Nilsequences, null-sequences, and multiple correlation sequences

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

November 5, 2019

Abstract

A \((d\text{-parameter})\) basic nilsequence is a sequence of the form \(\psi(n) = f(a^n x), \ n \in \mathbb{Z}^d\), where \(x\) is a point of a compact nilmanifold \(X\), \(a\) is a translation on \(X\), and \(f \in C(X)\); a nilsequence is a uniform limit of basic nilsequences. If \(X = G/\Gamma\) be a compact nilmanifold, \(Y\) is a sub-nilmanifold of \(X\), \((g(n))_{n \in \mathbb{Z}^d}\) is a polynomial sequence in \(G\), and \(f \in C(X)\), we show that the sequence \(\varphi(n) = \int g(n) Y f\) is the sum of a basic nilsequence and a sequence that converges to zero in uniform density (a null-sequence). We also show that an integral of a family of nilsequences is a nilsequence plus a null-sequence. We deduce that for any invertible finite measure preserving system \((W, \mathcal{B}, \mu, T)\), polynomials \(p_1, \ldots, p_k : \mathbb{Z}^d \rightarrow \mathbb{Z}\), and sets \(A_1, \ldots, A_k \in \mathcal{B}\), the sequence \(\varphi(n) = \mu(T^{p_1(n)} A_1 \cap \ldots \cap T^{p_k(n)} A_k), n \in \mathbb{Z}^d\), is the sum of a nilsequence and a null-sequence.

0. Introduction

Throughout the whole paper we will deal with “multiparameter sequences”, – we fix \(d \in \mathbb{N}\) and under “a sequence” will usually understand “a two-sided \(d\text{-parameter sequence}\)”, that is, a mapping with domain \(\mathbb{Z}^d\).

A (compact) \((r\text{-step})\) nilmanifold \(X\) is a factor space \(G/\Gamma\), where \(G\) is an \(r\text{-step}\) nilpotent (not necessarily connected) Lie group and \(\Gamma\) is a discrete co-compact subgroup of \(G\). Elements of \(G\) act on \(X\) by translations; an \((r\text{-step})\) nilsystem is an \((r\text{-step})\) nilmanifold \(X = G/\Gamma\) with a translation \(a \in G\) on it.

A basic \((r\text{-step})\) nilsequence is a sequence of the form \(\psi(n) = f(\eta(n)x), \ n \in \mathbb{Z}^d\), where \(x\) is a point of an \(r\text{-step}\) nilmanifold \(X = G/\Gamma\), \(\eta\) is a homomorphism \(\mathbb{Z}^d \rightarrow G\), and \(f \in C(X)\); an \((r\text{-step})\) nilsequence is a uniform limit of basic \(r\text{-step}\) nilsequences. The algebra of nilsequences is a natural generalization of Weyl’s algebra of almost periodic sequences, which are just 1-step nilsequences. An “inner” characterization of nilsequences, in terms of their properties, is obtained in [HKM]; see also [HK2].

The term “nilsequence” was introduced in [BHK], where it was proved that for

\[\text{Supported by NSF grant DMS-0901106.}\]
any ergodic finite measure preserving system \((W, \mathcal{B}, \mu, T)\), positive integer \(k\), and sets \(A_1, \ldots, A_k \in \mathcal{B}\) the multiple correlation sequence \(\varphi(n) = \mu(T^n A_1 \cap \cdots \cap T^{kn} A_k)\), \(n \in \mathbb{N}\), is a sum of a nilsequence and of a sequence that tends to zero in uniform density in \(\mathbb{Z}^d\), a null-sequence. Our goal in this paper is to generalize this result to multiparameter polynomial multiple correlation sequences and general (non-ergodic) systems. We prove (in Section 5):

**Theorem 0.1.** Let \((W, \mathcal{B}, \mu, T)\) be an invertible measure preserving system with \(\mu(W) < \infty\), let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \rightarrow \mathbb{Z}\), and let \(A_1, \ldots, A_k \in \mathcal{B}\). Then the “multiple polynomial correlation sequence” \(\varphi(n) = \mu(T^{p_1(n)} A_1 \cap \cdots \cap T^{p_k(n)} A_k), n \in \mathbb{Z}^d\), is a sum of a nilsequence and a null-sequence.

(In [L6] this theorem was proved in the case \(d = 1\) and ergodic \(T\).)

Based on the theory of nil-factors developed in [HK] and, independently, in [Z], it is shown in [L3] that nilsystems are characteristic for multiple polynomial correlation sequences induced by ergodic systems, in the sense that, up to a null-sequence and an arbitrarily small sequence, any such correlation sequence comes from a nilsystem. This reduces the problem of studying “ergodic” multiple polynomial correlation sequences to nilsystems.

Let \(X = G/\Gamma\) be a connected nilmanifold, let \(Y\) be a connected subnilmanifold of \(X\), and let \(g\) be a polynomial sequence in \(G\), that is, a mapping \(\mathbb{Z}^d \rightarrow G\) of the form \(g(n) = a_1^{p_1(n)} \cdots a_r^{p_r(n)}, n \in \mathbb{Z}^d\), where \(a_1, \ldots, a_r \in G\) and \(p_1, \ldots, p_r\) are polynomials \(\mathbb{Z}^d \rightarrow \mathbb{Z}\). We investigate (in Section 3) the behavior of the sequence \(g(n)Y\) of subnilmanifolds of \(X\): we show that there is a subnilmanifold \(Z\) of \(X\), containing \(Y\), such that the sequence \(g(n)\) only shifts \(Z\) along \(X\), without distorting it, whereas, outside of a null-set of \(n \in \mathbb{Z}^d\), \(g(n)Y\) becomes more and more “dense” in \(g(n)Z\):

**Proposition 0.2.** Assume (as we can) that the orbit \(g(n)Y, n \in \mathbb{Z}^d\), is dense in \(X\), and let \(Z\) be the normal closure (in the algebraic sense; see below) of \(Y\) in \(X\). Then for any \(f \in C(X)\), the sequence \(\lambda(n) = \int_{g(n)Y} f - \int_{g(n)Z} f, n \in \mathbb{Z}^d\), is a null-sequence.

We have \(\int_{g(n)Z} f = g(n)\hat{f}(g(n)e), n \in \mathbb{Z}^d\), where \(\hat{f} = E(f|X/Z)\) and \(e = Z/Z \in X/Z\). (Here and below, \(E(f|X')\) stands for the conditional expectation of a function \(f \in L^1(X)\) with respect to a factor \(X'\) of \(X\).) So, the sequence \(\int_{g(n)Z} f\) is a basic nilsequence, and we obtain:

**Theorem 0.3.** For any \(f \in C(X)\) the sequence \(\varphi(n) = \int_{g(n)Y} f, n \in \mathbb{Z}^d\), is the sum of a basic nilsequence and a null-sequence.

Applying this result to the diagonal \(Y\) of the power \(X^k\) of the nilmanifold \(X\), the polynomial sequence \(g(n) = (a_1^{p_1(n)}, \ldots, a_r^{p_r(n)}), n \in \mathbb{Z}^d, G^k\), and the function \(f = 1_{A_1} \otimes \cdots \otimes 1_{A_k}\), we obtain Theorem 0.1 in the ergodic case.

Our next step (Section 4) is to extend this result to the case of a non-ergodic \(T\). Using the ergodic decomposition \(W \rightarrow \Omega\) of \(T\) we obtain a measurable mapping from \(\Omega\) to the space of nilsequences–plus–null-sequences, which we then have to integrate over \(\Omega\). The integral of a family of null-sequences is a null-sequence, and creates no trouble. As for nilsequences, when we integrate them we arrive at the following problem: if \(X = G/\Gamma\) is a
nilmanifold, with \( \pi: G \rightarrow X \) being the factor mapping, and \( \rho(a) \), \( a \in G \), is a finite Borel measure on \( G \), what is the limiting behavior of the measures \( \pi_* (\rho(a^n)) \) on \( X \)? (This is the question corresponding to the case \( d = 1 \); for \( d \geq 2 \) it is slightly more complicated.) We show that this sequence of measures tends to a linear combination of Haar measures on (countably many) subnilmanifolds of \( X \), which are normal (and so travel, without distortion) in the closure of their orbits, and we again obtain:

**Proposition 0.4.** For any \( f \in C(X) \), the sequence \( \varphi(n) = \int_G f(\pi(a^n)) \, d\rho(a) \), \( n \in \mathbb{Z} \), is a sum of a basic nilsequence and a null-sequence.

(This proposition is a “nilpotent” extension of the following classical fact: if \( \rho \) is a finite Borel measure on the 1-dimensional torus \( \mathbb{T} \), then its Fourier transform \( \varphi(n) = \int_{\mathbb{T}} e^{-2\pi inx} \, d\rho(x) \) is the sum of an almost periodic sequence (a 1-step nilsequence; it corresponds to the atomic part of \( \rho \)) and a null-sequence (that corresponds to the non-atomic part of \( \rho \)).

As a corollary, we obtain the remaining ingredient of the proof of Theorem 0.1:

**Theorem 0.5.** Let \( \Omega \) be a measure space and let \( \varphi_\omega, \omega \in \Omega \), be an integrable family of nilsequences; then the sequence \( \varphi(n) = \int_\Omega \varphi_\omega(n) \) is a sum of a nilsequence and a null-sequence.

Let us also mention generalized (or bracket) polynomials, – the functions constructed from ordinary polynomials using the operations of addition, multiplication, and taking the integer part, \([\cdot]\). (For example, \( p_1[p_2[p_3 + p_4]] \), where \( p_i \) are ordinary polynomials, is a generalized polynomial.) Generalized polynomials (gps) appear quite often (for example, the fractional part, and the distance to the nearest integer, of an ordinary polynomial are gps); they were systematically studied in [Há1], [Há2], [BL], and [L7]. Because of their simple definition, gps are nice objects to deal with. On the other hand, similarly to nilsequences, gps come from nilsystems: bounded gps (on \( \mathbb{Z}^d \), in our case) are exactly the sequences of the form \( h(g(n)x), n \in \mathbb{Z}^d \), where \( h \) is a piecewise polynomial function on a nilmanifold \( X = G/\Gamma, x \in X \), and \( g \) is a polynomial sequence in \( G \) (see [BL] or [L7]). Since any continuous function is uniformly approximable by piecewise polynomial functions (this follows by an application of the Weierstrass theorem in the fundamental domain of \( X \)), nilsequences are uniformly approximable by generalized polynomials. We obtain as a corollary that any multiple polynomial correlation sequence is, up to a null-sequence, uniformly approximable by generalized polynomials.

**Acknowledgment.** I am grateful to Vitaly Bergelson for his interest and advice.

### 1. Nilmanifolds

In this section we collect the facts about nilmanifolds that we will need below; details and proofs can be found in [M], [L1], [L2], [L4].

Throughout the paper, \( X = G/\Gamma \) will be a compact nilmanifold, where \( G \) is a nilpotent Lie group and \( \Gamma \) is a discrete subgroup of \( G \), and \( \pi \) will denote the factor mapping \( G \rightarrow X \). By \( 1_X \) we will denote the point \( \pi(1_G) \) of \( X \). By \( \mu_X \) we will denote the normalized Haar measure on \( X \).
By $G^c$ we will denote the identity component of $G$. Note that if $G$ is disconnected, $X$ can be interpreted as a nilmanifold, $X = G' / \Gamma'$, in different ways: if, for example, $X$ is connected, then also $X = G^c / (\Gamma \cap G^c)$. If $X$ is connected and we study the action on $X$ of a sequence $g(n)$, $g: \mathbb{Z}^d \rightarrow G$, we may always assume that $G$ is generated by $G^c$ and the range $g(\mathbb{Z}^d)$ of $g$. Thus, we may (and will) assume that the group $G/G^c$ is finitely generated.

Every nilpotent Lie group $G$ is a factor of a torsion free nilpotent Lie group. (As such, a suitable “free nilpotent Lie group” $F$ can be taken. If $G^c$ has $k_1$ generators, $G/G^c$ has $k_2$ generators, and $G$ is $r$-step nilpotent, then $F = F/F_{r+1}$, where $F$ is the free product of $k_1$ copies of $\mathbb{R}$ and $k_2$ copies of $\mathbb{Z}$, and $F_{r+1}$ is the $(r+1)$st term of the lower central series of $F$.) Thus, we may always assume that $G$ is torsion-free. The identity component $G^c$ of $G$ is then an exponential Lie group, and for every element $a \in G^c$ there exists a (unique) one-parametric subgroup $a^t$ such that $a^1 = a$.

If $G$ is torsion free, it possesses a Malcev basis compatible with $\Gamma$, which is a finite set $\{e_1, \ldots, e_k\}$ of elements of $\Gamma$, with $e_1, \ldots, e_k \in G^c$ and $e_{k+1}, \ldots, e_k \notin G^c$, such that every element $a \in G$ can be uniquely written in the form $a = e_1^{u_1} \cdots e_k^{u_k}$ with $u_1, \ldots, u_k \in \mathbb{R}$ and $u_{k+1}, \ldots, u_k \in \mathbb{Z}$, and with $a \in \Gamma$ iff $u_1, \ldots, u_k \in \mathbb{Z}$; we call $u_1, \ldots, u_k$ the coordinates of $a$. Thus, Malcev coordinates define a homeomorphism $G \cong \mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$, $a \leftrightarrow (u_1, \ldots, u_k)$, which maps $\Gamma$ onto $\mathbb{Z}^k$.

The multiplication in $G$ is defined by the (finite) multiplication table for the Malcev basis of $G$, whose entries are integers; it follows that there are only countably many non-isomorphic nilpotent Lie groups with cocompact discrete subgroups, and countably many non-isomorphic compact nilmanifolds.

Let $X$ be connected. Then, under the identification $G^c \leftrightarrow \mathbb{R}^{k_1}$, the cube $[0,1]^{k_1}$ is the fundamental domain of $X$. We will call the closed cube $Q = [0,1]^{k_1}$ the fundamental cube of $X$ in $G^c$ and sometimes identify $X$ with $Q$. When $X$ is identified with its fundamental cube $Q$, the measure $\mu_X$ corresponds to the standard Lebesgue measure $\mu_Q$ on $Q$.

In Malcev coordinates, multiplication in $G$ is a polynomial operation: there are polynomials $q_1, \ldots, q_k$ in $2k$ variables with rational coefficients such that for $a = e_1^{u_1} \cdots e_k^{u_k}$ and $b = e_1^{v_1} \cdots e_k^{v_k}$ we have $ab = e_1^{q_1(u_1, v_1, \ldots, u_k, v_k)} \cdots e_k^{q_k(u_1, v_1, \ldots, u_k, v_k)}$. This implies that “life is polynomial” in nilpotent Lie groups: in coordinates, homomorphisms between these groups are polynomial mappings, and connected closed subgroups of such groups are images of polynomial mappings and are defined by systems of polynomial equations.

A subnilmanifold $Y$ of $X$ is a closed subset of the form $Y = Hx$, where $H$ is a closed subgroup of $G$ and $x \in X$. For a closed subgroup $H$ of $G$, the set $\pi(H) = H1_X$ is closed (and so, is a subnilmanifold) iff the subgroup $\Gamma \cap H$ is co-compact in $H$; we will call the subgroup $H$ with this property rational. Any subnilmanifold $Y$ of $X$ has the form $\pi(aH) = a\pi(H)$, where $H$ is a closed rational subgroup of $G$.

If $Y$ is a subnilmanifold of $X$ with $1_X \in Y$, then $Y = \pi(H) = H1_X$ for some closed subgroup $H$ of $G$. $H$ may not be the minimal subgroup with this property: if $Y$ is connected, then the identity component $H^c$ of $H$ also satisfies $\pi(H^c) = Y$.

The intersection of two subnilmanifolds is a finite disjoint union of subnilmanifolds.

Given a subnilmanifold $Y$ of $X$, by $\mu_Y$ we will denote the normalized Haar measure on $Y$. Translations of subnilmanifolds are measure preserving: we have $a_*\mu_Y = \mu_{aY}$ for all
\( \eta \in G. \)

Let \( Z \) be a subnilmanifold of \( X, Z = Lx \), where \( L \) is a closed subgroup of \( G \). We say that \( Z \) is normal if \( L \) is normal. In this case the nilmanifold \( \hat{X} = X/Z = G/(L\Gamma) \) is defined, and \( X \) splits into a disjoint union of fibers of the factor mapping \( X \rightarrow \hat{X} \). (Note that if \( L \) is normal in \( G^c \) only, then the factor \( X/Z = G^c/(L\Gamma) \) is also defined, but the elements of \( G \setminus G^c \) do not act on \( X \).)

One can show that a subgroup \( L \) is normal iff \( \gamma L \gamma^{-1} = L \) for all \( \gamma \in \Gamma \); hence, \( Z = \pi(L) \) is normal iff \( \gamma Z = Z \) for all \( \gamma \in \Gamma \).

If \( H \) is a closed rational subgroup of \( G \) then its normal closure \( L \) (the minimal normal subgroup of \( G \) containing \( H \)) is also closed and rational, thus \( Z = \pi(L) \) is a subnilmanifold of \( X \). We will call \( Z \) the normal closure of the subnilmanifold \( Y = \pi(H) \). If \( L \) is normal then the identity component of \( L \) is also normal; this implies that the normal closure of a connected subnilmanifold is connected.

If \( X \) is connected, the maximal factor-torus of \( X \) is the torus \([G^c,G^c]\backslash X\), and the nilmaximal factor-torus is \([G,G]\backslash X\). The nilmaximal factor-torus is a factor of the maximal one.

If \( \eta: \mathbb{Z}^d \rightarrow G \) is a homomorphism, then for any point \( x \in X \) the closure of the orbit \( \eta(\mathbb{Z}^d)x \) of \( x \) in \( X \) is a subnilmanifold \( V \) of \( X \) (not necessarily connected), and the sequence \( \eta(n)x, n \in \mathbb{Z}^d \), is well distributed in \( V \). (This means that for any function \( f \in C(V) \) and any Følner sequence \((\Phi_N)\) in \( \mathbb{Z}^d \), \( \lim_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(\eta(n)x) = \int_X f d\mu_V \).) If \( X \) is connected, the sequence \( \eta(n)x, n \in \mathbb{Z}^d \), is dense, and so, well distributed in \( X \) iff the image of this sequence is dense in the nil-maximal factor-torus of \( X \). All the same is true for the orbit of any subnilmanifold \( Y \) of \( X \): the closure of \( \bigcup_{n \in \mathbb{Z}^d} \eta(n)Y \) is a subnilmanifold \( W \) of \( X \); the sequence \( \eta(n)Y, n \in \mathbb{Z}^d \), is well distributed in \( W \) (this means that for any function \( f \in C(W) \) and any Følner sequence \((\Phi_N)\) in \( \mathbb{Z}^d \), \( \lim_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(\eta(n)x) = \int_Y f d\mu_Y \); and, in the case \( X \) is connected, the sequence \( \eta(n)Y \) is well distributed in \( X \) iff its image is dense in the nilmaximal factor-torus of \( X \).

A polynomial sequence in \( G \) is a sequence of the form \( g(n) = a_1^{p_1(n)} \cdots a_k^{p_k(n)}, \ n \in \mathbb{Z}^d \), where \( a_1, \ldots, a_k \in G \) and \( p_1, \ldots, p_k \) are polynomials \( \mathbb{Z}^d \rightarrow \mathbb{Z} \). Let \( g \) be a polynomial sequence in \( G \) and let \( x \in X \). Then the closure \( V \) of the orbit \( g(\mathbb{Z}^d)x \) is a finite disjoint union of connected subnilmanifolds of \( X \), and \( g(n)x \) visits these subnilmanifolds periodically: there exists \( l \in \mathbb{N} \) such that for any \( i \in \mathbb{Z}^d \), all the elements \( g(lm + i)x, m \in \mathbb{Z}^d \), belong to the same connected component of \( V \). If \( V \) is connected, then the sequence \( g(n)x, n \in \mathbb{Z}^d \), is well distributed in \( V \). In the case \( X \) is connected, the sequence \( g(n)x, n \in \mathbb{Z}^d \), is dense, and so, well distributed in \( X \) iff the image of this sequence is dense in the maximal factor-torus of \( X \). All the same is true for the orbit \( g(\mathbb{Z}^d)Y \) of any connected subnilmanifold \( Y \) of \( X \) under the action of \( g \): its closure \( W \) is a finite disjoint union of connected subnilmanifolds of \( X \), visited periodically; if \( W \) is connected, then the sequence \( g(n)Y, n \in \mathbb{Z}^d \), is well distributed in \( W \); and, if \( X \) is connected, the sequence \( g(n)Y \) is well distributed in \( X \) iff its image is dense in the maximal factor-torus of \( X \).

The following proposition, which is a corollary (of a special case) of the result obtained in [GT], says that “almost every” subnilmanifold of \( X \) is distributed in \( X \) "quite uniformly". (See Appendix in [L6] for details.)
Lemma 2.1. Let $X$ be connected. For any $C > 0$ and any $\varepsilon > 0$ there are finitely many subnilmanifolds $V_1, \ldots, V_r$ of $X$, connected and containing $1_X$, such that for any connected subnilmanifold $Y$ of $X$ with $1_X \in Y$, if $Y \not\subseteq V_i$ for all $i \in \{1, \ldots, r\}$, then $|\int_Y f \, d\mu_Y - \int_X f \, d\mu_X| < \varepsilon$ for all functions $f$ on $X$ with $\sup_{x \neq y} |f(x) - f(y)|/\text{dist}(x, y) \leq C$.

(This is in complete analogy with the situation on tori: if $X$ is a torus, for any $\varepsilon > 0$ there are only finitely many subtori $V_1, \ldots, V_r$ such that any subtorus $Y$ of $X$ that contains 0 and is not contained in $\bigcup_{i=1}^r V_i$ is $\varepsilon$-dense and “$\varepsilon$-uniformly distributed” in $X$.)

2. Nilsequences, null-sequences, and generalized polynomials

We will deal with the space $l^\infty = l^\infty(\mathbb{Z}^d)$ of bounded sequences $\varphi: \mathbb{Z}^d \to \mathbb{C}$, with the norm $\|\varphi\| = \sup_{n \in \mathbb{Z}^d} |\varphi(n)|$.

A basic $r$-step nilsequence is an element of $l^\infty$ of the form $\psi(n) = f(\eta(n)x)$, $n \in \mathbb{Z}^d$, where $x$ is a point of an $r$-step nilmanifold $X = G/\Gamma$, $\eta$ is a homomorphism $\mathbb{Z}^d \to G$, and $f \in C(X)$. We will denote the algebra of basic $r$-step nilsequences by $N_r^o$, and the algebra $\bigcup_{r \in \mathbb{N}} N_r^o$ of all basic nilsequences by $N^o$. We will denote the closure of $N_r^o$, $r \in \mathbb{N}$, in $l^\infty$ by $\bar{N}_r$, and the closure of $N^o$ by $\bar{N}$; the elements of these algebras will be called $r$-step nilsequences and, respectively, nilsequences.

Given a polynomial sequence $g(n) = a_1^{p_1(n)} \cdots a_k^{p_k(n)}$, $n \in \mathbb{Z}^d$, in a nilpotent group with $\deg p_i \leq s$ for all $i$, we will say that $g$ has naive degree $\leq s$. (The term “degree” was already reserved for another parameter of a polynomial sequence.) We will call a sequence of the form $\psi(n) = f(g(n)x)$, where $x$ is a point of an $r$-step nilmanifold $X = G/\Gamma$, $g$ is a polynomial sequence of naive degree $\leq s$ in $G$, and $f \in C(X)$, a basic polynomial $r$-step nilsequence of degree $\leq s$. We will denote the algebra of basic polynomial $r$-step nilsequences of degree $\leq s$ by $N^o_{r,s}$ and the closure of this algebra in $l^\infty$ by $\bar{N}_{r,s}$. It is shown in [L2] (see proof of Theorem B*) that any basic polynomial $r$-step nilsequence of degree $\leq s$ is a basic $l$-step nilsequence, where $l = 2rs$; we introduce this notion here only in order to keep track of the parameters $r, s$ of the “origination” of a nilsequence. So, for any $r$ and $s$, $N^o_{r,s} \subseteq N^o_{2rs}$; since also $N^o_r \subseteq N^o_{r,1}$, we have $\bigcup_{r,s \in \mathbb{N}} N^o_{r,s} = N^o$.

We will also need the following lemma, saying, informally, that the operation of “alternation” of sequences preserves the algebras of nilsequences:

Lemma 2.1. Let $k \in \mathbb{N}$ and let $\psi_i \in N^o_{r,s} \ (\text{respectively}, N^o_r)$, $i \in \{0, \ldots, k-1\}^d$. Then the sequence $\psi$ defined by $\psi(n) = \psi_i(m)$ for $n = km + i$ with $m \in \mathbb{Z}^d$, $i \in \{0, \ldots, k-1\}^d$, is also in $N^o_{r,s} \ (\text{respectively}, N^o_r)$.

Proof. Put $I = \{0, \ldots, k-1\}^d$. For each $i \in I$, let $X_i = G_i/\Gamma_i$ be the $r$-step nilmanifold, $g_i$ be the polynomial sequence in $G_i$ of naive degree $\leq s$, $x_i \in X_i$ be the point, and $f_i \in C(X_i)$ be the function such that $\psi_i(n) = f(g_i(n)x_i)$, $n \in \mathbb{Z}^d$. If, for some $i$, $G_i$ is not connected, it is a factor-group of a free $r$-step nilpotent group with both continuous and discrete generators, which, in its turn, is a subgroup of a free $r$-step nilpotent group with only continuous generators; thus after replacing, if needed, $X_i$ by a larger nilmanifold and extending $f_i$ to a continuous function on this nilmanifold we may assume that every $G_i$
then $\psi$ implies that every point returns to any its neighborhood regularly. It follows that if a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $G_i$, and thus the polynomial sequence $b^{p(n)}$ in $G_i$ makes sense even if a polynomial $p$ has non-integer rational coefficients. Thus, for each $i \in I$ we may construct a polynomial sequence $g_i^r$ in $G_i$, of the same naive degree as $g_i$, such that $g_i^r(km + i) = g_i(m)$ for all $m \in \mathbb{Z}^d$. Put $G = \mathbb{Z}^d \times \prod_{i \in I} G_i, X = (\mathbb{Z}/(k\mathbb{Z}))^d \times \prod_{i \in I} X_i, g(n) = (n, (g_i^r(n), i \in I))$ for $n \in \mathbb{Z}^d, x = (0, (x_i, i \in I)) \in X$, and $f(i, (y_i, i \in I)) = f_i(y_i)$ for $(i, (y_i, i \in I)) \in X$. Then $X$ is an $r$-step nilmanifold, $f \in C(X)$, and thus the sequence $\psi(n) = f(g(n)x) = f_i(g_i^r(n)x_i) = f_i(g_i(m)x_i) = \psi_i(m), n = km + i, m \in \mathbb{Z}^d, i \in I$, is in $\mathcal{N}_{r,s}$.

A set $S \subset \mathbb{Z}^d$ is said to be of uniform (or Banach) density zero if for any Følner sequence $(\Phi_N)_{N=1}^{\infty}$ in $\mathbb{Z}^d, \lim_{N \to \infty} |S \cap \Phi_N|/|\Phi_N| = 0$. A sequence $(\omega_n)_{n \in \mathbb{Z}^d}$ in a topological space $\Omega$ converges to $\omega \in \Omega$ in uniform density if for every neighborhood $U$ of $\omega$ the set $\{n \in \mathbb{Z}^d : \omega_n \notin U\}$ is of uniform density zero.

We will say that a sequence $\lambda \in l^\infty$ is a null-sequence if it tends to zero in uniform density. This is equivalent to $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)| = 0$ for any Følner sequence $(\Phi_N)_{N=1}^{\infty}$ in $\mathbb{Z}^d$, and is also equivalent to $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)|^2 = 0$ for any Følner sequence $(\Phi_N)_{N=1}^{\infty}$ in $\mathbb{Z}^d$. We will denote the set of (bounded) null-sequences by $\mathcal{Z}$.

The algebra $\mathcal{Z}$ is orthogonal to the algebra $\mathcal{N}$ in the following sense:

**Lemma 2.2.** For any $\psi \in \mathcal{N}$ and $\lambda \in \mathcal{Z}$, $\|\psi + \lambda\| \geq \|\psi\|$.

**Proof.** Let $c \geq \|\psi + \lambda\|$. Nilsystems are distal systems (see, for example, [L1]), which implies that every point returns to any its neighborhood regularly. It follows that if $|\psi(n)| > c$ for some $n$, then the set $\{n \in \mathbb{Z}^d : |\psi(n)| > c\}$ has positive lower density, and then $\psi(n) + \lambda(n) > c$ for some $n$, contradiction. Hence, $|\psi(n)| \leq c$ for all $n$.

It follows that $\mathcal{N} \cap \mathcal{Z} = 0$.

We will denote the algebras $\mathcal{N}^o_r + \mathcal{Z}, \mathcal{N}_r + \mathcal{Z}, \mathcal{N}^o_{r,s} + \mathcal{Z}, \mathcal{N}_{r,s} + \mathcal{Z}, \mathcal{N}^o + \mathcal{Z},$ and $\mathcal{N} + \mathcal{Z}$ by $\mathcal{M}^o, \mathcal{M}_r, \mathcal{M}^o_{r,s}, \mathcal{M}_{r,s}, \mathcal{M}^o$, and $\mathcal{M}$ respectively.

Lemma 2.2 implies:

**Lemma 2.3.** The algebras $\mathcal{M}, \mathcal{M}_r$, and $\mathcal{M}_{r,s}, r, s \in \mathbb{N}$, are all closed, and the projections $\mathcal{M} \to \mathcal{N}, \mathcal{M} \to \mathcal{Z}$ are continuous.

**Proof.** If a sequence $(\psi_n + \lambda_n)$ with $\psi_n \in \mathcal{N}, \lambda_n \in \mathcal{Z}$, converges to $\varphi \in l^\infty$, then since $\|\psi_n\| \leq \|\psi_n + \psi_n\|$ for all $n$, the sequence $(\psi_n)$ is Cauchy, and so converges to some $\psi \in \mathcal{N}$. Then $(\lambda_n)$ also converges, to some $\lambda \in \mathcal{Z}$, and so $\varphi = \psi + \lambda \in \mathcal{M}$. All the same is true for $\mathcal{M}_r$ and $\mathcal{M}_{r,s}$, instead of $\mathcal{M}$, for all $r$ and $s$.

For $\varphi_1 = \psi_1 + \lambda_1$ and $\varphi_2 = \psi_2 + \lambda_2$ with $\psi_1, \psi_2 \in \mathcal{N}$ and $\lambda_1, \lambda_2 \in \mathcal{Z}$ we have $\|\psi_1 - \psi_2\| \leq \|\varphi_1 - \varphi_2\|$, so the projection $\mathcal{M} \to \mathcal{N}, \psi + \lambda \mapsto \psi$, is continuous, and so the projection $\mathcal{M} \to \mathcal{Z}, \psi + \lambda \mapsto \lambda$, is also continuous.

Generalized polynomials on $\mathbb{Z}^d$ are the functions on $\mathbb{Z}^d$ constructed from ordinary polynomials using the operations of addition, multiplication, and the operation of taking the integer part. A function $h$ on a nilmanifold $X$ is said to be piecewise polynomial if it
can be represented in the form \( h(x) = q_i(x), x \in Q_i, i = 1, \ldots, k \), where \( X = \bigcup_{i=1}^k Q_i \) is a finite partition of \( X \) and, in Malcev coordinates on \( X \), for every \( i \) the set \( Q_i \) is defined by a system of polynomial inequalities and \( q_i \) is a polynomial function. (Since multiplication in a nilpotent Lie group is polynomial, this definition does not depend on the choice of \( \upsilon \) that a sequence \( v \in l^\infty \) is a generalized polynomial iff there is a nilmanifold \( X = G/\Gamma \), a piecewise polynomial function \( h \) on \( X \), a polynomial sequence \( g \) in \( G \), and a point \( x \in X \) such that \( v(n) = h(g(n)x), n \in \mathbb{Z}^d \).

Let \( \mathcal{P}^o \) be the algebra of bounded generalized polynomials on \( \mathbb{Z}^d \) and \( \mathcal{P} \) be the closure of \( \mathcal{P}^o \) in \( l^\infty \). Since (by the Weierstrass approximation theorem) any continuous function on a compact nilmanifold \( X \) is uniformly approximable by piecewise polynomial functions, any basic nilsequence is uniformly approximable by bounded generalized polynomials, and so, is contained in \( \mathcal{P} \). Hence, \( \mathcal{N} \subset \mathcal{P} \), and \( \mathcal{M} \subset \mathcal{P} + \mathcal{Z} \). The inverse inclusion does not hold, since not all piecewise polynomial functions are uniformly approximable by continuous functions; however, they are – on the complement of a set of arbitrarily small measure, which implies that generalized polynomials are also approximable by nilsequences, – in a certain weaker topology in \( l^\infty \).

### 3. Distribution of a polynomial sequence of subnilmanifolds

Let \( Y \) be a connected sub-nilmanifold of the (connected) nilmanifold \( X \), and let \( g(n), n \in \mathbb{Z}^d \), be a polynomials sequence in \( G \). We will investigate how the sequence \( g(n)Y \) of sunilmanifolds of \( X \) is distributed in \( X \).

**Proposition 3.1.** Let \( X = G/\Gamma \) be a connected nilmanifold, let \( Y \) be a connected sub-nilmanifold of \( X \), and let \( g: \mathbb{Z}^d \rightarrow G \) be a polynomial sequence in \( G \) with \( g(0) = 1_G \). Assume that \( g(\mathbb{Z}^d)Y \) is dense in \( X \), and that \( G \) is generated by \( G^c \) and the range \( g(\mathbb{Z}^d) \) of \( g \). Let \( Z \) be the normal closure of \( Y \) in \( X \); then for any \( f \in C(X), \lambda(n) = \int_{g(n)Y} f d\mu_g(n)Y - \int_{g(n)Z} f d\mu_g(n)Z, n \in \mathbb{Z}^d \), is a null-sequence.

**Proof.** Let \( f \in C(X) \) and let \( \varepsilon > 0 \); we have to show that the set \( \{ n \in \mathbb{Z}^d : \left| \int_{g(n)Y} f d\mu_g(n)Y - \int_{g(n)Z} f d\mu_g(n)Z \right| \geq \varepsilon \} \) has zero uniform density in \( \mathbb{Z}^d \). After replacing \( f \) by a close function we may assume that \( f \) is Lipschitz, so that \( C = \sup_{x \neq y} |f(x) - f(y)|/\text{dist}(x,y) \) is finite. Choose Malcev’s coordinates in \( G^c \), and let \( Q \subset G^c \) be the corresponding fundamental cube. Since \( Z \) is normal in \( X \), \( aZ = bZ \) whenever \( a = b \mod \Gamma \), and \( \bigcup_{a \in Q} aZ \) is a partition of \( X \).

We first want to determine for which \( a \in G \) one has \( \left| \int_{aY} f_{aY} d\mu_{aY} - \int_{aZ} f d\mu_{aZ} \right| \geq \varepsilon \). For every \( b \in Q \), by Proposition 1.1, applied to the nilmanifold \( bZ \), there exist proper sub-nilmanifolds \( V_{b,1}, \ldots, V_{b,r_b} \) of \( Z \) such that \( \left| \int_W f d\mu_W - \int_{bZ} f d\mu_{bZ} \right| < \varepsilon/2 \) whenever \( W \) is a sub-nilmanifold of \( bZ \) with \( b \in W \subsetneq bV_{b,i}, i = 1, \ldots, r_b \). By continuity, for each \( b \in Q \) there exists a neighborhood \( U_b \) of \( b \) such that for all \( a \in U_b \), \( \left| \int_W f d\mu_W - \int_{aZ} f d\mu_{aZ} \right| < \varepsilon \) whenever \( a \in W \subsetneq aZ \) and \( W \not\subsetneq aV_{b,i}, i = 1, \ldots, r_b \). Using the compactness of the closure of \( Q \), we can choose \( b_1, \ldots, b_l \in Q \) such that \( Q \subseteq \bigcup_{j=1}^l U_{b_j} \); let \( V = \bigcup_{i=1, \ldots, f_j} V_{b_j,i} \). Then for any \( b \in Q \), for any sub-nilmanifold \( W \) of \( bZ \) with \( b \in W \not\subsetneq V \) one has \( \left| \int_W f d\mu_W - \int_{aZ} f d\mu_{aZ} \right| < \varepsilon \).
\[ \int_{bZ} f \, d\mu_{bZ} < \varepsilon. \] Now let \( a \in G \), and let \( b \in Q \) be such that \( a = b \mod \Gamma \). Then \( aY \subseteq aZ = bZ \), thus, if \( aY \not\subseteq bV \), then \[ |\int_{aY} f \, d\mu_{aY} - \int_{aZ} f \, d\mu_{aZ}| < \varepsilon. \] Hence, \[ |\int_{aY} f \, d\mu_{aY} - \int_{aZ} f \, d\mu_{aZ}| \geq \varepsilon \] only if \( (a1X, aY) \subseteq (b1X, bV) \) for some \( b \in Q \).

Let \( N = \{(b1X, bV), b \in Q\} \); we have to prove that the set \( \{n \in \mathbb{Z}^d : (g(n)1X, g(n)Y) \subseteq N\} \) has zero uniform density in \( \mathbb{Z}^d \). For this purpose we are going to find the closure of the sequence \( \tilde{Y}_n = (g(n)1X, g(n)Y), n \in \mathbb{Z}^d \), (more exactly, of the union \( \bigcup_{n \in \mathbb{Z}^d} \tilde{Y}_n \)), the orbit of the subnilmanifold \( \tilde{Y} = (1X, Y) \) of \( X \times X \) under the polynomial action \( (g(n), g(n)), n \in \mathbb{Z}^d \). Assume for simplicity that the closure \( R \) of the orbit \( \{g(n)1X, n \in \mathbb{Z}^d\} \) is connected, and let \( P \) be the closed connected subgroup of \( G \) such that \( \pi(P) = R \). (If \( R \) is disconnected we pass to a sublattice of \( \mathbb{Z}^d \) and its cosets to deal with individual connected components of \( R \).) We will also assume that \( Y \ni 1_X \).

**Lemma 3.2.** The closure of the sequence \( \{\tilde{Y}_n\} \) is the subnilmanifold \( D = \{(a1_X, aZ), a \in P\} \cap (X \times X) \).

**Proof.** Let \( L \) be the closed connected subgroup of \( G \) such that \( \pi(L) = Z \), and let \( K = \{(a, au), a \in P, u \in L\} \); since \( L \) is normal in \( G \), \( K \) is a (closed rational) subgroup of \( G \times G \), and we have \( D = K/((\Gamma \times \Gamma) \cap K) \).

For any \( n \in \mathbb{Z}^d \) we have \( \tilde{Y}_n \subseteq D \) (since \( g(n)1_X \in R \), so \( g(n) \in P\Gamma \), so \( g(n)L \subseteq P\Gamma L = P\Gamma \)), and we have to show that the sequence \( \{\tilde{Y}_n\} \) is dense in \( D \). For this it suffices to prove that the image of this sequence is dense in the maximal torus \( T = [K, K] \setminus D \) of \( D \). Since \( L \) is normal, we have \( [K, K] = \{(a, au), a \in [P, P], u \in [P, L][L, L]\} \), and the torus \( T \) is the factor of the commutative group \( K/[K, K] \) by the image \( \Lambda \) in this group of the lattice \( \Gamma \times \Gamma \). Let \( H \) be the closed connected subgroup of \( G \) such that \( \pi(H) = Y \); then \( L = H[H, G] \), so \( K/[K, K] = \{(a, avw), a \in P, v \in H, w \in [H, G]\}/\{(a, au), a \in [P, P], u \in [P, L][L, L]\} \).

By assumption, \( G \) is generated by \( G^c \) and \( g \). Since the orbit \( \{g(n)Z, n \in \mathbb{Z}^d\} \) is dense in \( X \) and \( Z \) is normal, the orbit \( \{g(n)1_XZ, n \in \mathbb{Z}^d\} \) is dense in \( X/Z \), so \( P/(P \cap L) = G^c/L \), so \( G^c = PL \). Hence, \( [H, G] = [H, g][H, P][H, L] \). For any \( n \), \( g(n) = u_n\gamma_n \) for some \( u_n \in P \) and \( \gamma_n \in \Gamma \), thus, modulo \( [P, L][L, L] \), the group \( [H, G] \) is generated by \( \{(H, \gamma_n), n \in \mathbb{Z}^d\} \).

The closure \( B \) of the image of the sequence \( \{\tilde{Y}_n\} \) in \( T \) is a subtorus of \( T \). Since the sequence \( g(n)1_X \) is dense in \( R = \pi(P) \), the subtorus \( T_1 = \{(a, a), a \in P\}/([K, K]\Lambda) \) of \( T \) is the closure of the image of the sequence \( g(n)1_X, g(n)1_X \) and so, is contained in \( B \). Also, the subtorus \( T_2 = \{(1_X, u), u \in H\}/([K, K]\Lambda) \) is contained in \( B \). Finally, for \( n \in \mathbb{Z}^d \) and \( c \in H \) we have
\[
(g(n), g(n)c) = (u_n, u_n\gamma_n c) = (u_n1_X, u_n c[\gamma_n^{-1}][\gamma_n]).
\]

Taken modulo \( [K, K]\Lambda \), these elements of \( B \) generate \( T \) modulo \( T_1 + T_2 \), so \( B = T \). □

It follows that the sequence \( \tilde{Y}_n, n \in \mathbb{Z}^d \), is well distributed in \( D \). The set \( N = \{(b1_X, bV), b \in Q\} \) is a compact subset of \( D \) of zero measure, thus, the set \( \{n \in \mathbb{Z}^d : (g(n)1_X, g(n)Y) \subseteq N\} \) has zero uniform density in \( \mathbb{Z}^d \). □
Theorem 3.3. Let \( X = G/\Gamma \) be an \( r \)-step nilmanifold, let \( Y \) be a subnilmanifold of \( X \), let \( g \) be a polynomial sequence in \( G \) of naive degree \( \leq s \), let \( f \in C(X) \). Then the sequence \( \varphi(n) = \int_{g(n)Y} f \, d\mu_{g(n)Y}, \; n \in \mathbb{Z}^d \), is contained in \( M_{r,s}^o \).

Proof. We may assume that \( Y \supseteq 1_X \). After replacing \( f \) by \( f(g(0)x) \), \( x \in X \), we may assume that \( g(0) = 1_X \). We may also replace \( X \) by the closure of the orbit \( g(\mathbb{Z}^d)Y \), and we may assume that \( G \) is generated by \( G^c \) and the range of \( g \).

First, let \( X \) and \( Y \) be both connected. Let \( Z \) be the normal closure of \( Y \) in \( X \); then by Proposition 3.1, \( \varphi(n) = \int_{g(n)Z} f \, d\mu_{g(n)Z} + \lambda_n, \; n \in \mathbb{Z}^d \), with \( \lambda \in \mathbb{Z} \). Define \( \hat{X} = X/Z \), \( \hat{x} = \{Z\} \in \hat{X} \), and \( \hat{f} = E(f|\hat{X}) \in C(\hat{X}) \); then \( \int_{g(n)Z} f \, d\mu_{g(n)Z} = \hat{f}(g(n)\hat{x}), \; n \in \mathbb{Z}^d \), and the sequence \( \hat{f}(g(n)\hat{x}), \; n \in \mathbb{Z}^d \), is in \( M_{r,s}^o \), so \( \varphi \in M_{r,s}^o \).

Now assume that \( Y \) is connected but \( X \) is not. Then, by [L2], there exists \( k \in \mathbb{N} \) such that \( g(k\mathbb{Z}^d + i)Y \) is connected for every \( i \in \{0, \ldots, k-1\}^d \). Thus, for every \( i \in \{0, \ldots, k-1\}^d \), \( \varphi(kn+i) \in M_{r,s}^o \), and the assertion follows from Lemma 2.1.

Finally, if \( Y \) is disconnected and \( Y_1, \ldots, Y_r \) are the connected components of \( Y \), then \( \int_{g(n)Y} f \, d\mu_{g(n)Y} = \sum_{j=1}^r \int_{g(n)Y_j} f \, d\mu_{g(n)Y_j}, \; n \in \mathbb{Z}^d \), and the result holds since it holds for \( Y_1, \ldots, Y_r \).

4. Integrals of null- and of nil-sequences

On \( l^\infty \) and, thus, on \( N, \mathcal{Z} \) and \( M \) we will assume the Borel \( \sigma \)-algebra induced by the weak topology. We start with integration of null-sequences:

Lemma 4.1. Let \((\Omega, \nu)\) be a measurable space and let \( \Omega \longrightarrow \mathcal{Z}, \; \omega \mapsto \lambda_\omega \), be an integrable mapping. Then the sequence \( \lambda(n) = \int_{\Omega} \lambda_\omega(n) \, d\nu \) is in \( \mathcal{Z} \) as well.

(We say that a mapping \( \Psi: \Omega \longrightarrow l^\infty \) is integrable if it is measurable and \( \int_\Omega \|\Psi\| \, d\nu < \infty \).)

Proof. For each \( \omega \in \Omega \), \( \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| = 0 \) for any Følner sequence \((\Phi_N)_{N=1}^\infty \) in \( \mathbb{Z}^d \). By the dominated convergence theorem,

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)| = \limsup_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \left| \int_{\Omega} \lambda_\omega(n) \, d\nu \right| \\
\leq \lim_{N \to \infty} \int_{\Omega} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| \, d\nu = \int_{\Omega} \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| \, d\nu = 0.
\]

So, \( \lambda \in \mathcal{Z} \).

For nilsequences we have:

Proposition 4.2. Let \((\Omega, \nu)\) be a measure space and let \( \Omega \longrightarrow N, \; \omega \mapsto \varphi_\omega \), be an integrable mapping. Then the sequence \( \varphi(n) = \int_{\Omega} \varphi_\omega(n) \, d\nu \) belongs to \( M \). (If, for some \( r \), \( \varphi_\omega \in \mathcal{N}_r \) for all \( \omega \), then \( \varphi \in \mathcal{M}_r \).)
To simplify notation, let us start with the case $d = 1$. We are going to reduce Proposition 4.2 to a statement concerning a sequence of measures on a nilmanifold.\(^{(1)}\) Since $\mathcal{M}$ is closed in $l^\infty$, we are allowed to replace the mapping $\varphi_\omega$ from $\Omega$ to $\mathcal{N}$ by a close mapping $\varphi'_\omega$: we are done if for any $\varepsilon > 0$ we can find a mapping $\Omega \rightarrow \mathcal{N}$, $\omega \mapsto \varphi'_\omega$, with $\| \int_\Omega \|\varphi_\omega - \varphi'_\omega\|\|d\nu < \varepsilon$ and such that the assertion of Proposition 4.2 holds for $\varphi'_\omega$. Fix $\varepsilon > 0$. First, after replacing $\Omega$ by $\Omega'$ with $\nu(\Omega') < \infty$ such that $\int_{\Omega \setminus \Omega'} \|\varphi_\omega\|d\nu < \varepsilon$, we may assume that $\nu(\Omega) < \infty$. Next, since the set $\mathcal{N}^0$ of basic nilsequences is dense in $\mathcal{N}$, we may replace the nilsequences $\varphi_\omega$ by close basic nilsequences, if we manage to do this in a measurable way. We will, as we may, deal with $\mathbb{R}$-valued nilsequences. Let $X = G/\Gamma$ be a nilmanifold where $G$ is a simply connected nilpotent Lie group and $\Gamma$ is a lattice in $G$, and let $\pi: G \rightarrow X$ be the projection. We may assume that $G$ has the same number of connected components as $X$, then $G$ is homeomorphic to $\mathbb{R}^{\dim G} \times F$, where $F$ is a finite set, with $\Gamma$ corresponding to $\mathbb{Z}^{\dim G}$; this homeomorphism induces a natural metric on $G$ and on $X$. For $k \in \mathbb{N}$ let $Q_k$ be the set of elements of $G$ at the distance $\leq k$ from $1_G$ and let $L_k$ be the set of Lipschitz functions on $X$ with Lipshitz constant $k$ and of modulus $\leq k$. The subset $Q_k \times L_k$ of $G \times C(X)$ is compact; the “nilsequence reading” mapping $\Psi: G_k \times C(X_k) \rightarrow \mathcal{N}$, $\Psi(a, h)(n) = h(\pi(a^n))$, is continuous with respect to the weak topology on $\mathcal{N}$; thus the set $L_{X,k} = \Psi(Q_k \times L_k) \subset \mathcal{N}^0$ is compact in this topology. Fix a countable set $S$ dense in $L_{X,k}$ in the weak topology and enumerate it. Let $\varphi \in \mathcal{N}$. For each $j \in \mathbb{N}$ let $\psi_j$ be the element of $S$ for which

(i) the sum $\sum_{n=-j}^j |\varphi(n) - \psi_j(n)|$ is minimal;

(ii) among the elements of $S$ for which (i) holds, the vector $(\psi(0), \psi(-1), \psi(1), \ldots, \psi(-j), \psi(j))$ is minimal for $\psi = \psi_j$ with respect to the lexicographic order;

(iii) and among the elements of $S$ for which (i) and (ii) hold, $\psi_j$ has the minimal number under the ordering of $S$.

Put $\zeta_{X,k,j}(\varphi) = \psi_j$; then $\zeta_{X,k,j}$ is a measurable mapping $\mathcal{N} \rightarrow L_{X,k}$. For any $\varphi \in \mathcal{N}$ the sequence $\psi_j = \zeta_{X,k,j}(\varphi)$ converges in $L_k$: indeed, $L_{X,k}$ is compact, and any convergent subsequence of this sequence converges to the same element of $L_{X,k}$, namely, to $\psi \in Q_k$ which is closest to $\varphi$ in the $l^\infty$-norm, and among such, which is minimal with respect to the lexicographic order of its entries. Put $\zeta_{X,k}(\varphi) = \lim_{j \rightarrow \infty} \zeta_{X,k,j}(\varphi)$, $\varphi \in \mathcal{N}$; then $\zeta_{X,k}$ is a measurable mapping $\mathcal{N} \rightarrow L_{X,k}$ that maps each nilsequence to a closest in $l^\infty$-norm element of $L_{X,k}$. It also follows that the function $\partial_{X,k}(\varphi) = \min_{\psi \in L_{X,k}} ||\varphi - \psi||_\infty$ is measurable on $\mathcal{N}$.

In each class of isomorphic nilmanifolds choose a representative $X$ (along with $G$, $\Gamma$, a homeomorphism $G \rightarrow \mathbb{R}^{\dim G} \times F$, and a metric on $G$ and $X$); let $\mathcal{X}$ be the set of these representatives. Since there exists only countably many nonisomorphic nilmanifolds, $\mathcal{X}$ is countable. Introduce a well ordering of $\mathcal{X}$ satisfying $X' < X$ when $\dim X' < \dim X$ and when $\dim X' = \dim X$ and $X'$ has fewer connected components. For every $X \in \mathcal{X}$ put $\Omega_{X,k} = \{ \omega \in \Omega : \partial_{X,k}(\varphi_\omega) < \varepsilon/\nu(\Omega) \}$ and $\Omega_X = \bigcup_{k=1}^\infty \Omega_{X,k}$; these are measurable subsets of $\Omega$. The union $\bigcup_{X \in \mathcal{X}} \bigcup_{k=1}^\infty L_{X,k}$ is dense in $\mathcal{N}$, thus $\bigcup_{X \in \mathcal{X}} \bigcup_{k=1}^\infty \Omega_{X,k} = \Omega$. Next define $\Omega'_X = \Omega_X \setminus \bigcup_{X' \prec X} \Omega_{X'}$, $X \in \mathcal{X}$; these are disjoint sets that partition $\Omega$. Finally, for each

\(^{(1)}\) The argument that follows has been changed; I thank B. Host for pointing to me out a mistake in the previous version of the paper.
$X \in \mathcal{X}$ and $k \in \mathbb{N}$ put $\Omega_{X,k} = \Omega_X \setminus \bigcup_{k' < k} \Omega_{X,k'}$. Now, for $\omega \in \Omega$ define $\psi_\omega = \zeta_{X,k}(\varphi_\omega)$ when $\omega \in \Omega_{X,k}$, $X \in \mathcal{X}$, $k \in \mathbb{N}$; then $\omega \mapsto \psi_\omega$ is a measurable mapping $\Omega \rightarrow \mathcal{N}$ with $\|\psi_\omega - \varphi_\omega\|_\infty < \varepsilon/\nu(\Omega)$ for all $\omega \in \Omega$. We may now replace $\varphi_\omega$ by $\psi_\omega$, $\omega \in \Omega$; moreover, we may also deal with the sets $\Omega_X$ separately, and therefore assume that $\varphi_\omega$, $\omega \in \Omega$, are all read off the same nilmanifold $X = G/\Gamma$: $\varphi_\omega = \Psi(a,h)$ with $a \in G$ and $h$ being a Lipschitz function on $X$. (And, in addition, by our construction, $\varphi_\omega$ is not readable off any nilmanifold $X'$ with $X < X$.)

We now claim that for each $\omega \in \Omega$, $\varphi_\omega$ has only countably many preimages under this mapping; by Luzin’s theorem about the existence of a measurable section, this will imply that we can choose elements $a_\omega \in Q_k$, $h_\omega \in L_k$ with $\varphi_\omega = \Psi(a_\omega,h_\omega)$, $\omega \in \Omega$, such that the mapping $\omega \mapsto (a_\omega,h_\omega)$ is measurable. We get use of the following fact:

**Lemma 4.3.** Let nilmanifolds $X_i = G_i/\Gamma_i$, elements $a_i \in G_i$, and functions $h_i \in C(X_i)$, $i = 1,2$, be such that the orbits $\{\pi_i(a_i^n)\}_{n \in \mathbb{Z}}$, where $\pi_i$ are the projections $G_i \rightarrow X_i$, are dense in $X_i$, $i = 1,2$, and the triples $(X_1,a_1,h_1)$ and $(X_2,a_2,h_2)$ produce the same nilsequence: $\varphi(n) = h_1(\pi_1(a_1^n)) = h_2(\pi_2(a_2^n))$, $n \in \mathbb{Z}$. Then there exists a common factor $(\hat{X},\hat{a},\hat{h})$ of $(X_1,a_1,h_1)$ and $(X_2,a_2,h_2)$ such that $\varphi(n) = \hat{h}(\hat{\pi}(\hat{a}^n))$ (where $\hat{\pi}$ is the projection $\hat{G} \rightarrow \hat{X}$).

**Proof.** Let $\hat{G} = G_1 \times G_2$, $\hat{X} = X_1 \times X_2$, $\hat{\pi} = \pi_1 \times \pi_2$, $\hat{G} \rightarrow \hat{X}$. Let $\hat{a} = (a_1,a_2) \in \hat{G}$, and let $Y$ be the closure of the orbit of $\hat{a}$ in $\hat{X}$, $Y = \{\hat{\pi}(\hat{a}^n), n \in \mathbb{Z}\}$. Let $p_1$ and $p_2$ be the projections of $Y$ to $X_1$ and to $X_2$ respectively. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ be “the fibers” of the projections $p_2$ and $p_1$ respectively; $p_2^{-1}(X_2) = Y_1 \times \{1_{X_2}\}$ and $p_1^{-1}(X_1) = Y_2 \times \{1_{X_1}\}$. For any $n \in \mathbb{Z}$ we have $\varphi(n) = h_1(\pi_1(\hat{a}^n)) = h_2(\pi_2(\hat{a}^n))$; since the orbit $\{\hat{\pi}(\hat{a}^n)\}_{n \in \mathbb{Z}}$ is dense in $Y$, this implies that $h_1 \circ p_1 = h_2 \circ p_2$, so $h_1$ is constant on the fibers $b_1 Y_1, b_1 \in G_1$, of $p_2$ and $h_2$ is constant on the fibers $b_2 Y_2, b_2 \in G_2$, of $p_1$, and therefore we may factorize $X_1$ by $Y_1$ and $X_2$ by $Y_2$ (and $Y$ by $Y_1 \times Y_2$) to get the same result.

Now, assume that for some $\omega \in \Omega$, $\varphi_\omega = \Psi(a_1,h_1) = \Psi(a_2,h_2)$, $a_1,a_2 \in G$ and $h_1,h_2$ are Lipschitz functions on $X$. Let, by Lemma 4.3, $(\hat{X},\hat{a},\hat{h})$ be a common factor of $(X,a_1,h_1)$ and $(X,a_2,h_2)$ such that $\varphi_\omega = \Psi(\hat{a},\hat{h})$. Since $\varphi_\omega$ cannot be read off a nilmanifold $\hat{X}$ with $\dim \hat{X} < \dim X$, there must be $\dim \hat{X} = \dim X$. However, for any pair $(\hat{X},\hat{a})$, “nilmanifold with a translation”, there are only countably many (up to isomorphism) pairs $(X,a)$ extending $(\hat{X},\hat{a})$ and with $\dim X = \dim \hat{X}$.

Thus, we arrive at the following situation: we have a nilmanifold $X = G/\Gamma$ and a measurable function $H : \Omega \rightarrow G \times C(X)$, $\omega \mapsto (a_\omega,h_\omega)$, such that for every $\omega \in \Omega$ one has $\varphi_\omega(n) = h_\omega(\pi(a_\omega^n))$, $n \in \mathbb{Z}$. Let $H(\omega,x) = h_\omega(x)$, $\omega \in \Omega$, $x \in X$; then $\varphi(n) = \int_\Omega H(\omega, \pi(a_\omega^n)) d\nu(\omega)$, $n \in \mathbb{Z}$. Choose a basis $f_1, f_2, \ldots$ in $C(X)$; the function $H$ is representable in the form $H(\omega,x) = \sum_{i = 1}^\infty \theta_i(\omega)f_i(x)$, where convergence is uniform with respect to $x$ for any $\omega$; we are done if we prove the assertion for the functions $\theta_i(\omega)f_i(x)$ instead of $H$. So, let $\theta \in L^1(\Omega)$ and $f \in C(X)$; we have to show that the sequence $\varphi(n) = \int_\Omega \theta(\omega)f(\pi(a_\omega^n)) d\nu(\omega)$ is in $\mathcal{M}$. We may also assume that $\theta \geq 0$. Let $\tau : \Omega \rightarrow G$ be the mapping defined by $\tau(\omega) = a_\omega$, and let $\rho = \tau_*(\theta \nu)$; then $\rho$ is a finite measure on $G$ and $\varphi(n) = \int_G f(\pi(a_\omega^n)) d\rho(a)$. Thus, Proposition 4.2 will follow from the following:
Proposition 4.4. Let \( X = G/\Gamma \) be a nilmanifold, let \( \rho \) be a finite Borel measure on \( G \), and let \( f \in C(X) \). Then the sequence \( \varphi(n) = \int_G f(\pi(a^n)) \, d\rho(a) \), \( n \in \mathbb{Z} \), is in \( \mathcal{M}^\rho \). (If \( X \) is an \( r \)-step nilmanifold, then \( \varphi \in \mathcal{M}^\rho_r \).)

Proof. We may and will assume that \( X \) is connected. Let \( \tilde{\rho} = \pi_\ast(\rho) \); we decompose \( \tilde{\rho} \) in the following way:

Lemma 4.5. There exists an at most countable collection \( \mathcal{V} \) of connected subnilmanifolds of \( X \) (which may include \( X \) itself and singletons) and finite Borel measures \( \rho_V \), \( V \in \mathcal{V} \), on \( G \) such that \( \rho = \sum_{V \in \mathcal{V}} \rho_V \) and for every \( V \in \mathcal{V} \), supp(\( \rho_V \)) \( \subseteq \) \( V \) and \( \tilde{\rho}_V(W) = 0 \) for any proper subnilmanifold \( W \) of \( V \), where \( \tilde{\rho}_V = \pi_\ast(\rho_V) \).

Proof. Let \( \mathcal{V}_0 \) be the (at most countable) set of the singletons \( V = \{ x \} \) in \( X \) (connected 0-dimensional subnilmanifolds of \( X \)) for which \( \tilde{\rho}(V) > 0 \). For each \( V \in \mathcal{V}_0 \) let \( \rho_V \) be the restriction of \( \rho \) to \( \pi^{-1}(V) \) (that is, \( \tilde{\rho}_V(A) = \rho(A \cap \pi^{-1}(V)) \) for measurable subsets \( A \) of \( G \)), and let \( \rho_1 = \rho - \sum_{V \in \mathcal{V}_0} \rho_V \) and \( \tilde{\rho}_1 = \pi_\ast(\rho_1) \). Now let \( \mathcal{V}_1 \) be the (at most countable) set of connected 1-dimensional subnilmanifolds of \( X \) for which \( \tilde{\rho}_1(V) > 0 \), for each \( V \in \mathcal{V}_1 \) let \( \rho_V \) be the restriction of \( \rho_1 \) to \( \pi^{-1}(V) \), and \( \rho_2 = \rho - \sum_{V \in \mathcal{V}_1} \rho_V \), \( \tilde{\rho}_2 = \pi_\ast(\rho_2) \). (Note that for \( V_1, V_2 \in \mathcal{V}_1 \), the subnilmanifold \( V_1 \cap V_2 \), if nonempty, has dimension 0, so \( \tilde{\rho}_1(V_1 \cap V_2) = 0 \).)

And so on, by induction on the dimension of the subnilmanifolds; at the end, we put \( \mathcal{V} = \bigcup_{i=0}^{\dim X} \mathcal{V}_i \).

By Lemma 2.3, it suffices to prove the assertion for each of \( \rho_V \) instead of \( \rho \). So, we will assume that the measure \( \rho \) is supported by a connected subnilmanifold \( V \) of \( X \) and \( \rho(W) = 0 \) for any proper subnilmanifold \( W \) of \( X \).

First, let \( V = X \):

Lemma 4.6. Let \( \rho \) be a finite Borel measure on \( G \) such that for \( \tilde{\rho} = \pi_\ast(\rho) \) one has \( \tilde{\rho}(W) = 0 \) for any proper subnilmanifold \( W \) of \( X \). Then for any \( f \in C(X) \) the sequence \( \varphi(n) = \int_G f(\pi(a^n)) \, d\rho(a) \), \( n \in \mathbb{Z} \), converges to \( \int_X f \, d\mu_X \) in uniform density.

Proof. We may assume that \( \int_X f \, d\mu_X = 0 \); we then have to show that \( \varphi \) is a null-sequence. Let \( (\Phi_N) \) be a Følner sequence in \( \mathbb{Z} \). By the dominated convergence theorem we have

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_G f(\pi(a^n)) \, d\rho(a) \int_G \tilde{f}(\pi(b^n)) \, d\rho(b) \\
= \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{G \times G} f(\pi(a^n)) \tilde{f}(\pi(b^n)) \, d(\rho \times \rho)(a,b) \\
= \int_{G \times G} \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi \times \pi^2)(a^n, b^n) \, d\rho^\times \times^2(a,b) \\
= \int_{G \times G} F(a,b) \, d\rho^\times \times^2(a,b),
\]

where \( \pi \times \pi \), \( \rho \times \rho \), \( \rho \times \rho \), and \( F(a,b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi \times \pi^2)(a^n, b^n) \), \( a, b \in G \). For \( a, b \in G \), if the sequence \( u_n = \pi \times \pi^2(a^n, b^n) \), \( n \in \mathbb{Z} \), is well distributed in \( X \times X \) then \( F(a,b) = \int_{X \times X} f \otimes \tilde{f} \, d\mu_{X \times X} = \int_X f \, d\mu \int_X \tilde{f} \, d\mu = 0 \). So, \( F(a,b) \neq 0 \) only.
if the sequence \((u_n)\) is not well distributed in \(X \times X\), which only happens if the point \(\pi^{x^2}(a, b)\) is contained in a proper subnilmanifold \(D\) of \(X \times X\) with \(1_{X \times X} \in D\). So,

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 \leq \sum_{D \in \mathcal{D}} \int_{(\pi^{x^2})^{-1}(D)} |F(a, b)| \, d\rho^{x^2}(a, b),
\]

where \(\mathcal{D}\) is the (countable) set of proper subnilmanifolds of \(X \times X\) containing \(1_{X \times X}\). Let \(D \in \mathcal{D}\); then either for any \(x \in X\) the fiber \(W'_x = \{y \in X : (x, y) \in D\}\) of \(D\) over \(x\) is a proper subnilmanifold of \(X\), or for any \(y \in X\) the fiber \(W''_y = \{x \in X : (x, y) \in D\}\) of \(D\) over \(y\) is a proper subnilmanifold of \(X\), (or both). Since, by our assumption, \(\tilde{\rho}(W) = 0\) for any proper subnilmanifold \(W\) of \(X\), in either case \(\tilde{\rho}^{x^2}(D) = 0\), so \(\rho^{x^2}((\pi^{x^2})^{-1}(D)) = 0\). Hence, \(\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = 0\), which means that \(\varphi \in \mathcal{Z}\). 

Thus, in this case, \(\varphi\) is a constant plus a null-sequence, that is, \(\varphi \in \mathcal{M}^o\).

Let now \(V\) be of the form \(V = cY\), where \(Y\) is a (proper) connected subnilmanifold of \(X\) with \(1_X \in Y\) and \(c \in G^o\). We may and will assume that the orbit \((c^nY, n \in \mathbb{Z})\) of \(Y\) is dense in \(X\). Let \(Z\) be the normal closure of \(Y\) in \(X\). In this situation the following generalization of Lemma 4.6 does the job:

**Lemma 4.7.** Let \(Z\) be a normal subnilmanifold of \(X\) and let \(c \in G\) be such that \((c^nZ, n \in \mathbb{Z})\) is dense in \(X\). Let \(\rho\) be a finite Borel measure on \(G\) such that for \(\tilde{\rho} = \pi_+^*(\rho)\) one has \(\text{supp}(\tilde{\rho}) \subseteq c\mathbb{Z}\) and \(\tilde{\rho}(cW) = 0\) for any proper normal subnilmanifold \(W\) of \(Z\). Let \(\varphi(n) = \int_G f(\tilde{\rho}(c^n)) \, d\rho(a), n \in \mathbb{Z}\), let \(\tilde{X}\) be the fiber \(X/\mathbb{Z}\), and let \(\hat{f} = E(f|\tilde{X})\). Then \(\varphi - \hat{f}(\pi(c^n)) \in \mathcal{Z}\).

**Proof.** After replacing \(f\) by \(f - \hat{f}\) we will assume that \(E(f|\tilde{X}) = 0\); we then have to prove that \(\varphi\) is a null-sequence. Let \(L = \pi^{-1}(Z)\). Let \((\Phi_N)\) be a Følner sequence in \(\mathbb{Z}\); as in Lemma 4.6, we obtain

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \int_{G \times G} F(a, b) \, d\rho^x(a, b) = \int_{(cL) \times (cL)} F(a, b) \, d\rho^x(a, b),
\]

where \(F(a, b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \hat{f}(\pi^{x^2}(a^n, b^n)), a, b \in L\). Let us “shift” \(\rho\) to the origin, by replacing it by \(c_*^{-1} \rho(a), a \in G\), so that now \(\text{supp}(\rho) \subseteq L, \text{supp}(\tilde{\rho}) \subseteq \mathbb{Z}\), and

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \int_{L \times L} F(a, b) \, d\rho^x(a, b),
\]

where \(F(a, b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \hat{f}(\pi^{x^2}((ca)^n, (cb)^n)), a, b \in L\).

For \(a, b \in L\), the sequence \(u_n = \pi^{x^2}((ca)^n, (cb)^n), n \in \mathbb{Z}\), is contained in \(X \times \tilde{X}\). If this sequence is well distributed in \(X \times \tilde{X}\), then \(F(a, b) = \int_{X \times \tilde{X}} f \otimes f \, d\mu_{X \times \tilde{X}} = \int_{\tilde{X}} E(f|\tilde{X})E(f|\tilde{X}) \, d\mu_{\tilde{X}} = 0\). So, \(F(a, b) \neq 0\) only if the sequence \((u_n)\) is not well distributed in \(X \times \tilde{X}\), which only happens if the image \((\tilde{u}_n)\) of \((u_n)\) is not well distributed in the nil-maximal factor-torus \(T\) of \(X \times \tilde{X}\). Using additive notation on \(T\) we have \(\tilde{u}_n = n\tilde{c} + n\tilde{a} + n\tilde{b}, n \in \mathbb{Z}\), where \(T\)
contains the direct sum $S \oplus S$ of two copies of a torus $S$, \( \tilde{a} \in S \oplus \{0\} \), \( \tilde{b} \in \{0\} \oplus S \), and the sequence \((n \tilde{c})\) is dense in the factor-torus $T/(S \oplus S)$. The sequence \((\tilde{a}_n)\) is not dense in $T$ only if the point \((\tilde{a}, \tilde{b})\) is contained in a proper subtorus $R$ of $S \oplus S$, and either for each $\tilde{x} \in S$ the fiber \( \{ \tilde{y} \in S : (\tilde{x}, \tilde{y}) \in R \} \) is a proper subtorus of $S$, or for each $\tilde{y} \in S$ the fiber \( \{ \tilde{x} \in S : (\tilde{x}, \tilde{y}) \in R \} \) is a proper subtorus of $S$ (or both). Without loss of generality, assume that the first possibility holds. Then, returning back to $X \times \hat{X}$, we obtain that the sequence \( (u_n) \) is not well distributed in this space only if the point $\pi^2(a, b)$ is contained in a subnilmanifold $D$ (the preimage of the torus $R$) in $Z \times Z$ with $D \ni 1_{X \times \hat{X}}$ such that for every $x \in Z$ the fiber $W_x = \{ y \in Z : (x, y) \in D \}$ is a proper normal subnilmanifold of $Z$. Since, by our assumption, $\tilde{\rho}(W_x) = 0$ for all $x$, we have $\tilde{\rho}^2(D) = 0$, so $\rho^2((\pi^2)^{-1}(D)) = 0$. The function $F(a, b)$ may only be nonzero on the union of a countable collection of the subnilmanifolds $D$ like this, so $\int_{L \times L} F(a, b) d\rho^2(a, b) = 0$. Hence, $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = 0$, which means that $\varphi \in Z$.

Since, in the notation of Lemma 4.7, the sequence $\hat{f}(\pi(e^n)) = \hat{f}(e^n 1_X)$, $n \in \mathbb{Z}$, is a basic nilsequence, $\varphi$ is a sum of a nilsequence and a null-sequence, so $\varphi \in M^o$ in this case as well.

The proof of Proposition 4.2 in the case $d \geq 2$ is not much harder than in the case $d = 1$, and we will only sketch it. For each $\omega \in \Omega$, instead of a single element $a_\omega \in G_\omega$ we now have $d$ commuting elements $a_{\omega, 1}, \ldots, a_{\omega, d} \in G_\omega$. After passing to a single nilmanifold $X = G/\Gamma$, we obtain $d$ mappings $\tau_i: \Omega \to G$, $\omega \mapsto a_{\omega, i}$, $i = 1, \ldots, d$, and so, the mapping $\tau = (\tau_1, \ldots, \tau_d): \Omega \to G^d$. We define a measure $\rho$ on $G^d$ by $\rho = \tau_* (\theta \nu)$; then Proposition 4.2 follows from the following modification of Proposition 4.4:

**Proposition 4.8.** Let $X = G/\Gamma$ be a nilmanifold, let $\rho$ be a finite Borel measure on $G^d$, and let $f \in C(X)$. Then the sequence $\varphi(n_1, \ldots, n_d) = \int_{G^d} f(\pi(a_1^{n_1} \ldots a_d^{n_d})) d\rho(a_1, \ldots, a_d)$, $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, is in $M^o$. (If $X$ is an $r$-step nilmanifold, then $\varphi \in M^o_{r,v}$.)

The proof of this proposition is the same as of Proposition 4.4, with $a^n$ replaced by $a_1^{n_1} \ldots a_d^{n_d}$, and the mapping $G \to X$, $a \mapsto \pi(a)$, replaced by the mapping $G^d \to X$, $(a_1, \ldots, a_d) \mapsto \pi(a_1 \ldots a_d)$.

Uniting Proposition 4.2 with Lemma 4.1, we obtain:

**Theorem 4.9.** Let $(\Omega, \nu)$ be a measure space and let $\Omega \to M$, $\omega \mapsto \varphi_\omega$, be an integrable mapping. Then the sequence $\varphi(n) = \int_{\Omega} \varphi_\omega(n) d\nu$ is in $M$ as well. If, for some $r$, $\varphi_\omega \in M_r$ for all $\omega$, then $\varphi \in M^o_r$.

5. Multiple polynomial correlation sequences and nilsequences

Now let $(W, \mathcal{B}, \mu)$ be a probability measure space and let $T$ be an ergodic invertible measure preserving transformation of $W$. Let $p_1, \ldots, p_k$ be polynomials $\mathbb{Z}^d \to \mathbb{Z}$. Let $A_1, \ldots, A_k \in \mathcal{B}$ and let $\varphi(n) = \mu(T^{p_1(n)} A_1 \cap \ldots \cap T^{p_k(n)} A_k)$, $n \in \mathbb{Z}^d$; or, more generally, let $f_1, \ldots, f_k \in L^\infty(W)$ and $\varphi(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k d\mu$, $n \in \mathbb{Z}^d$. Then, given $\varepsilon > 0$, there exist an $r$-step nilsystem $(X, a)$, $X = G/\Gamma$, $a \in G$, and functions $\tilde{f}_1, \ldots, \tilde{f}_k \in L^\infty(X)$
such that, for $\phi(n) = \int_X a^{p_1(n)} \tilde{f}_1 \cdots a^{p_k(n)} \tilde{f}_k d\mu_X$, $n \in \mathbb{Z}^d$, the set \( \{ n \in \mathbb{Z}^d : |\phi(n) - \varphi(n)| > \varepsilon \} \) has zero uniform density. Moreover, there is a universal integer $r$ that works for all systems $(W, B, \mu, T)$, functions $h_i$, and $\varepsilon$, and depends only on the polynomials $p_i$; for the minimal such $r$, the integer $c = r - 1$ is called the complexity of the system \( \{p_1, \ldots, p_k\} \) (see [L5]).

(Here is a sketch of the proof, for completeness; for more details see [HK] and [BHK]. By [L3], there exists $c \in \mathbb{N}$, which only depends on the polynomials $p_i$, such that, if $(V, \nu, S)$ is an ergodic probability measure preserving system and $Z_c(V)$ is the $c$-th Host-Kra-Ziegler factor of $V$ and $h_1, \ldots, h_k \in L^\infty(V)$ are such that $E(h_i | Z_c(V)) = 0$ for some $i$, then for any Følner sequence $(\Phi_N)$ in $\mathbb{Z}^d$ one has $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_V S^{p_1(n)} h_1 \cdots S^{p_k(n)} h_k d\nu = 0$.

Applying this to the ergodic components of the system $(W \times W, \mu \times \mu, T \times T)$ and the functions $h_i = f_i \otimes \tilde{f}_i$, $i = 1, \ldots, k$, we obtain that for any Følner sequence $(\Phi_N)$ in $\mathbb{Z}^d$,

$$\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \left| \int_{W \times W} T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu \right|^2 = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{W \times W} T^{p_1(n)} f_1(x) \cdots T^{p_k(n)} f_k(y) \cdots \cdot T^{p_1(n)} f_k(x) \cdot T^{p_k(n)} f_k(y) d(\mu(x) \times \mu(y)) = 0$$

whenever, for some $i$, the function $f_i \otimes \tilde{f}_i$ has zero conditional expectation with respect to almost all ergodic components of $Z_c(W \times W)$. This is so if $E(f_i | Z_{c+1}(W)) = 0$, and we obtain that the sequence $\int_W T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu$ tends to zero in uniform density whenever $E(f_i | Z_r(W)) = 0$ for some $i$, where $r = c + 1$. It follows that for any $f_1, \ldots, f_k \in L^\infty(W)$ the sequence

$$\int_W T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu - \int_{Z_r(W)} T^{p_1(n)} E(f_1 | Z_r(W)) \cdots T^{p_k(n)} E(f_k | Z_r(W)) d\mu_{Z_r(W)}$$

tends to zero in uniform density. Now, $Z_r(W)$ has the structure of the inverse limit of a sequence of $r$-step nilmanifolds on which $T$ acts as a translation; given $\varepsilon > 0$, we can therefore find an $r$-step nilmanifold factor $X$ of $W$ such that $\|E(f_i | Z_r(W)) - E(f_i | X)\|_{L^\infty(W)} < \varepsilon / \prod_{j=1}^k \|f_j\|_{L^\infty(W)}$ for all $i$. Putting $\tilde{f}_i = E(f_i | X)$, $i = 1, \ldots, k$, and denoting the translation induced by $T$ on $X$ by $a$, we then have

$$\left| \int_{Z_r(W)} T^{p_1(n)} E(f_1 | Z_r(W)) \cdots T^{p_k(n)} E(f_k | Z_r(W)) d\mu_{Z_r(W)} \right| - \int_X a^{p_1(n)} \tilde{f}_1 \cdots a^{p_k(n)} \tilde{f}_k d\mu_X < \varepsilon$$

for all $n$, which implies the assertion.)

So, there exists $\lambda \in \mathcal{Z}$ such that $\|\varphi - (\psi + \lambda)\| < \varepsilon$. After replacing $\tilde{f}_i$ by $L^1$-close continuous functions, we may assume that $\tilde{f}_1, \ldots, \tilde{f}_k \in C(X)$, and still $\|\varphi - (\psi + \lambda)\| < \varepsilon$. Applying Theorem 3.3 to the nilmanifold $X^k = G^k / \Gamma^k$, the diagonal subnilmanifold $Y = \{ (x, \ldots, x), x \in X \} \subseteq X^k$, the polynomial sequence $g(n) = (a^{p_1(n)}), a^{p_k(n)}$, $n \in \mathbb{Z}^d$, in $G^k$, and the function $f(x_1, \ldots, x_k) = \tilde{f}_1(x_1) \cdots \tilde{f}_k(x_k) \in C(X^k)$, we obtain that $\psi \in \mathcal{M}^\alpha(r, s)$, so also $\psi + \lambda \in \mathcal{M}^\alpha(r, s)$. Since $\varepsilon$ is arbitrary and, by Lemma 2.3, $\mathcal{M}_{r,s}$ is the closure of $\mathcal{M}^\alpha_{r,s}$, we obtain:
Proposition 5.1. Let \((W, \mathcal{B}, \mu, T)\) be an ergodic invertible probability measure preserving system, let \(f_1, \ldots, f_k \in L^\infty(W)\), and let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \to \mathbb{Z}\). Then the sequence \(\varphi(n) = \int_W T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu, n \in \mathbb{Z}^d\), is in \(\mathcal{M}\). If the complexity of the system \(\{p_1, \ldots, p_k\}\) is \(c\) and \(\deg p_i \leq s\) for all \(i\), then \(\varphi_n \in \mathcal{M}_{c+1,s}\).

Let now \((W, \mathcal{B}, \mu, T)\) be a non-ergodic (or, rather, not necessarily ergodic) system. Let \(\mu = \int_\Omega \mu_\omega d\nu(\omega)\) be the ergodic decomposition of \(\mu\). For each \(\omega \in \Omega\), let \(\varphi_\omega(n) = \int_W T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu_\omega, n \in \mathbb{Z}^d\); then \(\omega \mapsto \varphi_\omega\) is a measurable mapping \(\Omega \to l^\infty\), and \(\varphi(n) = \int_\Omega \varphi_\omega(n) d\nu(\omega), n \in \mathbb{Z}^d\). By Proposition 5.1, for each \(\omega \in \Omega\) we have \(\varphi_\omega \in \mathcal{M}_{c+1,s}\subseteq \mathcal{M}_l\), where \(l = 2(c + 1)s\). By Theorem 4.9 we obtain:

Theorem 5.2. Let \((W, \mathcal{B}, \mu, T)\) be an invertible probability measure preserving system, let \(f_1, \ldots, f_k \in L^\infty(W)\), and let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \to \mathbb{Z}\). Then the sequence \(\varphi(n) = \int_W T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k d\mu, n \in \mathbb{Z}^d\), is in \(\mathcal{M}\). If the complexity of the system \(\{p_1, \ldots, p_k\}\) is \(c\) and \(\deg p_i \leq s\) for all \(i\), then \(\varphi_n \in \mathcal{M}_l\), where \(l = 2(c + 1)s\).

Since \(\mathcal{M} \subseteq \mathcal{P} + \mathcal{Z}\), where \(\mathcal{P}\) is the closure in \(l^\infty\) of the algebra of bounded generalized polynomials (see the last paragraph of Section 2), we get as a corollary:

Corollary 5.3. Up to a null-sequence, the sequence \(\varphi\) is uniformly approximable by generalized polynomials.

Bibliography


