_i}}$ with $n_1, \ldots, n_{r_i} \in \mathbb{Z}$. The mapping $\mathbb{Z}^{r_1+\ldots+r_l} \longrightarrow G$, $(n_{1,1}, \ldots, n_{l,r_l}) \mapsto \prod_{i=1}^l \prod_{j=1}^{r_i} a_{i,j}^{n_{i,j}}$ is therefore a polynomial mapping onto $A_1 \ldots A_l$. By Theorem B, $\overline{A_1 \ldots A_l x}$ has form $\bigcup_{j=1}^k Hx_j$, and $A_1 \ldots A_l x$ is well distributed in the components of this union (with respect to any Følner sequence in $\mathbb{Z}^{r_1+\ldots+r_l}$). ∎

1.9. Theorem B also remains true if, instead of the orbit of a point in X, one considers the orbit of a sub-nilmanifold of X. Let us say that a family $\{Z_n\}_{n\in B}$, $B \subseteq \mathbb{Z}^d$, of sub-nilmanifolds of X is *well distributed* in a sub-nilmanifold Y of X if $Z_n \subseteq Y$ for all $n \in B$ and for any $f \in C(Y)$ and any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$ one has $\lim\limits_{N\to\infty} \frac{1}{|\Phi_N\cap B|} \sum\limits_{n\in\Phi_N\cap B} \int_{Z_n} f\, d\mu_{Z_n} = \int_Y f\, d\mu_Y$. In particular, $\overline{\bigcup_{n\in B} Z_n} = Y$ in this case.

Corollary. *Let $g\colon \mathbb{Z}^d \longrightarrow G$ be a polynomial mapping and let Z be a connected sub-nilmanifold of X. There exist a connected closed subgroup H of G and points $x_1, x_2, \ldots, x_k \in X$ such that $\overline{\bigcup_{n\in\mathbb{Z}^d} g(n)Z} = \bigcup_{j=1}^k Hx_j$, and for each $j = 1, \ldots, k$, $\{g(n)Z\}_{n:g(n)Z\subseteq Hx_j}$ is well distributed in Hx_j.*

Proof. Let $x \in Z$ and let $a \in G$ be such that $\{a^l x\}_{l\in\mathbb{N}}$ is well distributed in Z. (Letting F be a closed subgroup of G such that $Z = Fx$, take any $a \in F$ such that the projection of $\{a^l x\}_{l\in\mathbb{N}}$ is well distributed in the maximal factor-torus of Z; see 1.10 below.) Consider the polynomial sequence $h(n,l) = g(n)a^l$, $n \in \mathbb{Z}^d$, $l \in \mathbb{Z}$. Then $\overline{\bigcup_{n\in\mathbb{Z}^d} g(n)Z} = \overline{\{h(n,l)x\}_{(n,l)\in\mathbb{Z}^{d+1}}}$ and by Theorem B, $\overline{\bigcup_{n\in\mathbb{Z}^d} g(n)Z} = \bigcup_{j=1}^k Hx_j$ for suitable H and $x_1, \ldots, x_k$.

For $j \in \{1, \ldots, k\}$ let $B_j = \big\{n \in \mathbb{Z}^d : g(n)Z \subseteq Hx_j\big\}$ and $C_j = B_j \times \mathbb{Z}$. Now let $\{\Phi_N\}_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{Z}^d$; given $f \in C(Hx_j)$ consider a Følner sequence $\Psi_N = \Phi_N \times \{1, \ldots, p_N\}$, $N \in \mathbb{N}$, in $\mathbb{Z}^{d+1}$. Then, if the integers p_N tend to infinity fast enough, one has $\lim\limits_{N\to\infty} \frac{1}{|\Phi_N\cap B_j|} \sum\limits_{n\in\Phi_N\cap B_j} \int_{g(n)Z} f\, d\mu_{g(n)Z} = \lim\limits_{N\to\infty} \frac{1}{|\Psi_N\cap C_j|} \sum\limits_{(n,l)\in\Psi_N\cap C_j} f(h(n,l)x) = \int_{Hx_j} f\, d\mu_{Hx_j}$. ∎

1.10. It follows from Theorem B that if the orbit $\{g(n)x\}_{n\in\mathbb{Z}^d}$ of a point $x\in X$ is dense in X then it is well distributed in X. In the case where X is connected we have a simple criterion of this situation. Let G^o be the identity component of G; if X is connected, then X is a homogeneous space of G^o, $X=G^o/(\Gamma\cap G^o)$. The factor $T=[G^o,G^o]\backslash X=G^o/\bigl((\Gamma\cap G^o)[G^o,G^o]\bigr)$ of X is a compact connected abelian Lie group, which we will call *the maximal factor-torus of* X.

Theorem C. *Let X be connected, let T be the maximal factor-torus of X, let $p\colon X\longrightarrow T$ be the factorization mapping and let g be a polynomial mapping $\mathbb{Z}^d\longrightarrow G$. The orbit $\{g(n)x\}_{n\in\mathbb{Z}^d}$ of $x\in X$ is dense X iff $\{g(n)p(x)\}_{n\in\mathbb{Z}^d}$ is dense in T.*

1.11. Let $\{z_n\}_{n\in\mathbb{Z}^d}$ be a (multiparameter) sequence in a topological space X. A point z_m of this sequence is called *recurrent* if for in any neighborhood U of z_m the set $\bigl\{n\in\mathbb{Z}^d:z_n\in U\bigr\}$ is infinite. If $g\colon\mathbb{Z}^d\longrightarrow G$ is a polynomial mapping and $x\in X$, it follows from Theorem B that every point of $\{g(n)x\}_{n\in\mathbb{Z}^d}$ is recurrent.

Actually, a stronger fact holds. The set of finite sums of distinct elements of a sequence in $\mathbb{Z}^d$ is called *an IP-set*; a subset of $\mathbb{Z}^d$ that has nonempty intersection with any IP-set is called *an IP*-set*. IP*-sets are "regular" and "large"; in particular, any IP*-set is syndetic, that is, has bounded gaps. (See [F], ch. 9.) Given a (multiparameter) sequence $\{z_n\}_{n\in\mathbb{Z}^d}$ in a topological space X, following [F] we say that a point z_m, $m\in\mathbb{Z}^d$, is *IP*-recurrent* if for any neighborhood U of z_m the set $\bigl\{n\in\mathbb{Z}^d:z_n\in U\bigr\}$ is IP*.

Theorem D. *Let $g\colon\mathbb{Z}^d\longrightarrow G$ be a polynomial mapping and let $x\in X$. The point $g(0)x$ is IP*-recurrent for $\{g(n)x\}_{n\in\mathbb{Z}^d}$.*

2. Proofs

2.1. By $[a,b]$ we will denote $a^{-1}b^{-1}ab$. If B is a subset of a group G, we will denote by $\langle B\rangle$ the subgroup of G generated by B. Given a group G, by G_2 we will denote the derived subgroup $[G,G]$ of G.

When G is a nilpotent Lie group we will denote by G^o the identity component of G. Any connected nilpotent Lie group is exponential and so, for any $a\in G^o$ there exists a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}\subseteq G^o$ with $\alpha(1)=a$. We will denote $\alpha(t)$ by a^t (ignoring the fact that a^t may not be uniquely defined).

2.2. Let $\mathcal{F}$ be the free group generated by continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$, that is, the group of words in the alphabet $\bigl\{a_1^{t_1},\ldots,a_l^{t_l},e_1^{k_1},\ldots,e_m^{k_m}\bigr\}_{\substack{t_i\in\mathbb{R}\\ k_j\in\mathbb{Z}}}$. Let $\mathcal{F}=\mathcal{F}_1\supseteq\mathcal{F}_2\supseteq\ldots$ be the lower central series of $\mathcal{F}$: $\mathcal{F}_{i+1}=[\mathcal{F}_i,\mathcal{F}]$, $i\in\mathbb{N}$. Let $r\in\mathbb{N}$; we will call the nilpotent Lie group $F=\mathcal{F}/\mathcal{F}_{r+1}$ *the free nilpotent Lie group (of class r, with continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$).* The discrete subgroup of F generated by the set $\{a_1,\ldots,a_l,e_1,\ldots,e_m\}$ is uniform in F; we will denote it by $\Gamma(F)$.

2.3. Lemma. *Let G be a nilpotent Lie group of class $\leq r$ and let F be a free nilpotent Lie group of class r with continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$. Any mapping $\eta\colon\{a_1,\ldots,a_l,$ $e_1,\ldots,e_m\}\longrightarrow G$ with $\eta\bigl(\{a_1,\ldots,a_l\}\bigr)\subseteq G^o$ extends to a homomorphism $F\longrightarrow G$.*

Proof. Put $\eta(a_i^t)=\bigl(\eta(a_i)\bigr)^t$, $t\in\mathbb{R}$, $i=1,\ldots,l$, then η extends to a homomorphism $\eta\colon\mathcal{F}\longrightarrow G$ from the free group $\mathcal{F}$ generated by $\{a_1^{t_1},\ldots,a_l^{t_l},e_1,\ldots,e_m\}_{t_i\in\mathbb{R}}$. Since $\eta(\mathcal{F}_{r+1})\subseteq G_{r+1}=\{\mathbf{1}_G\}$, η factors to a homomorphism $F\longrightarrow G$. ■

2.4. Let G be a nilpotent Lie group such that G/G^o is finitely generated. Then G is generated by a set of the form $\bigl\{a_1^{t_1},\ldots,a_l^{t_l},e_1,\ldots,e_m\bigr\}_{t_i\in\mathbb{R}}$, where $a_1^{t_1},\ldots,a_l^{t_l}$ generate G^o and $e_1,\ldots,e_m$ generate G/G^o. It follows from Lemma 2.3 that G is a factor of the free nilpotent Lie group of the same nilpotency class as G with continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$.

2.5. Lemma. *Let G be a nilpotent group and let H be a subgroup of G such that $HG_2=G$. Then $H=G$.*

Proof. Let $G=G_1\supseteq G_2\supseteq\ldots\supseteq G_r\supseteq G_{r+1}=\{\mathbf{1}_G\}$ be the lower central series of G. By induction on r, $HG_r=G$, and it is only to be checked that $G_r\subseteq H$. G_r is generated by elements of the form $[b,a]$ with $a\in G$ and $b\in G_{r-1}$. Let $c\in H$ be such that $cG_2=aG_2$ and $d\in H\cap G_{r-1}$ be such that $dG_r=bG_r$. Then $[d,c]\in H$ and $[d,c]=[b,a]$. ■

2.6. Lemma. *Let F be a free nilpotent Lie group and let a self-homomorphism τ of F be such that the induced self-homomorphism of F/F_2 is invertible. Then τ is also invertible.*

Proof. Since $\tau(F)F_2 = F$, $\tau(F) = F$ by Lemma 2.5. It follows from Lemma 2.3 that there exists a homomorphism $\sigma\colon F \longrightarrow F$ such that $\tau\circ\sigma = \mathrm{Id}_F$. Since σ induces an automorphism of F/F_2, σ is also surjective. Hence, $\sigma = \tau^{-1}$. ∎

2.7. We say that an automorphism τ of a group G is *unipotent* if the mapping $\xi\colon G \longrightarrow G$ defined by $\xi(a) = \tau(a)a^{-1}$, $a \in G$, satisfies $\xi^{\circ q} \equiv \mathbf{1}_G$ for $q \in \mathbb{N}$ large enough.

2.8. Proposition. *Let G be a nilpotent group and let $\tau_1, \dots, \tau_k$ be automorphisms of G such that the automorphisms induced by $\tau_1, \dots, \tau_k$ on G/G_2 are unipotent and commute. Then the group extension of G by $\tau_1, \dots, \tau_k$ is nilpotent. In particular, $\tau_1, \dots, \tau_k$ generate a nilpotent group.*

Proof. Let $\mathcal{T}$ be the group of automorphisms of G generated by $\tau_1, \dots, \tau_k$. For $\delta \in \mathcal{T}$ let $\xi_\delta\colon G \longrightarrow G$ be defined by $\xi_\delta(a) = \delta(a)a^{-1}$, $a \in G$. Since $\tau_1, \dots, \tau_k$ are unipotent and commuting on G/G_2, there exists $q \in \mathbb{N}$ such that $\xi_\delta^{\circ q}(G) \subseteq G_2$ for any $\delta \in \mathcal{T}$. For $j = 0, 1, \dots$ let $A_{1,j}$ be the subgroup of G generated by G_2 and the set $\big\{\xi_{\delta_1}\circ\dots\circ\xi_{\delta_j}(G),\ \delta_1, \dots, \delta_j \in \mathcal{T}\big\}$. We then have a $\mathcal{T}$-invariant series $G = A_{1,0} \supseteq A_{1,1} \supseteq \dots \supseteq A_{1,q-1} \supseteq A_{1,q} = G_2$ such that for any $j < q$, $a \in A_{1,j}$ and $\delta \in \mathcal{T}$ one has $\delta(a) = ca$ with $c \in A_{1,j+1}$.

Let $G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_r \supseteq G_{r+1} = \{\mathbf{1}_G\}$ be the lower central series of G. For each $s = 2, \dots, r$ and $j \geq 0$ let $A_{s,j}$ be the subgroup of G_s generated by G_{s+1} and $\big\{[A_{s-1,l}, A_{1,m}] : l+m = j\big\}$. Then $A_{s,sq} = G_{s+1}$ and we get the $\mathcal{T}$-invariant series $G_s = A_{s,0} \supseteq A_{s,1} \supseteq \dots \supseteq A_{s,sq-1} \supseteq A_{s,sq} = G_{s+1}$.

Lemma. *For any $s \leq r$, $j < sq$, $a \in A_{s,j}$ and $\delta \in \mathcal{T}$ one has $\delta(a) = ba$ with $b \in A_{s,j+1}$.*

Proof. Let $a = [v, u]$ where $u \in A_{1,l}$ and $v \in A_{s-1,m}$ with $l + m = j$. Then $\delta(u) = cu$ with $c \in A_{1,l+1}$, and, by induction on s, $\delta(v) = dv$ with $d \in A_{s-1,m+1}$. Thus $\delta(a) = \delta([v, u]) = [dv, cu] = [d, c][v, c][d, u]w[v, u]$ with $w \in G_{s+1}$, and so, $\delta(a) = ba$ where $b = [d, c][v, c][d, u]w \in A_{s,j+1}$. ∎

Let us now consider the "long" series

$$\begin{aligned} G = A_{1,0} \supseteq A_{1,1} \supseteq \dots \supseteq A_{1,q-1} \supseteq A_{1,q} = A_{2,0} \supseteq A_{2,1} \supseteq \dots \supseteq A_{2,q-1} \supseteq A_{2,2q} = A_{3,0} \supseteq \dots \\ \dots \supseteq A_{r-1,(r-1)q} = A_{r,0} \supseteq A_{r,1} \supseteq \dots \supseteq A_{r,rq-1} \supseteq A_{r,rq} = \{\mathbf{1}_G\}. \end{aligned}$$

Denote the distinct terms of this series by $A_1, A_2, \dots, A_p$ so that $G = A_1 \supseteq A_2 \supseteq \dots \supseteq A_p = \{\mathbf{1}_G\}$ is a $\mathcal{T}$-invariant central series in G such that for any $j < p$, $a \in A_j$ and $\delta \in \mathcal{T}$ one has $\delta(a) = ba$ with $b \in A_{j+1}$. Also, define $A_{p+1} = A_{p+2} = \dots = \{\mathbf{1}_G\}$. Let $\mathcal{T} = \mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \dots$ be the lower central series of $\mathcal{T}$.

Lemma. *For any $l, j \in \mathbb{N}$, $\tau \in \mathcal{T}_l$ and $a \in A_j$ one has $\tau(a) = ca$ with $c \in A_{j+l}$.*

Proof. We will use induction on l. Assume that the statement is true for some l; let $\tau \in \mathcal{T}_l$, $\delta \in \mathcal{T}$, $a \in A_j$, $\delta(a) = ba$ and $\tau(a) = ca$ with $b \in A_{j+1}$ and $c \in A_{j+l}$. Then $\delta^{-1}(a) = \delta^{-1}(b^{-1})a$ and $\tau^{-1}(a) = \tau^{-1}(c^{-1})a$. Also, we have $\tau(b) \equiv b \bmod A_{j+l+1}$ and $\delta(c) \equiv c \bmod A_{j+l+1}$. Performing calculations modulo A_{j+l+1} we obtain

$$\begin{aligned} [\tau, \delta](a) &\equiv \tau^{-1}\delta^{-1}\tau\delta(a) \equiv \tau^{-1}\delta^{-1}\tau(ba) = \tau^{-1}\delta^{-1}(bca) \equiv \tau^{-1}\big(\delta^{-1}(b)c\delta^{-1}(b^{-1})a\big) \\ &\equiv \tau^{-1}\big(\delta^{-1}(b)c\delta^{-1}(b^{-1})\big)\tau^{-1}(c^{-1})a \equiv \tau^{-1}\big(\big[\delta^{-1}(b^{-1}), c^{-1}\big]\big)a \equiv a \bmod A_{j+l+1}. \end{aligned}$$

∎

It follows that $\tau(a) = a$ for all $\tau \in \mathcal{T}_p$ and $a \in G$. Hence, $\mathcal{T}_p$ is trivial and $\mathcal{T}$ is nilpotent.

Now let $\widehat{G}$ be the extension of G by $\mathcal{T}$, that is, $\widehat{G} = \big\{(a, \delta),\ a \in G,\ \delta \in \mathcal{T}\big\}$ with $(a_1, \delta_1)(a_2, \delta_2) = \big(a_2\delta_1(a_2), \delta_1\delta_2\big)$. We will identify G and $\mathcal{T}$ with their images in $\widehat{G}$; then $(a, \delta) = a\delta$, $\delta(a) = \delta a\delta^{-1}$ and $[a, \delta] = a^{-1}\delta^{-1}(a)$. Given $j \in \mathbb{N}$, $b \in A_j$, $\tau \in \mathcal{T}_j$, $a \in G$ and $\delta \in \mathcal{T}$, one has $\delta b\delta^{-1} = \delta(b) \in A_j$, $[b, \delta] = b^{-1}\delta^{-1}(b) \in A_{j+1}$ and $[\tau, a] = \tau^{-1}(a^{-1})a \in A_{j+1}$. Thus,

$$[b\tau, a\delta] = \big(\tau^{-1}[b, \delta]\tau\big)\big(\tau^{-1}\delta^{-1}[b, a]\delta\tau\big)\big([\tau, \delta]\delta^{-1}[\tau, a]\delta[\tau, \delta]^{-1}\big)[\tau, \delta] \in A_{j+1}\mathcal{T}_{j+1}.$$

Hence $\widehat{G} = A_1\mathcal{T}_1 \supseteq A_2\mathcal{T}_2 \supseteq \dots \supseteq A_p\mathcal{T}_p = \{\mathbf{1}_{\widehat{G}}\}$ is a central series in $\widehat{G}$ and $\widehat{G}$ is nilpotent. ∎

2.9. We want also to mention the following fact. Assume that $\eta\colon\widetilde{G}\longrightarrow G$ is a group homomorphism and $\widetilde{\Gamma}$ and Γ are closed subgroups of $\widetilde{G}$ and G respectively such that $\eta(\widetilde{\Gamma})\subseteq\Gamma$. Then η factors to a homomorphism of homogeneous spaces $\widetilde{X}=\widetilde{G}/\widetilde{\Gamma}\xrightarrow{\eta}X=G/\Gamma$. Assume that the set $\{\varphi(a)\}_{a\in A}$ of values of a mapping $\varphi\colon A\longrightarrow\widetilde{X}$ from an amenable group A is well distributed in $\widetilde{X}$. Then $\bigl\{\eta(\varphi(a))\bigr\}_{a\in A}$ is well distributed in $Y=\eta(\widetilde{X})$. Indeed, for any $f\in C(Y)$ and any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in A we have

$$\lim_{N\to\infty}\tfrac{1}{|\Phi_N|}\sum_{a\in\Phi_N}f\bigl(\eta\circ\varphi(a)\bigr)=\lim_{N\to\infty}\tfrac{1}{|\Phi_N|}\sum_{a\in\Phi_N}f\circ\eta\bigl(\varphi(a)\bigr)=\int_{\widetilde{X}}f\circ\eta\,d\mu_{\widetilde{X}}=\int_Y f\,d(\eta_*\mu_{\widetilde{X}}),$$

where $\mu_{\widetilde{X}}$ is the $\widetilde{G}$-invariant probability measure on $\widetilde{X}$. $\eta_*(\mu_{\widetilde{X}})$ is therefore the $\eta(\widetilde{G})$-invariant probability measure on Y, that is, μ_Y.

2.10. From now on let G be a nilpotent Lie group with identity component G^o, Γ be a closed uniform subgroup of G and $X=G/\Gamma$. We may and will assume that G/G^o is finitely generated and that Γ is discrete; see [L] for more detail.

The group G possesses a *basis*, that is, a system $a_1,\ldots,a_l\in G^o$, $e_1,\ldots,e_m\in G$, such that any element of G is representable in the form $a_1^{t_1}\ldots a_l^{t_l}e_1^{k_1}\ldots e_m^{k_m}$ with $t_1,\ldots,t_l\in\mathbb{R}$ and $k_1,\ldots,k_m\in\mathbb{Z}$, and Γ is a subgroup of finite index in $\langle a_1,\ldots,a_l,e_1,\ldots,e_m\rangle$. (For the case of a connected G see [M], for the general case see [L].) We will refer to $a_1,\ldots,a_l$ as to *continuous generators* and to $e_1,\ldots,e_m$ as to *discrete generators*. The multiplication in a nilpotent group is polynomial; this implies that any polynomial mapping $g\colon\mathbb{Z}^d\longrightarrow G$ can be written in the basis $\{a_1,\ldots,a_l,e_1,\ldots,e_m\}$ in the form $g(n)=a_1^{p_1(n)}\ldots a_l^{p_l(n)}e_1^{q_1(n)}\ldots e_m^{q_m(n)}$, where $p_1,\ldots,p_l$ are polynomial mappings $\mathbb{Z}^d\longrightarrow\mathbb{R}$ and $q_1,\ldots,q_m$ are polynomial mappings $\mathbb{Z}^d\longrightarrow\mathbb{Z}$.

2.11. Proof of Theorem B*. Let $g\colon\mathbb{Z}^d\longrightarrow G$ be a polynomial mapping and let $x\in X$. Our plan is to represent X as a factor, $\widetilde{\widetilde{X}}\xrightarrow{\tilde{\tilde{\eta}}}X$, of a "larger" nilmanifold $\widetilde{\widetilde{X}}=\widehat{\widehat{G}}/\widehat{\widehat{\Gamma}}$ and find a homomorphism $\varphi\colon\mathbb{Z}^d\longrightarrow\widehat{\widehat{G}}$ so that the "polynomial orbit" $\{g(n)x\}_{n\in\mathbb{Z}^d}$ would be the projection, $g(n)x=\tilde{\tilde{\eta}}(\varphi(n)\tilde{\tilde{x}})$, of the "linear orbit" $\{\varphi(n)\tilde{\tilde{x}}\}_{n\in\mathbb{Z}^d}$ in $\widetilde{\widetilde{X}}$. (Let us remark that in the construction that follows G is not a factor-group of $\widehat{\widehat{G}}$ and g is not a projection of φ.) This will allow us to derive Theorem B* from Theorem 1.4.

Let $\pi\colon G\longrightarrow X$ be the factorization mapping and let $a\in\pi^{-1}(x)$. Choose a basis $\{a_1,\ldots,a_K\}$ in G, where some of a_k may be continuous and some may be discrete, and $s\in\mathbb{N}$ such that $c^s\in\Gamma$ for any $c\in\langle a_1,\ldots,a_K\rangle$. Write $g(n)a=a_1^{p_1(n)}\ldots a_K^{p_K(n)}$, where p_k, $k=1,\ldots,K$, is a polynomial mapping $\mathbb{Z}^d\longrightarrow\mathbb{Z}$ if a_k is a discrete generator and $\mathbb{Z}^d\longrightarrow\mathbb{R}$ if a_k is a continuous generator. Any polynomial mapping $p\colon\mathbb{Z}^d\longrightarrow\mathbb{R}$ is representable in the form $p(n_1,\ldots,n_d)=\sum_j\lambda_j\bigl(\prod_{i=1}^d\binom{n_i}{r_{j,i}}\bigr)$, $r_{i,j}\in\{0,1,\ldots\}$, $\lambda_j\in\mathbb{R}$, with all $\lambda_j\in\mathbb{Z}$ if $p(\mathbb{Z}^d)\subseteq\mathbb{Z}$. This allows us to write

$$g(n)a=\prod_{j=1}^{J}a_{k_j}^{\lambda_j\prod_{i=1}^d\binom{n_i}{r_{j,i}}},\quad n=(n_1,\ldots,n_d)\in\mathbb{Z}^d,$$

where $\lambda_j\in\mathbb{R}$ if a_{k_j} is a continuous generator and $\lambda_j\in\mathbb{Z}$ if a_{k_j} is a discrete generator. For each $j=1,\ldots,J$ define $V_j=\prod_{i=1}^d\{0,\ldots,r_{j,i}\}$, and let $\widetilde{G}$ be the free nilpotent Lie group of same nilpotency class as G, with generators $\alpha_{j,v}$, $v\in V_j$, $j\in\{1,\ldots,J\}$, such that $\alpha_{j,v}$ is continuous if a_{k_j} is continuous, and discrete if a_{k_j} is discrete. Define an epimorphism $\eta\colon\widetilde{G}\longrightarrow G$ by

$$\eta(\alpha_{j,v})=\begin{cases}a_{k_j}\ \text{if}\ v=(r_{j,1},\ldots,r_{j,d})\\ \mathbf{1}_G\ \text{otherwise}\end{cases},\quad v\in V_j,\ j\in\{1,\ldots,J\}.$$

Let $\Gamma(\widetilde{G})$ be the lattice $\bigl\langle\alpha_{j,v},\ j\in\{1,\ldots,J\},\ v\in V_j\bigr\rangle$ in $\widetilde{G}$ and let $\widetilde{\Gamma}=\bigl\langle\gamma^s,\ \gamma\in\Gamma(\widetilde{G})\bigr\rangle$. Then $\widetilde{\Gamma}$ is a discrete uniform subgroup of $\widetilde{G}$ invariant under all automorphisms of $\Gamma(\widetilde{G})$ and $\eta(\widetilde{\Gamma})\subseteq\Gamma$. Define $\widetilde{X}=\widetilde{G}/\widetilde{\Gamma}$ and let $\widetilde{\pi}\colon\widetilde{G}\longrightarrow\widetilde{X}$ be the factorization mapping; then η factors to a mapping $\widetilde{X}\longrightarrow X$ so that $\eta\circ\widetilde{\pi}=\pi\circ\eta$.

We will now define automorphisms $\tau_1,\ldots,\tau_d$ of $\widetilde{G}$. For each $i=1,\ldots,d$ let ϵ_i be the i-th basis vector $(0,\ldots,0,1,0,\ldots,0)$ in $\mathbb{Z}^d$ and let

$$\tau_i(\alpha_{j,v})=\begin{cases}\alpha_{j,v}\alpha_{j,v+\epsilon_i}\ \text{if}\ v=(v_1,\ldots,v_d)\ \text{with}\ v_i<r_{j,i}\\ \alpha_{j,v}\ \text{if}\ v_i=r_{j,i}\end{cases},\quad v\in V_j,\ j\in\{1,\ldots,K\}.$$

(The following diagram shows how $\alpha^{-1}\tau_i(\alpha)$ act; here $d=2$, $r_{j,1}=2$, $r_{j,2}=3$, "$\rightarrow$" stands for $\alpha^{-1}\tau_1(\alpha)$ and "$\downarrow$" stands for $\alpha^{-1}\tau_2(\alpha)$:

$$\begin{array}{lllll}
\alpha_{j,(0,0)} & \rightarrow & \alpha_{j,(1,0)} & \rightarrow & \alpha_{j,(2,0)} \\
\downarrow & & \downarrow & & \downarrow \\
\alpha_{j,(0,1)} & \rightarrow & \alpha_{j,(1,1)} & \rightarrow & \alpha_{j,(2,1)} \\
\downarrow & & \downarrow & & \downarrow \\
\alpha_{j,(0,2)} & \rightarrow & \alpha_{j,(1,2)} & \rightarrow & \alpha_{j,(2,2)} \\
\downarrow & & \downarrow & & \downarrow \\
\alpha_{j,(0,3)} & \rightarrow & \alpha_{j,(1,3)} & \rightarrow & \alpha_{j,(2,3)}.)
\end{array}$$

By Lemmas 2.3 and 2.6, $\tau_1,\dots,\tau_d$ are extendible to automorphisms of $\widetilde{G}$. One checks that for any $j\in\{1,\dots,J\}$, $i\in\{1,\dots,d\}$, $v=(v_1,\dots,v_d)$ with $v_i=0$, $\lambda\in\mathbb{R}$ if a_{k_j} is continuous and $\lambda\in\mathbb{Z}$ if a_{k_j} is discrete one has $\tau_i^n(\alpha_{j,v}^{\lambda})=\prod_{m=0}^{r_{j,i}}\alpha_{j,v+m\epsilon_i}^{\lambda\binom{n}{m}}$, $n\in\mathbb{Z}$. It follows that for any $(n_1,\dots,n_d)\in\mathbb{Z}^d$ one has $\tau_1^{n_1}\cdots\tau_d^{n_d}(\alpha_{j,(0,\dots,0)}^{\lambda})=\prod_{m_d=0}^{r_{j,d}}\cdots\prod_{m_1=0}^{r_{j,1}}\alpha_{j,(m_1,\dots,m_d)}^{\lambda\binom{n_1}{m_1}\cdots\binom{n_d}{m_d}}$, and $\eta\big((\prod_{i=1}^d\tau_i^{n_i})(\alpha_{j,(0,\dots,0)}^{\lambda})\big)=a_{k_j}^{\lambda\prod_{i=1}^d\binom{n_i}{r_{j,i}}}$. Define $\alpha=\prod_{j=1}^J\alpha_{j,(0,\dots,0)}^{\lambda_j}$, then

$$\eta\Big(\big(\prod_{i=1}^d\tau_i^{n_i}\big)(\alpha)\Big)=\prod_{j=1}^J a_{k_j}^{\lambda_j\prod_{i=1}^d\binom{n_i}{r_{j,i}}}=g(n)a,\quad n=(n_1,\dots,n_d)\in\mathbb{Z}^d. \tag{2.1}$$

The automorphisms induced by $\tau_1,\dots,\tau_d$ on $\widetilde{G}/\widetilde{G}_2$ are unipotent and commute. (The automorphisms τ_i themselves do not commute: $\tau_1\tau_2\alpha_{j,(0,0)}=\alpha_{j,(0,0)}\alpha_{j,(1,0)}\alpha_{j,(0,1)}\alpha_{j,(1,1)}$, whereas $\tau_1\tau_2\alpha_{j,(0,0)}=\alpha_{j,(0,0)}\alpha_{j,(0,1)}\alpha_{j,(1,0)}\alpha_{j,(1,1)}$.) Let $\widehat{G}$ be the extension of $\widetilde{G}$ by the discrete group of automorphisms generated by $\tau_1,\dots,\tau_d$; by Proposition 2.8, $\widehat{G}$ is nilpotent. $\widetilde{G}$ is normal in $\widehat{G}$, so $\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha\big(\prod_{i=1}^d\tau_i^{n_i}\big)^{-1}\in\widetilde{G}$ and by (2.1),

$$\eta\Big(\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha\big(\prod_{i=1}^d\tau_i^{n_i}\big)^{-1}\Big)=g(n)a,\quad n=(n_1,\dots,n_d)\in\mathbb{Z}^d. \tag{2.2}$$

Define $\widehat{\Gamma}=\langle\widetilde{\Gamma},\tau_1,\dots,\tau_k\rangle$; since τ_i preserve $\widetilde{\Gamma}$ one has $\widehat{\Gamma}\cap\widetilde{G}=\widetilde{\Gamma}$. Hence, $\widehat{\Gamma}$ is a discrete subgroup in $\widehat{G}$ and $\widehat{G}/\widehat{\Gamma}\simeq\widetilde{G}/\widetilde{\Gamma}=\widetilde{X}$; let $\widehat{\pi}\colon\widehat{G}\longrightarrow\widetilde{X}$ be the factorization mapping. We get the commutative diagram

$$\begin{tikzcd}
\widetilde{G} \arrow[r, phantom, "\subseteq"] \arrow[d, "\eta"'] \arrow[dr, "\widetilde{\pi}"] & \widehat{G} \arrow[d, "\widehat{\pi}"] \\
G \arrow[dr, "\pi"] & \widetilde{X} \arrow[d, "\eta"] \\
 & X
\end{tikzcd}$$

Let $\widetilde{x}=\widehat{\pi}(\alpha)\in\widetilde{X}$ and $\widetilde{g}(n)=\prod_{i=1}^d\tau_i^{n_i}$, $n=(n_1,\dots,n_d)\in\mathbb{Z}^d$. Then, since $\prod_{i=1}^d\tau_i^{n_i}\in\widehat{\Gamma}$, we have by (2.2):

$$\widetilde{g}(n)\widetilde{x}=\widehat{\pi}\Big(\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha\Big)=\widehat{\pi}\Big(\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha\big(\prod_{i=1}^d\tau_i^{n_i}\big)^{-1}\Big)\xrightarrow{\ \eta\ }\pi\big(g(n)a\big)=g(n)x, \tag{2.3}$$

$n=(n_1,\dots,n_d)\in\mathbb{Z}^d$.

The polynomial mapping $\widetilde{g}\colon\mathbb{Z}^d\longrightarrow\widehat{G}$ is not a homomorphism since τ_i do not commute. We will now repeat the procedure described above. Let $\widetilde{\widehat{G}}$ be the free nilpotent group of same nilpotency class as $\widehat{G}$ with generators $\alpha_{j,v}$ for $v\in V_j$, $j\in\{1,\dots,J\}$, and discrete generators τ_i and δ_i for $i=1,\dots,d$. Define an epimorphism $\widetilde{\eta}\colon\widetilde{\widehat{G}}\longrightarrow\widehat{G}$ by

$$\begin{gathered}
\widetilde{\eta}(\alpha_{j,v})=\alpha_{j,v},\quad v\in V_j,\ j\in\{1,\dots,J\},\\
\widetilde{\eta}(\tau_i)=\tau_i \text{ and } \widetilde{\eta}(\delta_i)=\mathbf{1}_{\widetilde{G}},\quad i=1,\dots,d.
\end{gathered}$$

Define automorphisms σ_i, $i=1,\dots,d$, of $\widetilde{\widehat{G}}$ by

$$\begin{gathered}
\sigma_i(\alpha_{j,v})=\alpha_{j,v},\quad v\in V_j,\ j\in\{1,\dots,J\},\\
\sigma_i(\tau_l)=\tau_l,\ \sigma_i(\delta_l)=\begin{cases}\delta_l \text{ if } l\neq i\\ \delta_i\tau_i \text{ if } l=i\end{cases},\quad l=1,\dots,d.
\end{gathered}$$

$\sigma_1,\dots,\sigma_d$ commute and the automorphisms induced by $\sigma_1,\dots,\sigma_d$ on $\widetilde{\widehat{G}}/\widetilde{\widehat{G}}_2$ are unipotent. For any i one has $\sigma_i^n(\delta_i)=\delta_i\tau_i^n$, $n\in\mathbb{Z}$. Put $\delta=(\prod_{i=1}^d\delta_i)\alpha$, then $(\prod_{i=1}^d\sigma_i^{n_i})(\delta)=(\prod_{i=1}^d\delta_i\tau_i^{n_i})\alpha$ and

$$\widetilde{\eta}\Big(\big(\prod_{i=1}^d\sigma_i^{n_i}\big)(\delta)\Big)=\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha,\quad (n_1,\dots,n_d)\in\mathbb{Z}^d. \tag{2.4}$$

Now let $\widehat{\widehat{G}}$ be the extension of $\widetilde{\widehat{G}}$ by the discrete group generated by $\sigma_1,\dots,\sigma_d$. By Proposition 2.8, $\widehat{\widehat{G}}$ is nilpotent. $\widetilde{\widehat{G}}$ is normal in $\widehat{\widehat{G}}$, so $(\prod_{i=1}^d\sigma_i^{n_i})\delta(\prod_{i=1}^d\sigma_i^{n_i})^{-1}\in\widetilde{\widehat{G}}$ and by (2.4),

$$\widetilde{\eta}\Big(\big(\prod_{i=1}^d\sigma_i^{n_i}\big)\delta\big(\prod_{i=1}^d\sigma_i^{n_i}\big)^{-1}\Big)=\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha,\quad (n_1,\dots,n_d)\in\mathbb{Z}^d. \tag{2.5}$$

Let $\widetilde{\widehat{\Gamma}}=\langle\widehat{\Gamma},\delta_1,\dots,\delta_d\rangle\subseteq\widetilde{\widehat{G}}$ and $\widehat{\widehat{\Gamma}}=\langle\widetilde{\widehat{\Gamma}},\sigma_1,\dots,\sigma_d\rangle\subseteq\widehat{\widehat{G}}$. Then $\widetilde{\widehat{\Gamma}}$ and $\widehat{\widehat{\Gamma}}$ are discrete uniform subgroups of $\widetilde{\widehat{G}}$ and $\widehat{\widehat{G}}$ respectively, and $\widetilde{\widetilde{X}}:=\widetilde{\widehat{G}}/\widetilde{\widehat{\Gamma}}\simeq\widehat{\widehat{G}}/\widehat{\widehat{\Gamma}}$; let $\widehat{\widehat{\pi}}\colon\widehat{\widehat{G}}\longrightarrow\widetilde{\widetilde{X}}$ be the factorization mapping. We have the commutative diagram

$$\begin{tikzcd}
\widetilde{\widehat{G}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\tilde{\eta}"'] \arrow[dr] & \widehat{\widehat{G}} \arrow[d, "\hat{\hat{\pi}}"] \\
\widehat{G} \arrow[dr, "\hat{\pi}"'] & \widetilde{\widetilde{X}} \arrow[d, "\tilde{\eta}"] \\
 & \widetilde{X}
\end{tikzcd}$$

Put $\widetilde{\widetilde{x}}=\widehat{\widehat{\pi}}(\delta)$ and define $\varphi(n)=\prod_{i=1}^d\sigma_i^{n_i}$, $n=(n_1,\dots,n_d)\in\mathbb{Z}^d$. Then, since $\prod_{i=1}^d\sigma_i^{n_i}\in\widehat{\widehat{\Gamma}}$, we have by (2.5):

$$\varphi(n)\widetilde{\widetilde{x}}=\widehat{\widehat{\pi}}\Big(\big(\prod_{i=1}^d\sigma_i^{n_i}\big)\delta\Big)=\widehat{\widehat{\pi}}\Big(\big(\prod_{i=1}^d\sigma_i^{n_i}\big)\delta\big(\prod_{i=1}^d\sigma_i^{n_i}\big)^{-1}\Big)\xrightarrow{\widetilde{\eta}}\widehat{\pi}\Big(\big(\prod_{i=1}^d\tau_i^{n_i}\big)\alpha\Big)=\widetilde{g}(n)\widetilde{x},$$

$n=(n_1,\dots,n_d)\in\mathbb{Z}^d$. Combining this with (2.3) we get $\eta\circ\widetilde{\eta}(\varphi(n)\widetilde{\widetilde{x}})=g(n)x$, $n\in\mathbb{Z}^d$.

Since $\sigma_1,\dots,\sigma_d$ commute, $\varphi\colon\mathbb{Z}^d\longrightarrow\widehat{\widehat{G}}$ is a group homomorphism. By Theorem 1.4, there exists a closed subgroup E of $\widehat{\widehat{G}}$ such that $\varphi(\mathbb{Z}^d)\subseteq E$ and $\{\varphi(n)\widetilde{\widetilde{x}}\}_{n\in\mathbb{Z}^d}$ is well distributed in Ex. Let $\widetilde{\widetilde{H}}$ be the identity component of E; since $\widehat{\widehat{G}}/\widetilde{\widehat{G}}$ is discrete, $\widetilde{\widetilde{H}}\subseteq\widetilde{\widehat{G}}$. $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}$ is a connected component of $E\widetilde{\widetilde{x}}$; since $E\widetilde{\widetilde{x}}$ is compact it consists of finitely many translates of $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}$ and so, the stabilizer $\mathrm{Stab}(\widetilde{\widetilde{H}}\widetilde{\widetilde{x}})$ of $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}$ has finite index in E. Let W be the finite group $\mathbb{Z}^d/\varphi^{-1}\big(\mathrm{Stab}(\widetilde{\widetilde{H}}\widetilde{\widetilde{x}})\big)$, let $\omega\colon\mathbb{Z}^d\longrightarrow W$ be the factorization mapping, for each $w\in W$ let $n_w\in\mathbb{Z}^d$ be a representative of w and $\widetilde{\widetilde{x}}_w=\varphi(n_w)\widetilde{\widetilde{x}}$. Then $Ex=\bigcup_{w\in W}\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w$, $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w$ is closed and $\{\varphi(n)\widetilde{\widetilde{x}}\}_{n\in\omega^{-1}(w)}$ is well distributed in $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w$ for any $w\in W$.

Now let $\widetilde{H}=\widetilde{\eta}(\widetilde{\widetilde{H}})\subseteq\widehat{G}$. Since $\widetilde{H}$ is connected, $\widetilde{H}\subseteq\widetilde{G}$; let $H=\eta(\widetilde{H})$. Let $x_w=\eta\circ\widetilde{\eta}(\widetilde{\widetilde{x}}_w)$, $w\in W$. For each $w\in W$, since $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w$ is compact, $Hx_w=\eta\circ\widetilde{\eta}(\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w)$ is closed in X. Let $b\in\pi^{-1}(Hx_0)$; since Γ is discrete, Hb is a connected component of the closed set $\pi^{-1}\big(Hx_0\big)$, and thus H is closed in G. By 2.9, $\{g(n)x\}_{n\in\omega^{-1}(w)}=\big\{\eta\circ\widetilde{\eta}(\varphi(n)\widetilde{\widetilde{x}})\big\}_{n\in\omega^{-1}(w)}$ is well distributed in Hx_w for any $w\in W$. ■

2.12. Remark. Note that the components Hx_w of $\overline{\{g(n)x\}}_{n\in\mathbb{Z}^d}$ do not have to be distinct though $\widetilde{\widetilde{H}}\widetilde{\widetilde{x}}_w$ are all distinct. Here is a simple example: let $G=\mathbb{R}$, $\Gamma=\mathbb{Z}$, $x=0$, $d=1$, $g(n)=\frac{n^2}{3}\in\mathbb{R}$; then $H=0$, $x_0=0$ and $x_1=x_2=\frac{1}{3}$, so that $\{g(n)x\}_{n\in\mathbb{Z}}=\big\{0,\frac{1}{3}\big\}$.

2.13. Proof of Theorem D. In the notation of 2.11, the action of $\mathbb{Z}^d$ on X by $x\mapsto\varphi(n)x$, $x\in X$, $n\in\mathbb{Z}^d$, is distal. (See, for example, [L].) It follows that the point $\varphi(0)\widetilde{\widetilde{x}}$ is IP*-recurrent for the sequence $\{\varphi(n)\widetilde{\widetilde{x}}\}_{n\in\mathbb{Z}^d}$. ([F], Theorem 9.11.) Hence, the point $g(0)x$ is IP*-recurrent for the sequence $\{g(n)x\}_{n\in\mathbb{Z}^d}=\big\{\eta\circ\widetilde{\eta}(\varphi(n)\widetilde{\widetilde{x}})\big\}_{n\in\mathbb{Z}^d}$. ■

2.14. Proof of Theorem C. Let X be connected and let $g\colon \mathbb{Z}^d \longrightarrow G$ be a polynomial mapping. Let $x \in X$ and let, by Theorem B, H be a connected closed subgroup of G such that $\overline{\{g(n)x\}}_{n\in\mathbb{Z}^d} = \bigcup_{j=1}^{k} Hx_j$ for some $x_1, \ldots, x_k \in X$.

Let $T = [G^o, G^o]\backslash X$ and $p\colon X \longrightarrow T$ be the factorization mapping. Assume that $\{g(n)p(x)\}_{n\in\mathbb{Z}^d}$ is dense in T. Then $T = \bigcup_{j=1}^{k} Hp(x_j)$, and since T is connected, $Hp(x_j) = T$ for some j. Thus $H[G^o, G^o](\Gamma \cap G^o) = G^o$, and since Γ is countable, $H[G^o, G^o] = G^o$. By Lemma 2.5, $H = G^o$, so $\overline{\{g(n)x\}}_{n\in\mathbb{Z}^d} = Hx_1 = X$. ■

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