Nilsequences, null-sequences, and multiple correlation sequences

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

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Abstract

A \((d\text{-parameter})\) basic nilsequence is a sequence of the form \(\psi(n) = f(a^n x), n \in \mathbb{Z}^d\), where \(x\) is a point of a compact nilmanifold \(X\), \(a\) is a translation on \(X\), and \(f \in C(X)\); a nilsequence is a uniform limit of basic nilsequences. If \(X = G/\Gamma\) be a compact nilmanifold, \(Y\) is a subnilmanifold of \(X\), \((g(n))_{n \in \mathbb{Z}^d}\) is a polynomial sequence in \(G\), and \(f \in C(X)\), we show that the sequence \(\varphi(n) = \int_{g(n)Y} f\) is the sum of a basic nilsequence and a sequence that converges to zero in uniform density (a null-sequence). We also show that an integral of a family of nilsequences is a nilsequence plus a null-sequence. We deduce that for any invertible finite measure preserving system \((W, \mathcal{B}, \mu, T)\), polynomials \(p_1, \ldots, p_k: \mathbb{Z}^d \to \mathbb{Z}\), and sets \(A_1, \ldots, A_k \in \mathcal{B}\), the sequence \(\varphi(n) = \mu(T^{p_1(n)}A_1 \cap \ldots \cap T^{p_k(n)}A_k), n \in \mathbb{Z}^d\), is the sum of a nilsequence and a null-sequence.

0. Introduction

Throughout the whole paper we will deal with “multiparameter sequences”, – we fix \(d \in \mathbb{N}\) and under “a sequence” will usually understand “a two-sided \(d\text{-parameter sequence}\), that is, a mapping with domain \(\mathbb{Z}^d\).

A (compact) \(r\text{-step}\) nilmanifold \(X\) is a factor space \(G/\Gamma\), where \(G\) is an \(r\text{-step}\) nilpotent (not necessarily connected) Lie group and \(\Gamma\) is a discrete co-compact subgroup of \(G\). Elements of \(G\) act on \(X\) by translations; an \((r\text{-step})\) nilsystem is an \((r\text{-step})\) nilmanifold \(X = G/\Gamma\) with a translation \(a \in G\) on it.

A basic \(r\text{-step}\) nilsequence is a sequence of the form \(\psi(n) = f(\eta(n)x), n \in \mathbb{Z}^d\), where \(x\) is a point of an \(r\text{-step}\) nilmanifold \(X = G/\Gamma\), \(\eta\) is a homomorphism \(\mathbb{Z}^d \to G\), and \(f \in C(X)\); an \(r\text{-step}\) nilsequence is a uniform limit of basic \(r\text{-step}\) nilsequences. The algebra of nilsequences is a natural generalization of Weyl’s algebra of almost periodic sequences, which are just 1-step nilsequences. An “inner” characterization of nilsequences, in terms of their properties, is obtained in [HKM]; see also [HK2].

The term “nilsequence” was introduced in [BHK], where it was proved that for any ergodic finite measure preserving system \((W, \mathcal{B}, \mu, T)\), positive integer \(k\), and sets

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Let $(W, \mathcal{B}, \mu, T)$ be an invertible measure preserving system with $\mu(W) < \infty$, let $p_1, \ldots, p_k$ be polynomials $\mathbb{Z}^d \to \mathbb{Z}$, and let $A_1, \ldots, A_k \in \mathcal{B}$. Then the “multiple polynomial correlation sequence” $\varphi(n) = \mu(T^{p_1(n)}A_1 \cap \ldots \cap T^{p_k(n)}A_k)$, $n \in \mathbb{Z}^d$, is a sum of a nilsequence and a null-sequence.

(In [L6] this theorem was proved in the case $d = 1$ and ergodic $T$.)

Based on the theory of nil-factors developed in [HK1] and, independently, in [Z], it is shown in [L3] that nilsystems are characteristic for multiple polynomial correlation sequences induced by ergodic systems, in the sense that, up to a nil-sequence and an arbitrarily small sequence, any such correlation sequence comes from a nilsystem. This reduces the problem of studying “ergodic” multiple polynomial correlation sequences to nilsystems.

Let $X = G/\Gamma$ be a connected nilmanifold, let $Y$ be a connected subnilmanifold of $X$, and let $g$ be a polynomial sequence in $G$, that is, a mapping $\mathbb{Z}^d \to G$ of the form $g(n) = a_1^{p_1(n)} \ldots a_r^{p_r(n)}$, $n \in \mathbb{Z}^d$, where $a_1, \ldots, a_r \in G$ and $p_1, \ldots, p_r$ are polynomials $\mathbb{Z}^d \to \mathbb{Z}$. We investigate (in Section 3) the behavior of the sequence $g(n)Y$ of subnilmanifolds of $X$: we show that there is a subnilmanifold $Z$ of $X$, containing $Y$, such that the sequence $g(n)$ only shifts $Z$ along $X$, without distorting it, whereas, outside of a null-set of $n \in \mathbb{Z}^d$, $g(n)Y$ becomes more and more “dense” in $g(n)Z$:

Proposition 0.2. Assume (as we can) that the orbit $g(n)Y$, $n \in \mathbb{Z}^d$, is dense in $X$, and let $Z$ be the normal closure (in the algebraic sense; see below) of $Y$ in $X$. Then for any $f \in C(X)$, the sequence $\lambda(n) = \int_{g(n)Y} f - \int_{g(n)Z} f$, $n \in \mathbb{Z}^d$, is a null-sequence.

We have $\int_{g(n)Z} f = g(n) \hat{f}(g(n)e)$, $n \in \mathbb{Z}^d$, where $\hat{f} = E(f|X/Z)$ and $e = Z/Z \in X/Z$. (Here and below, $E(f|X')$ stands for the conditional expectation of a function $f \in L^1(X)$ with respect to a factor $X'$ of $X$.) So, the sequence $\int_{g(n)Z} f$ is a basic nilsequence, and we obtain:

Theorem 0.3. For any $f \in C(X)$ the sequence $\varphi(n) = \int_{g(n)Y} f$, $n \in \mathbb{Z}^d$, is the sum of a basic nilsequence and a null-sequence.

Applying this result to the diagonal $Y$ of the power $X^k$ of the nilmanifold $X$, the polynomial sequence $g(n) = (a_1^{p_1(n)}, \ldots, a_r^{p_r(n)})$, $n \in \mathbb{Z}^d$, in $G^k$, and the function $f = 1_{A_1} \otimes \ldots \otimes 1_{A_k}$, we obtain Theorem 0.1 in the ergodic case.

Our next step (Section 4) is to extend this result to the case of a non-ergodic $T$. Using the ergodic decomposition $W \to \Omega$ of $T$ we obtain a measurable mapping from $\Omega$ to the space of nilsequences–plus–null-sequences, which we then have to integrate over $\Omega$. The integral of a family of null-sequences is a null-sequence, and creates no trouble. As for nilsequences, when we integrate them we arrive at the following problem: if $X = G/\Gamma$ is a nilmanifold, with $\pi: G \to X$ being the factor mapping, and $\rho(a)$, $a \in G$, is a finite Borel
measure on $G$, what is the limiting behavior of the measures $\pi_*(\rho(a^n))$ on $X$? (This is the question corresponding to the case $d = 1$; for $d \geq 2$ it is slightly more complicated.) We show that this sequence of measures tends to a linear combination of Haar measures on (countably many) subnilmanifolds of $X$, which are normal (and so travel, without distortion) in the closure of their orbits, and we again obtain:

**Proposition 0.4.** For any $f \in C(X)$, the sequence $\varphi(n) = \int_G f(\pi(a^n)) \, d\rho(a), \, n \in \mathbb{Z}$, is a sum of a basic nilsequence and a null-sequence.

(This proposition is a “nilpotent” extension of the following classical fact: if $\rho$ is a finite Borel measure on the 1-dimensional torus $\mathbb{T}$, then its Fourier transform $\varphi(n) = \int_\mathbb{T} e^{-2\pi i n x} \, d\rho(x)$ is the sum of an almost periodic sequence (a 1-step nilsequence; it corresponds to the atomic part of $\rho$) and a null-sequence (that corresponds to the non-atomic part of $\rho$).)

As a corollary, we obtain the remaining ingredient of the proof of Theorem 0.1:

**Theorem 0.5.** Let $\Omega$ be a measure space and let $\varphi_\omega, \, \omega \in \Omega$, be an integrable family of nilsequences; then the sequence $\varphi(n) = \int_\Omega \varphi_\omega(n)$ is a sum of a nilsequence and a null-sequence.

Let us also mention generalized (or bracket) polynomials, – the functions constructed from ordinary polynomials using the operations of addition, multiplication, and taking the integer part, \([\cdot]\). (For example, $p_1[p_2[p_3] + p_4]$, where $p_i$ are ordinary polynomials, is a generalized polynomial.) Generalized polynomials (gps) appear quite often (for example, the fractional part, and the distance to the nearest integer, of an ordinary polynomial are gps); they were systematically studied in [Hå1], [Hå2], [BL], and [L7]. Because of their simple definition, gps are nice objects to deal with. On the other hand, similarly to nilsequences, gps come from nilsystems: bounded gps (on $\mathbb{Z}^d$, in our case) are exactly the sequences of the form $h(g(n)x), \, n \in \mathbb{Z}^d$, where $h$ is a piecewise polynomial function on a nilmanifold $X = G/\Gamma, \, x \in X$, and $g$ is a polynomial sequence in $G$ (see [BL] or [L7]). Since any continuous function is uniformly approximable by piecewise polynomial functions (this follows by an application of the Weierstrass theorem in the fundamental domain of $X$), nilsequences are uniformly approximable by generalized polynomials. We obtain as a corollary that any multiple polynomial correlation sequence is, up to a null-sequence, uniformly approximable by generalized polynomials.

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### 1. Nilmanifolds

In this section we collect the facts about nilmanifolds that we will need below; details and proofs can be found in [M], [L1], [L2], [L4].

Throughout the paper, $X = G/\Gamma$ will be a compact nilmanifold, where $G$ is a nilpotent Lie group and $\Gamma$ is a discrete subgroup of $G$, and $\pi$ will denote the factor mapping $G \longrightarrow X$. By $1_X$ we will denote the point $\pi(1_G)$ of $X$. By $\mu_X$ we will denote the normalized Haar measure on $X$.

By $G^c$ we will denote the identity component of $G$. Note that if $G$ is disconnected,
X can be interpreted as a nilmanifold, \( X = G'/\Gamma' \), in different ways: if, for example, X is connected, then also \( X = G^c/(\Gamma \cap G^c) \). If \( X \) is connected and we study the action on \( X \) of a sequence \( g(n), g: \mathbb{Z}^d \rightarrow G \), we may always assume that \( G \) is generated by \( G^c \) and the range \( g(\mathbb{Z}^d) \) of \( g \). Thus, we may (and will) assume that the group \( G/G^c \) is finitely generated.

Every nilpotent Lie group \( G \) is a factor of a torsion free nilpotent Lie group. (As such, a suitable “free nilpotent Lie group” \( F \) can be taken. If \( G^c \) has \( k_1 \) generators, \( G/G^c \) has \( k_2 \) generators, and \( G \) is \( r \)-step nilpotent, then \( F = F_1 \cdots F_r \), where \( F_r \) is the free product of \( k_1 \) copies of \( \mathbb{R} \) and \( k_2 \) copies of \( \mathbb{Z} \), and \( F_r \) is the \((r+1)\)st term of the lower central series of \( F \).) Thus, \( G \) may always be assumed to be torsion-free. The identity component \( G^c \) of \( G \) is then an exponential Lie group, and for every element \( a \in G^c \) there exists a (unique) one-parametric subgroup \( a^t \) such that \( a^1 = a \).

If \( G \) is torsion free, it possesses a Malcev basis compatible with \( \Gamma \), which is a finite set \( \{e_1, \ldots, e_k\} \) of elements of \( \Gamma \), with \( e_1, \ldots, e_{k_1} \in G^c \) and \( e_{k_1 + 1}, \ldots, e_k \notin G^c \), such that every element \( a \in G \) can be uniquely written in the form \( a = e_{k_1}^{u_1} \cdots e_k^{u_k} \) with \( u_1, \ldots, u_k \in \mathbb{R} \). Thus, \( G \) may always be assumed that the group \( G/G^c \) is torsion-free. The identity component \( G^c \) of \( G \) is then an exponential Lie group, and for every element \( a \in G^c \) there exists a (unique) one-parametric subgroup \( a^t \) such that \( a^1 = a \).

The multiplication in \( G \) is defined by the (finite) multiplication table for the Malcev basis of \( G \), whose entries are integers; it follows that there are only countably many non-isomorphic nilpotent Lie groups with cocompact discrete subgroups, and countably many non-isomorphic compact nilmanifolds.

Let \( X \) be connected. Then, under the identification \( G^c \leftrightarrow \mathbb{R}^{k_1} \), the cube \([0,1]^{k_1}\) is the fundamental domain of \( X \). We will call the closed cube \( Q = [0,1]^{k_1} \) the fundamental cube of \( X \) in \( G^c \) and sometimes identify \( X \) with \( Q \). When \( X \) is identified with its fundamental cube \( Q \), the measure \( \mu_X \) corresponds to the standard Lebesgue measure \( \mu_Q \) on \( Q \).

In Malcev coordinates, multiplication in \( G \) is a polynomial operation: there are polynomials \( q_1, \ldots, q_k \) in \( 2k \) variables with rational coefficients such that for \( a = e_1^{u_1} \cdots e_k^{u_k} \) and \( b = e_1^{v_1} \cdots e_k^{v_k} \) we have \( ab = e_1^{q_1(u_1,v_1,\ldots,u_k,v_k)} \cdots e_k^{q_k(u_1,v_1,\ldots,u_k,v_k)} \). This implies that “life is polynomial” in nilpotent Lie groups: in coordinates, homomorphisms between these groups are polynomial mappings, and connected closed subgroups of such groups are images of polynomial mappings and are defined by systems of polynomial equations.

A subnilmanifold \( Y \) of \( X \) is a closed subset of the form \( Y = Hx \), where \( H \) is a closed subgroup of \( G \) and \( x \in X \). For a closed subgroup \( H \) of \( G \), the set \( \pi(H) = H1_X \) is closed (and so, is a subnilmanifold) iff the subgroup \( \Gamma \cap H \) is co-compact in \( H \); we will call the subgroup \( H \) with this property rational. Any subnilmanifold \( Y \) of \( X \) has the form \( \pi(aH) = a\pi(H) \), where \( H \) is a closed rational subgroup of \( G \).

If \( Y \) is a subnilmanifold of \( X \) with \( 1_X \in Y \), then \( H = \pi^{-1}(Y) \) is a closed subgroup of \( G \), and \( Y = \pi(H) = H1_X \). \( H \), however, does not have to be the minimal subgroup with this property: if \( Y \) is connected, then the identity component \( H^c \) of \( H \) also satisfies \( \pi(H^c) = Y \).

The intersection of two subnilmanifolds is a subnilmanifold (if nonempty).

Given a subnilmanifold \( Y \) of \( X \), by \( \mu_Y \) we will denote the normalized Haar measure on \( Y \). Translations of subnilmanifolds are measure preserving: we have \( a_*\mu_Y = \mu_{aY} \) for all
Let $Z$ be a subnilmanifold of $X$, $Z = Lx$, where $L$ is a closed subgroup of $G$. We say that $Z$ is normal if $L$ is normal. In this case the nilmanifold $\hat{X} = X/Z = G/(LG)$ is defined, and $X$ splits into a disjoint union of fibers of the factor mapping $X \rightarrow \hat{X}$. (Note that if $L$ is normal in $G^c$ only, then the factor $X/Z = G^c/(LG)$ is also defined, but the elements of $G \setminus G^c$ do not act on it.)

One can show that a subgroup $L$ is normal iff $\gamma L \gamma^{-1} = L$ for all $\gamma \in \Gamma$; hence, $Z = \pi(L)$ is normal iff $\gamma Z = Z$ for all $\gamma \in \Gamma$.

If $H$ is a closed rational subgroup of $G$ then its normal closure $L$ (the minimal normal subgroup of $G$ containing $H$) is also closed and rational, thus $Z = \pi(L)$ is a subnilmanifold of $X$. We will call $Z$ the normal closure of the subnilmanifold $Y = \pi(H)$. If $L$ is normal then the identity component of $L$ is also normal; this implies that the normal closure of a connected subnilmanifold is connected.

If $X$ is connected, the maximal factor-torus of $X$ is the torus $[G^c, G^c]\setminus X$, and the nil-maximal factor-torus is $[G, G]\setminus X$. The nil-maximal factor-torus is a factor of the maximal one.

If $\eta: \mathbb{Z}^d \rightarrow G$ is a homomorphism, then for any point $x \in X$ the closure of the orbit $\eta(\mathbb{Z}^d)x$ of $x$ in $X$ is a subnilmanifold $V$ of $X$ (not necessarily connected), and the sequence $\eta(n)x$, $n \in \mathbb{Z}^d$, is well distributed in $V$. (This means that for any function $f \in C(V)$ and any Følner sequence $(\Phi_N)$ in $\mathbb{Z}^d$, \(\lim_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(\eta(n)x) = \int_V f \, d\mu_V\).) If $X$ is connected, the sequence $\eta(n)x$, $n \in \mathbb{Z}^d$, is dense, and so, well distributed in $X$ iff the image of this sequence is dense in the nil-maximal factor-torus of $X$. All the same is true for the orbit of any subnilmanifold $Y$ of $X$: the closure of $\bigcup_{n \in \mathbb{Z}^d} \eta(n)Y$ is a subnilmanifold $W$ of $X$; the sequence $\eta(n)Y$, $n \in \mathbb{Z}^d$, is well distributed in $W$ (this means that for any function $f \in C(W)$ and any Følner sequence $(\Phi_N)$ in $\mathbb{Z}^d$, \(\lim_{N \rightarrow \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{\eta(n)Y} f(x) \, d\mu_{\eta(n)}Y = \int_{Y} f \, d\mu_V\)); and, in the case $X$ is connected, the sequence $\eta(n)Y$ is well distributed in $X$ iff its image is dense in the nil-maximal factor-torus of $X$.

A polynomial sequence in $G$ is a sequence of the form $g(n) = a_1^{p_1(n)} \cdots a_k^{p_k(n)}$, $n \in \mathbb{Z}^d$, where $a_1, \ldots, a_k \in G$ and $p_1, \ldots, p_k$ are polynomials $\mathbb{Z}^d \rightarrow \mathbb{Z}$. Let $g$ be a polynomial sequence in $G$ and let $x \in X$. Then the closure $V$ of the orbit $g(\mathbb{Z}^d)x$ is a finite disjoint union of connected subnilmanifolds of $X$, and $g(n)x$ visits these subnilmanifolds periodically: there exists $l \in \mathbb{N}$ such that for any $i \in \mathbb{Z}^d$, all the elements $g(lm + i)x$, $m \in \mathbb{Z}^d$, belong to the same connected component of $V$. If $V$ is connected, then the sequence $g(n)x$, $n \in \mathbb{Z}^d$, is well distributed in $V$. In the case $X$ is connected, the sequence $g(n)x$, $n \in \mathbb{Z}^d$, is dense, and so, well distributed in $X$ iff the image of this sequence is dense in the maximal factor-torus of $X$. All the same is true for the orbit $g(\mathbb{Z}^d)Y$ of any connected subnilmanifold $Y$ of $X$ under the action of $g$: its closure $W$ is a finite disjoint union of connected subnilmanifolds of $X$, visited periodically; if $W$ is connected, then the sequence $g(n)Y$, $n \in \mathbb{Z}^d$, is well distributed in $W$; and, if $X$ is connected, the sequence $g(n)Y$ is well distributed in $X$ iff its image is dense in the maximal factor-torus of $X$.

The following proposition, which is a corollary (of a special case) of the result obtained in [GT], says that “almost every” subnilmanifold of $X$ is distributed in $X$ “quite uniformly”. (See Appendix in [L6] for details.)
Proposition 1.1. Let $X$ be connected. For any $C > 0$ and any $\varepsilon > 0$ there are finitely many subnilmanifolds $V_1, \ldots, V_r$ of $X$, connected and containing $1_X$, such that for any connected subnilmanifold $Y$ of $X$ with $1_X \in Y$, if $Y \not\subset V_i$ for all $i \in \{1, \ldots, r\}$, then $|\int_Y f \, d\mu_Y - \int_X f \, d\mu_X| < \varepsilon$ for all functions $f$ on $X$ with $\sup_{x \not= y} |f(x) - f(y)|/\text{dist}(x,y) \leq C$.

(This is in complete analogy with the situation on tori: if $X$ is a torus, for any $\varepsilon > 0$ there are only finitely many subtori $V_1, \ldots, V_r$ such that any subtorus $Y$ of $X$ that contains $0$ and is not contained in $\bigcup_{i=1}^r V_i$ is $\varepsilon$-dense and “$\varepsilon$-uniformly distributed” in $X$.)

2. Nilsequences, null-sequences, and generalized polynomials

We will deal with the space $l^\infty = l^\infty(\mathbb{Z}^d)$ of bounded sequences $\varphi : \mathbb{Z}^d \to \mathbb{C}$, with the norm $\|\varphi\| = \sup_{n \in \mathbb{Z}^d} |\varphi(n)|$.

A basic $r$-step nilsequence is an element of $l^\infty$ of the form $\psi(n) = f(\eta(n)x)$, $n \in \mathbb{Z}^d$, where $x$ is a point of an $r$-step nilmanifold $X = G/\Gamma$, $\eta$ is a homomorphism $\mathbb{Z}^d \to G$, and $f \in C(X)$. We will denote the algebra of basic $r$-step nilsequences by $\mathcal{N}^\circ_r$, and the algebra $\bigcup_{r \in \mathbb{N}} \mathcal{N}^\circ_{r}$ of all basic nilsequences by $\mathcal{N}^\circ$. We will denote the closure of $\mathcal{N}^\circ_r$, $r \in \mathbb{N}$, in $l^\infty$ by $\mathcal{N}^\circ_r$, and the closure of $\mathcal{N}^\circ$ by $\mathcal{N}^\circ$; the elements of these algebras will be called $r$-step nilsequences and, respectively, nilsequences.

Given a polynomial sequence $g(n) = a_{1}^{p_1(n)} \cdots a_{k}^{p_k(n)}$, $n \in \mathbb{Z}^d$, in a nilpotent group with $\text{deg} p_i \leq s$ for all $i$, we will say that $g$ has naive degree $\leq s$. (The term “degree” was already reserved for another parameter of a polynomial sequence.) We will call a sequence of the form $\psi(n) = f(g(n)x)$, where $x$ is a point of an $r$-step nilmanifold $X = G/\Gamma$, $g$ is a polynomial sequence of naive degree $\leq s$ in $G$, and $f \in C(X)$, a basic polynomial $r$-step nilsequence of degree $\leq s$. We will denote the algebra of basic polynomial $r$-step nilsequences of degree $\leq s$ by $\mathcal{N}^\circ_{r,s}$ and the closure of this algebra in $l^\infty$ by $\mathcal{N}^\circ_{r,s}$. It is shown in [L2] (see proof of Theorem B*) that any basic polynomial $r$-step nilsequence of degree $\leq s$ is a basic l-step nilsequence, where $l = 2rs$; we introduce this notion here only in order to keep trace of the parameters $r$, $s$ of the “origination” of a nilsequence. So, for any $r$ and $s$, $\mathcal{N}^\circ_{r,s} \subseteq \mathcal{N}^\circ_{2rs}$; since also $\mathcal{N}^\circ_r \subseteq \mathcal{N}^\circ_{r,1}$, we have $\bigcup_{r,s \in \mathbb{N}} \mathcal{N}^\circ_{r,s} = \mathcal{N}^\circ$.

We will also need the following lemma, saying, informally, that the operation of “alternation” of sequences preserves the algebras of nilsequences:

Lemma 2.1. Let $k \in \mathbb{N}$ and let $\psi_i \in \mathcal{N}^\circ_{r,s}$ (respectively, $\mathcal{N}^\circ_{r}$), $i \in \{0, \ldots, k-1\}^d$. Then the sequence $\psi$ defined by $\psi(n) = \psi_i(m)$ for $n = km + i$ with $m \in \mathbb{Z}^d$, $i \in \{0, \ldots, k-1\}^d$, is also in $\mathcal{N}^\circ_{r,s}$ (respectively, $\mathcal{N}^\circ_r$).

Proof. Put $I = \{0, \ldots, k-1\}^d$. For each $i \in I$, let $X_i = G_i/\Gamma_i$ be the $r$-step nilmanifold, $g_i$ be the polynomial sequence in $G_i$ of naive degree $\leq s$, $x_i \in X_i$ be the point, and $f_i \in C(X_i)$ be the function such that $\psi_i(n) = f_i(g_i(n)x_i)$, $n \in \mathbb{Z}^d$. If, for some $i$, $G_i$ is not connected, it is a factor-group of a free $r$-step nilpotent group with both continuous and discrete generators, which, in its turn, is a subgroup of a free $r$-step nilpotent group with only continuous generators; thus after replacing, if needed, $X_i$ by a larger nilmanifold and extending $f_i$ to a continuous function on this nilmanifold we may assume that every $G_i$
is connected and simply-connected. In this case for any element \( b \in G_i \) and any \( l \in \mathbb{N} \) an \( l \)-th root \( b^{1/l} \) exists in \( G_i \), and thus the polynomial sequence \( p^{(n)} \) in \( G_i \) makes sense even if a polynomial \( p \) has non-integer rational coefficients. Thus, for each \( i \in I \) we may construct a polynomial sequence \( g_i^{(n)} \) in \( G_i \), of the same naive degree as \( g_i \), such that \( g_i^{(n)}(km+i) = g_i(m) \) for all \( m \in \mathbb{Z} \). Put \( G = \mathbb{Z}^d \times \prod_{i \in I} G_i \), \( X = (\mathbb{Z}/(k\mathbb{Z}))^d \times \prod_{i \in I} X_i \), \( g(n) = (n, (g_i^{(n)}(n), i \in I)) \) for \( n \in \mathbb{Z}^d \), \( x = (0, (x_i, i \in I)) \in X \), and \( f(i, (y_i, i \in I)) = f_i(y_i) \) for \( (i, (y_i, i \in I)) \in X \). Then \( X \) is an \( r \)-step nilmanifold, \( f \in C(X) \), and thus the sequence \( \psi(n) = f(g(n)x) = f_i(g_i^{(n)}(n)x_i) = f_i(g_i(m)x_i) = \psi_i(m) \), \( n = km+i, m \in \mathbb{Z}^d, i \in I \), is in \( \mathcal{N}_{r,s} \).

A set \( S \subset \mathbb{Z}^d \) is said to be of uniform (or Banach) density zero if for any Følner sequence \( (\Phi_N)_{N=1}^{\infty} \) in \( \mathbb{Z}^d \), \( \lim_{N \to \infty} |S \cap \Phi_N|/|\Phi_N| = 0 \). A sequence \((\omega_n)_{n \in \mathbb{Z}^d}\) in a topological space \( \Omega \) converges to \( \omega \in \Omega \) in uniform density if for every neighborhood \( U \) of \( \omega \) the set \((\{n \in \mathbb{Z}^d : \omega_n \notin U\})\) is of uniform density zero.

We will say that a sequence \( \lambda \in \mathbb{Z}^\infty \) is a null-sequence if it tends to zero in uniform density. This is equivalent to \( \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)| = 0 \) for any Følner sequence \( (\Phi_N)_{N=1}^{\infty} \) in \( \mathbb{Z}^d \), and is also equivalent to \( \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)|^2 = 0 \) for any Følner sequence \( (\Phi_N)_{N=1}^{\infty} \) in \( \mathbb{Z}^d \). We will denote the set of (bounded) null-sequences by \( \mathcal{Z} \). \( \mathcal{Z} \) is a closed ideal in \( \mathbb{Z}\).

The algebra \( \mathcal{Z} \) is orthogonal to the algebra \( \mathcal{N} \) in the following sense:

**Lemma 2.2.** For any \( \psi \in \mathcal{N} \) and \( \lambda \in \mathcal{Z} \), \( \| \psi + \lambda \| \geq \| \psi \| \).

**Proof.** Let \( c \geq \| \psi + \lambda \| \). Nil-systems are distal systems (see, for example, [L1]), which implies that every point returns to any its neighborhood regularly. It follows that if \( |\psi(n)| > c \) for some \( n \), then the set \( \{n \in \mathbb{Z}^d : |\psi(n)| > c\} \) has positive lower density, and then \( \psi(n) + \lambda(n) > c \) for some \( n \), contradiction. Hence, \( |\psi(n)| \leq c \) for all \( n \).

It follows that \( \mathcal{N} \cap \mathcal{Z} = 0 \).

We will denote the algebras \( \mathcal{N}^{o} + \mathcal{Z}, \mathcal{N}^{o} + \mathcal{Z}, \mathcal{N}^{o} + \mathcal{Z}, \mathcal{N}^{o} + \mathcal{Z}, \mathcal{N}^{o} + \mathcal{Z}, \) and \( \mathcal{N} + \mathcal{Z} \) by \( \mathcal{M}_{o}^{r}, \mathcal{M}_{r}, \mathcal{M}_{o,r,s}, \mathcal{M}_{r,s}, \mathcal{M}_{r}, \) and \( \mathcal{M} \) respectively.

Lemma 2.2 implies:

**Lemma 2.3.** The algebras \( \mathcal{M}, \mathcal{M}_{r}, \) and \( \mathcal{M}_{r,s}, r,s \in \mathbb{N} \), are all closed, and the projections \( \mathcal{M} \to \mathcal{N}, \mathcal{M} \to \mathcal{Z} \) are continuous.

**Proof.** If a sequence \( (\psi_{n} + \lambda_{n}) \) with \( \psi_{n} \in \mathcal{N}, \lambda_{n} \in \mathcal{Z}, \) converges to \( \varphi \in \mathbb{Z}^\infty \), then since \( \| \psi_{n} + \lambda_{n} \| \) for all \( n \), the sequence \( (\psi_{n}) \) is Cauchy, and so converges to some \( \psi \in \mathcal{N} \). Then \( (\lambda_{n}) \) also converges, to some \( \lambda \in \mathcal{Z} \), and so \( \varphi = \psi + \lambda \in \mathcal{M} \). All the same is true for \( \mathcal{M}_{r} \) and \( \mathcal{M}_{r,s} \), instead of \( \mathcal{M} \), for all \( r \) and \( s \).

For \( \varphi_{1} = \psi_{1} + \lambda_{1} \) and \( \varphi_{2} = \psi_{2} + \lambda_{2} \) with \( \psi_{1}, \psi_{2} \in \mathcal{N} \) and \( \lambda_{1}, \lambda_{2} \in \mathcal{Z} \) we have \( \| \psi_{1} - \psi_{2} \| \leq \| \varphi_{1} - \varphi_{2} \| \), so the projection \( \mathcal{M} \to \mathcal{N}, \psi + \lambda \mapsto \psi \), is continuous, and so the projection \( \mathcal{M} \to \mathcal{Z}, \psi + \lambda \mapsto \lambda \), is also continuous.

**Generalized polynomials** on \( \mathbb{Z}^d \) are the functions on \( \mathbb{Z}^d \) constructed from ordinary polynomials using the operations of addition, multiplication, and the operation of taking the integer part. A function \( h \) on a nilmanifold \( X \) is said to be **piecewise polynomial** if it
can be represented in the form \( h(x) = q_i(x), \ x \in Q_i, \ i = 1, \ldots, k, \) where \( X = \bigcup_{i=1}^k Q_i \) is a finite partition of \( X \) and, in Malcev coordinates on \( X \), for every \( i \) the set \( Q_i \) is defined by a system of polynomial inequalities and \( q_i \) is a polynomial function. (Since multiplication in a nilpotent Lie group is polynomial, this definition does not depend on the choice of coordinates on \( X \); see [BL].) It was shown in [BL] (and also, in a simpler way, in [L7]), that a sequence \( v \in l^\infty \) is a generalized polynomial iff there is a nilmanifold \( X = G/\Gamma \), a piecewise polynomial function \( h \) on \( X \), a polynomial sequence \( g \) in \( G \), and a point \( x \in X \) such that \( v(n) = h(g(n)x), \ n \in \mathbb{Z}^d \).

Let \( \mathcal{P}^o \) be the algebra of bounded generalized polynomials on \( \mathbb{Z}^d \) and \( \mathcal{P} \) be the closure of \( \mathcal{P}^o \) in \( l^\infty \). Since (by the Weierstrass approximation theorem) any continuous function on a compact nilmanifold \( X \) is uniformly approximable by piecewise polynomial functions, any basic nilsequence is uniformly approximable by bounded generalized polynomials, and so, is contained in \( \mathcal{P} \). Hence, \( \mathcal{N} \subset \mathcal{P} \), and \( \mathcal{M} \subset \mathcal{P} + \mathcal{Z} \). The inverse inclusion does not hold, since not all piecewise polynomial functions are uniformly approximable by continuous functions; however, they are – on the complement of a set of arbitrarily small measure, which implies that generalized polynomials are also approximable by nilsequences, – in a certain weaker topology in \( l^\infty \).

3. Distribution of a polynomial sequence of subnilmanifolds

Let \( Y \) be a connected subnilmanifold of the (connected) nilmanifold \( X \), and let \( g(n), n \in \mathbb{Z}^d \), be a polynomials sequence in \( G \). We will investigate how the sequence \( g(n)Y \) of subnilmanifolds of \( X \) is distributed in \( X \).

Proposition 3.1. Let \( X = G/\Gamma \) be a connected nilmanifold, let \( Y \) be a connected sub-nilmanifold of \( X \), and let \( g: \mathbb{Z}^d \rightarrow G \) be a polynomial sequence in \( G \) with \( g(0) = 1_G \). Assume that \( g(\mathbb{Z}^d)Y \) is dense in \( X \), and that \( G \) is generated by \( G^c \) and the range \( g(\mathbb{Z}^d) \) of \( g \). Let \( Z \) be the normal closure of \( Y \) in \( X \); then for any \( f \in C(X) \), \( \lambda(n) = \int g(n)Y f d\mu_{g(n)Y} - \int g(n)Z f d\mu_{g(n)Z}, \ n \in \mathbb{Z}^d \), is a null-sequence.

Proof. Let \( f \in C(X) \) and let \( \varepsilon > 0 \); we have to show that the set \( \{ n \in \mathbb{Z}^d : \left| \int g(n)Y f d\mu_{g(n)Y} - \int g(n)Z f d\mu_{g(n)Z} \right| \geq \varepsilon \} \) has zero uniform density in \( \mathbb{Z}^d \). After replacing \( f \) by a close function we may assume that \( f \) is Lipschitz, so that \( C = \sup_{x \neq y} |f(x) - f(y)|/\text{dist}(x,y) \) is finite. Choose Malcev’s coordinates in \( G^c \), and let \( Q \subset G^c \) be the corresponding fundamental cube. Since \( Z \) is normal in \( X \), \( aZ = bZ \) whenever \( a = b \mod \Gamma \), and \( \bigcup_{a \in Q} aZ \) is a partition of \( X \).

We first want to determine for which \( a \in G \) one has \( \left| \int aY f aY d\mu_{aY} - \int aZ f d\mu_{aZ} \right| \geq \varepsilon \). For every \( b \in Q \), by Proposition 1.1, applied to the nilmanifold \( bZ \), there exist proper sub-nilmanifolds \( V_{b_1}, \ldots, V_{b_{r_b}} \) of \( Z \) such that \( \left| \int W f d\mu_W - \int bZ f d\mu_{bZ} \right| < \varepsilon/2 \) whenever \( W \) is a sub-nilmanifold of \( bZ \) with \( b \in W \not\subseteq bV_{b_i}, i = 1, \ldots, r_b \). By continuity, for each \( b \in Q \) there exists a neighborhood \( U_b \) of \( b \) such that for all \( a \in U_b \), \( \left| \int W f d\mu_W - \int aZ f d\mu_{aZ} \right| < \varepsilon \) whenever \( a \in W \not\subseteq aV_{b_i}, i = 1, \ldots, r_b \). Using the compactness of the closure of \( Q \), we can choose \( b_1, \ldots, b_l \in Q \) such that \( Q \subseteq \bigcup_{j=1}^l U_{b_{j,i}} \); let \( V = \bigcup_{j=1, \ldots, l} V_{b_{j,i}} \). Then for any \( b \in Q \), for any sub-nilmanifold \( W \) of \( bZ \) with \( b \in W \not\subseteq V \) one has \( \left| \int W f d\mu_W - \right| \geq \varepsilon \).
Since \( T \) is the factor of the commutative group \( G_{K/L} \) in \( \Gamma \) and \( \gamma \),

Proof. Let \( N = \{(b1_X, bV) \mid b \in Q\} \); we have to prove that the set \( \{n \in \mathbb{Z}^d : (g(n)1_X, g(n)Y) \subseteq N\} \) has zero uniform density in \( \mathbb{Z}^d \). For this purpose we are going to find the closure of the sequence \( \tilde{Y}_n = (g(n)1_X, g(n)Y) \), \( n \in \mathbb{Z}^d \), and \( \gamma \). Hence, \( \tilde{Y}_n \) is in every torus generated by the orbit \( \{g(n)1_X, n \in \mathbb{Z}^d\} \) is connected, and let \( P \) be the closed connected subgroup of \( G \) such that \( \pi(P) = R \). (If \( R \) is disconnected we pass to a sublattice of \( \mathbb{Z}^d \) and its cosets to deal with individual connected components of \( R \).) We will also assume that \( Y \ni 1_X \).

**Lemma 3.2.** The closure of the sequence \( \tilde{Y}_n \) is the subnilmanifold \( D = \{(a1_X, aZ) \mid a \in P \} = \{(a1_X, aZ) \mid a \in \pi(P \cap Q) \} \) of \( X \times X \).

Proof. Let \( L \) be the closed connected subgroup of \( G \) such that \( \pi(L) = Z \), and let \( K = \{(a, au) \mid a \in P, u \in L\} \); since \( L \) is normal in \( G \), \( K \) is a (closed rational) subgroup of \( G \times G \), and we have \( D = K/(\Gamma \times \Gamma \cap K) \).

For any \( n \in \mathbb{Z}^d \) we have \( \tilde{Y}_n \subseteq D \) (since \( g(n)1_X \in R \), so \( g(n) \in P\Gamma \), so \( g(n)L \subseteq P\Gamma L = PL\Gamma \)), and we have to show that the sequence \( \tilde{Y}_n \) is dense in \( D \). For this it suffices to prove that the image of this sequence is dense in the maximal torus \( T = [K, K]\) of \( D \).

Since \( L \) is normal, we have \( [K, K] = \{(a, au) \mid a \in [P, P], u \in [P, L][L, L]\} \), and the torus \( T \) is the factor of the commutative group \( K/[K, K] \) by the image \( \Lambda \) in this group of the lattice \( \Gamma \times \Gamma \). Let \( H \) be the closed connected subgroup of \( G \) such that \( \pi(H) = Y \); then \( L = H[H, G] \), so

\[
K/[K, K] = \{(a, auv) \mid a \in P, v \in H, w \in [H, G]\}/\{(a, au) \mid a \in [P, P], u \in [P, L][L, L]\}.
\]

By assumption, \( G \) is generated by \( G^c \) and \( g \). Since the orbit \( \{g(n)Z \mid n \in \mathbb{Z}^d\} \) is dense in \( X \) and \( Z \) is normal, the orbit \( \{g(n)1_X/Z \mid n \in \mathbb{Z}^d\} \) is dense in \( X/Z \), so \( P/(P \cap L) = G^c/L \), so \( G^c = PL \). Hence, \( [H, G] = [H, g][H, P][H, L] \). For any \( n, g(n) = u_n\gamma_n \) for some \( u_n \in P \) and \( \gamma_n \in \Gamma \), thus, modulo \( [P, L][L, L] \), the group \( [H, G] \) is generated by \( \{[H, \gamma_n] \mid n \in \mathbb{Z}^d\} \).

The closure of \( B \) of the image of the sequence \( \tilde{Y}_n \) in \( T \) is a subtorus of \( T \). Since the sequence \( g(n)1_X \) is dense in \( \pi(P) \), the subtorus \( T_1 = \{(a, a) \mid a \in \pi(P)\}/[K, K] \Lambda \) of \( T \) is the closure of the image of the sequence \( \{g(n)1_X, n \in \mathbb{Z}^d\} \) and so, is contained in \( B \). Also, the subtorus \( T_2 = \{(1_X, u) \mid u \in \pi(H)\}/[K, K] \Lambda \) is contained in \( B \). Finally, for \( n \in \mathbb{Z}^d \) and \( c \in H \) we have

\[
(g(n), g(n)c) = (u_n, u_n\gamma_n c) = (u_n1_X, u_n c[\gamma_n^{-1}]\gamma_n).
\]

Taken modulo \( [K, K] \Lambda \), these elements of \( B \) generate \( T \) modulo \( T_1 + T_2 \), so \( B = T \).

It follows that the sequence \( \tilde{Y}_n \), \( n \in \mathbb{Z}^d \), is well distributed in \( D \). The set \( N = \{(b1_X, bV) \mid b \in Q\} \) is a compact subset of \( D \) of zero measure, thus, the set \( \{n \in \mathbb{Z}^d : (g(n)1_X, g(n)Y) \subseteq N\} \) has zero uniform density in \( \mathbb{Z}^d \).
Theorem 3.3. Let $X = G/\Gamma$ be an r-step nilmanifold, let $Y$ be a subnilmanifold of $X$, let $g$ be a polynomial sequence in $G$ of naive degree $\leq s$, let $f \in C(X)$. Then the sequence

$$\varphi(n) = \int_{g(n)Y} f \, d\mu_g(n)Y, \quad n \in \mathbb{Z}^d,$$

is contained in $\mathcal{M}^o_{r,s}$.

**Proof.** We may assume that $Y \ni 1_X$. After replacing $f$ by $f(g(0)x)$, $x \in X$, we may assume that $g(0) = 1_X$. We may also replace $X$ by the closure of the orbit $g(\mathbb{Z}^d)Y$, and we may assume that $G$ is generated by $G^o$ and the range of $g$.

First, let $X$ and $Y$ be both connected. Let $Z$ be the normal closure of $Y$ in $X$; then by Proposition 3.1, $\varphi(n) = \int_{g(n)Z} f \, d\mu_g(n)Z + \lambda_n, \quad n \in \mathbb{Z}^d$, with $\lambda \in \mathbb{Z}$. Define $\hat{X} = X/Z$, $\hat{x} = \{Z\} \in \hat{X}$, and $\hat{f} = E(f|\hat{X}) \in C(\hat{X})$; then $\int_{g(n)Z} f \, d\mu_g(n)Z = \hat{f}(g(n)\hat{x}), \quad n \in \mathbb{Z}^d$, and the sequence $\hat{f}(g(n)\hat{x}), \quad n \in \mathbb{Z}^d$, is in $\mathcal{N}^o_{r,s}$, so $\varphi \in \mathcal{M}^o_{r,s}$.

Now assume that $Y$ is connected but $X$ is not. Then, by [L2], there exists $k \in \mathbb{N}$ such that $g(k\mathbb{Z}^d + i)\mathbb{Y}$ is connected for every $i \in \{0, \ldots, k - 1\}^d$. Thus, for every $i \in \{0, \ldots, k - 1\}^d$, $\varphi(kn + i) \in \mathcal{M}^o_{r,s}$, and the assertion follows from Lemma 2.1.

Finally, if $Y$ is disconnected and $Y_1, \ldots, Y_r$ are the connected components of $Y$, then

$$\int_{g(n)Y} f \, d\mu_g(n)Y = \sum_{j=1}^r \int_{g(n)Y_j} f \, d\mu_g(n)Y_j, \quad n \in \mathbb{Z}^d,$$

and the result holds since it holds for $Y_1, \ldots, Y_r$. \hfill \Box

4. Integrals of null- and of nil-sequences

On $l^\infty$ and, thus, on $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{M}$ we will assume the Borel $\sigma$-algebra induced by the weak topology.

We start with integration of null-sequences:

**Lemma 4.1.** Let $(\Omega, \nu)$ be a measurable space and let $\Omega \to \mathcal{Z}, \omega \mapsto \lambda_\omega$, be an integrable mapping. Then the sequence $\lambda(n) = \int_\Omega \lambda_\omega(n) \, d\nu$ is in $\mathcal{Z}$ as well.

(We say that a mapping $\Psi: \Omega \to l^\infty$ is integrable if it is measurable and $\int_\Omega \|\Psi\| \, d\nu < \infty$.)

**Proof.** For each $\omega \in \Omega$, $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| = 0$ for any Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{Z}^d$. By the dominated convergence theorem,

$$\lim \sup_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda(n)| = \lim \sup_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \left| \int_\Omega \lambda_\omega(n) \, d\nu \right| \leq \lim_{N \to \infty} \int_\Omega \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| \, d\nu = \int_\Omega \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\lambda_\omega(n)| \, d\nu = 0.$$

So, $\lambda \in \mathcal{Z}$. \hfill \Box

For nilsequences we have:

**Proposition 4.2.** Let $(\Omega, \nu)$ be a measure space and let $\Omega \to \mathcal{N}, \omega \mapsto \varphi_\omega$, be an integrable mapping. Then the sequence $\varphi(n) = \int_\Omega \varphi_\omega(n) \, d\nu$ belongs to $\mathcal{M}$. (If, for some $r$, $\varphi_\omega \in \mathcal{N}_r$ for all $\omega$, then $\varphi \in \mathcal{M}_r$.)

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To simplify notation, let us start with the case $d = 1$. We are going to reduce Proposition 4.2 to a statement concerning a sequence of measures on a nilmanifold. Since $\mathcal{M}$ is closed in $l^\infty$, we are allowed to replace the mapping $\varphi_\omega$ from $\Omega$ to $N$ by a close mapping $\varphi_\omega'$: we are done if for any $\varepsilon > 0$ we can find a mapping $\Omega \rightarrow N$, $\omega \mapsto \varphi_\omega'$, with $\| \int_\Omega \| \varphi_\omega - \varphi_\omega' \|_\infty d\nu < \varepsilon$ and such that the assertion of Proposition 4.2 holds for $\varphi_\omega'$. Fix $\varepsilon > 0$. First, after replacing $\Omega$ by $\Omega'$ with $\nu(\Omega') < \infty$ such that $\int_{\Omega \setminus \Omega'} \| \varphi_\omega \|_\infty d\nu < \varepsilon$, we may assume that $\nu(\Omega') < \infty$. Next, since the set $N^0$ of basic nilsequences is dense in $\mathcal{N}$, we may replace the nilsequences $\varphi_\omega$ by close basic nilsequences, if we manage to do this in a measurable way. We will, as we may, deal with $\mathbb{R}$-valued nilsequences. Let $X = G/\Gamma$ be a nilmanifold where $G$ is a simply connected nilpotent Lie group and $\Gamma$ is a lattice in $G$, and let $\pi: G \rightarrow X$ be the projection. We may assume that $G$ has the same number of connected components as $X$, then $G$ is homeomorphic to $\mathbb{R}^{\dim G} \times F$, where $F$ is a finite set, with $\Gamma$ corresponding to $\mathbb{Z}^{\dim G}$; this homeomorphism induces a natural metric on $G$ and on $X$. For $k \in \mathbb{N}$ let $Q_k$ be the set of elements of $G$ at the distance $\leq k$ from $1_G$ and let $L_k$ be the set of Lipschitz functions on $X$ with Lipschitz constant $k$ and of modulus $\leq k$. The subset $Q_k \times L_k$ of $G \times C(X)$ is compact; the "nilsequence reading" mapping $\Psi: G \times C(X) \rightarrow N$, $\Psi(a, h)(n) = h(\pi(a_n))$, is continuous with respect to the weak topology on $N$; thus the set $\mathcal{L}_{X,k} = \Psi(Q_k \times L_k) \subset N^0$ is compact in this topology.

Fix a countable set $S$ dense in $\mathcal{L}_{X,k}$ in the weak topology and enumerate it. Let $\varphi \in \mathcal{N}$. For each $j \in \mathbb{N}$ let $\psi_j$ be the element of $S$ for which

(i) the sum $\sum_{n=-j}^j |\varphi(n) - \psi_j(n)|$ is minimal;
(ii) among the elements of $S$ for which (i) holds, the vector $(\psi(0), \psi(-1), \psi(1), \ldots, \psi(-j), \psi(j))$ is minimal for $\psi = \psi_j$ with respect to the lexicographic order;
(iii) and among the elements of $S$ for which (i) and (ii) hold, $\psi_j$ has the minimal number under the ordering of $S$.

Put $\zeta_{X,j}(\varphi) = \psi_j$; then $\zeta_{X,k,j}$ is a measurable mapping $\mathcal{N} \rightarrow \mathcal{L}_{X,k}$. For any $\varphi \in \mathcal{N}$ the sequence $\psi_j = \zeta_{X,k,j}(\varphi)$ converges in $\mathcal{L}_k$: indeed, $\mathcal{L}_{X,k}$ is compact, and any convergent subsequence of this sequence converges to the same element of $\mathcal{L}_{X,k}$, namely, to $\psi \in Q_k$ which is closest to $\varphi$ in the $l^\infty$-norm, and among such, which is minimal with respect to the lexicographic order of its entries. Put $\zeta_{X,k}(\varphi) = \lim_{j \rightarrow \infty} \zeta_{X,j,k}(\varphi)$, $\varphi \in \mathcal{N}$; then $\zeta_{X,k}$ is a measurable mapping $\mathcal{N} \rightarrow \mathcal{L}_{X,k}$ that maps each nilsequence to a closest in $l^\infty$-norm element of $\mathcal{L}_{X,k}$. It also follows that the function $\partial_{X,k}(\varphi) = \min_{\psi \in \mathcal{L}_{X,k}} \| \varphi - \psi \|_{l^\infty}$ is measurable on $\mathcal{N}$.

In each class of isomorphic nilmanifolds choose a representative $X$ (along with $G$, $\Gamma$, a homeomorphism $G \rightarrow \mathbb{R}^{\dim G} \times F$, and a metric on $G$ and $X$); let $\mathcal{X}$ be the set of these representatives. Since there exists only countably many nonisomorphic nilmanifolds, $\mathcal{X}$ is countable. Introduce a well ordering of $\mathcal{X}$ satisfying $X' < X$ when $\dim X' < \dim X$.

For every $X \in \mathcal{X}$ put $\Omega_{X,k} = \{ \omega \in \Omega : \partial_{X,k}(\varphi_\omega) < \varepsilon/\nu(\Omega) \}$ and $\Omega_X = \bigcup_{k=1}^\infty \Omega_{X,k}$; these are measurable subsets of $\Omega$. The union $\bigcup_{X \in \mathcal{X}} \bigcup_{k=1}^\infty \mathcal{L}_{X,k}$ is dense in $\mathcal{N}$, thus $\bigcup_{X \in \mathcal{X}} \bigcup_{k=1}^\infty \Omega_{X,k} = \Omega$. Next define $\Omega_X = \Omega_X \setminus \bigcup_{X' < X} \Omega_{X'}$, $X \in \mathcal{X}$; these are disjoint sets that partition $\Omega$. Finally, for each $X \in \mathcal{X}$ and $k \in \mathbb{N}$ put $\Omega_{X,k} = \Omega_X \setminus \bigcup_{k' < k} \Omega_{X,k'}$.

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1 The argument that follows has been changed; I thank B. Host for pointing to me out a mistake in the previous version of the paper.
Now, for $\omega \in \Omega$ define $\psi_\omega = \zeta_{X,k}(\varphi_\omega)$ when $\omega \in \Omega'_{X,k}$, $X \in \mathcal{X}$, $k \in \mathbb{N}$; then $\omega \mapsto \psi_\omega$ is a measurable mapping $\Omega \rightarrow \mathcal{N}^0$ with $\|\psi_\omega - \varphi_\omega\|_{L^1} < \varepsilon/\nu(\Omega)$ for all $\omega \in \Omega$. We may now replace $\varphi_\omega$ by $\psi_\omega$, $\omega \in \Omega$; moreover, we may also deal with the sets $\Omega'_X$ separately, and therefore assume that $\varphi_\omega$, $\omega \in \Omega$, are all read off the same nilmanifold $X = G/\Gamma$: $\varphi_\omega = \Psi(a,h)$ with $a \in G$ and $h$ being a Lipschitz function on $X$. (And, in addition, by our construction, $\varphi_\omega$ is not readable off any nilmanifold $X'$ with $X' < X$.)

We now claim that for each $\omega \in \Omega$, $\varphi_\omega$ has only countably many preimages under this mapping; by Lusin’s theorem about the existence of a measurable section, this will imply that we can choose elements $a_\omega \in Q_k$, $h_\omega \in L_k$ with $\varphi_\omega = \Psi(a_\omega, h_\omega)$, $\omega \in \Omega$, such that the mapping $\omega \mapsto (a_\omega, h_\omega)$ is measurable. We get use of the following fact:

**Lemma 4.2a.** Let nilmanifolds $X_i = G_i/\Gamma_i$, elements $a_i \in G_i$, and functions $h_i \in C(X_i)$, $i = 1, 2$, be such that the orbits $\{\pi_i(a^n_i)\}_{n \in \mathbb{Z}}$, where $\pi_i$ are the projections $G_i \rightarrow X_i$, are dense in $X_i$, $i = 1, 2$, and the triples $(X_1, a_1, h_1)$ and $(X_2, a_2, h_2)$ produce the same nisequence: $\varphi(n) = h_1(\pi_1(a^n_1)) = h_2(\pi_2(a^n_2))$, $n \in \mathbb{Z}$. Then there exists a common factor $(\widehat{X}, \widehat{a}, \widehat{h})$ of $(X_1, a_1, h_1)$ and $(X_2, a_2, h_2)$ such that $\varphi(n) = \widehat{h}(\widehat{\pi}(\widehat{a}^n))$ (where $\widehat{\pi}$ is the projection $\widehat{G} \rightarrow \widehat{X}$).

**Proof.** Let $\widehat{G} = G_1 \times G_2$, $\widehat{X} = X_1 \times X_2$, $\widehat{\pi} = \pi_1 \times \pi_2$: $\widehat{G} \rightarrow \widehat{X}$. Let $\widehat{a} = (a_1, a_2) \in \widehat{G}$, and let $Y$ be the closure of the orbit of $\widehat{a}$ in $\widehat{X}$, $Y = \{\widehat{\pi}(\widehat{a}^n), n \in \mathbb{Z}\}$. Let $p_1$ and $p_2$ be the projections of $Y$ to $X_1$ and to $X_2$ respectively. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ be “the fibers” of the projections $p_2$ and $p_1$ respectively: $p_2^{-1}(1_{X_2}) = Y_1 \times \{1_{X_2}\}$ and $p_1^{-1}(1_{X_1}) = Y_2 \times \{1_{X_1}\}$. For any $n \in \mathbb{Z}$ we have $\varphi(n) = h_1(p_1(\pi_1(\widehat{a}^n))) = h_2(p_2(\pi_2(\widehat{a}^n)))$: since the orbit $\{\pi(\widehat{a}^n)\}_{n \in \mathbb{Z}}$ is dense in $Y$, this implies that $h_1 \circ p_1 = h_2 \circ p_2$; so $h_1$ is constant on the fibers $b_1 Y_1$, $b_1 \in G_1$, of $p_2$ and $h_2$ is constant on the fibers $b_2 Y_2$, $b_2 \in G_2$, of $p_1$, and therefore we may factorize $X_1$ by $Y_1$ and $X_2$ by $Y_2$ (and $Y$ by $Y_1 \times Y_2$) to get the same factor $(\widehat{X}, \widehat{a}, \widehat{h})$.

Now, assume that for some $\omega \in \Omega$, $\varphi_\omega = \Psi(a_1, h_1) = \Psi(a_2, h_2)$, $a_1, a_2 \in G$ and $h_1, h_2$ are Lipschitz functions on $X$. Let, by Lemma 4.2a, $(\widehat{X}, \widehat{a}, \widehat{h})$ be a common factor of $(X, a_1, h_1)$ and $(X, a_2, h_2)$ such that $\varphi_\omega = \Psi(\widehat{a}, \widehat{h})$. Since $\varphi_\omega$ cannot be read off a nilmanifold $\widehat{X}$ with $\dim \widehat{X} < \dim X$, there must be $\dim \widehat{X} = \dim X$. However, for any pair $(\widehat{X}, \widehat{a})$, “nilmanifold with a translation”, there are only countably many (up to isomorphism) pairs $(X, a)$ extending $(\widehat{X}, \widehat{a})$ and with $\dim X = \dim \widehat{X}$.

Thus, we arrive at the following situation: we have a nilmanifold $X = G/\Gamma$ and a measurable function $\Omega \rightarrow G \times C(X)$, $\omega \mapsto (a_\omega, h_\omega)$, such that for every $\omega \in \Omega$ one has $\varphi_\omega(n) = h_\omega(\pi(a^n_\omega))$, $n \in \mathbb{Z}$. Let $H(\omega, x) = h_\omega(x)$, $\omega \in \Omega$, $x \in X$; then $\varphi(n) = \int_H H(\omega, \pi(a^n_\omega))d\nu(\omega)$, $n \in \mathbb{Z}$. Choose a basis $f_1, f_2, \ldots \in C(X)$; the function $H$ is representable in the form $H(\omega, x) = \sum \theta_1(\omega)f_1(x)$, where convergence is uniform with respect to $x$ for any $\omega$; we are done if we prove the assertion for the functions $\theta_1(\omega)f_1(x)$ instead of $H$. So, let $\theta \in L^1(\Omega)$ and $f \in C(X)$; we have to show that the sequence $\varphi(n) = \int_{\Omega} \theta(\omega)f(\pi(a^n_\omega))d\nu(\omega)$ is in $\mathcal{M}$. We may also assume that $\theta \geq 0$. Let $\tau: \Omega \rightarrow G$ be the mapping defined by $\tau(\omega) = a_\omega$, and let $\rho = \tau_*(\theta)$; then $\rho$ is a finite measure on $G$ and $\varphi(n) = \int_G \tau(f(\pi(a^n_\omega))d\rho(a)$. Thus, Proposition 4.2 will follow from the following:

**Proposition 4.3.** Let $X = G/\Gamma$ be a nilmanifold, let $\rho$ be a finite Borel measure on $G$, and let $f \in C(X)$. Then the sequence $\varphi(n) = \int_G f(\pi(a^n_\omega))d\rho(a)$, $n \in \mathbb{Z}$, is in $\mathcal{M}$. (If $X$
is an $r$-step nilmanifold, then $\varphi \in \mathcal{M}_r^\omega$.)

**Proof.** We may and will assume that $X$ is connected. Let $\tilde{\rho} = \pi_*(\rho)$; we decompose $\tilde{\rho}$ in the following way:

**Lemma 4.4.** There exists an at most countable collection $\mathcal{V}$ of connected subnilmanifolds of $X$ (which may include $X$ itself and singletons) and finite Borel measures $\rho_V$, $V \in \mathcal{V}$, on $G$ such that $\rho = \sum_{V \in \mathcal{V}} \rho_V$ and for every $V \in \mathcal{V}$, $\text{supp}(\tilde{\rho}_V) \subseteq V$ and $\tilde{\rho}_V(W) = 0$ for any proper subnilmanifold $W$ of $V$, where $\tilde{\rho}_V = \pi_*(\rho_V)$.

**Proof.** Let $\mathcal{V}_0$ be the (at most countable) set of the singletons $V = \{x\}$ in $X$ (connected $0$-dimensional subnilmanifolds of $X$) for which $\tilde{\rho}(V) > 0$. For each $V \in \mathcal{V}_0$, let $\rho_V$ be the restriction of $\rho$ to $\pi^{-1}(V)$ (that is, $\tilde{\rho}_V(A) = \rho(A \cap \pi^{-1}(V))$ for measurable subsets $A$ of $G$), and let $\rho_1 = \rho - \sum_{V \in \mathcal{V}_0} \rho_V$ and $\tilde{\rho}_1 = \pi_*(\rho_1)$. Now let $\mathcal{V}_1$ be the (at most countable) set of connected $1$-dimensional subnilmanifolds of $X$ for which $\tilde{\rho}_1(V) > 0$, for each $V \in \mathcal{V}_1$, let $\rho_V$ be the restriction of $\rho_1$ to $\pi^{-1}(V)$, and $\rho_2 = \rho - \sum_{V \in \mathcal{V}_1} \rho_V$, $\tilde{\rho}_2 = \pi_*(\rho_2)$. (Note that for $V_1, V_2 \in \mathcal{V}_1$, the subnilmanifold $V_1 \cap V_2$, if nonempty, has dimension 0, so $\tilde{\rho}_1(V_1 \cap V_2) = 0$.) And so on, by induction on the dimension of the subnilmanifolds; at the end, we put $\mathcal{V} = \bigcup_{i=0}^{\dim X} \mathcal{V}_i$.

By Lemma 2.3, it suffices to prove the assertion for each of $\rho_V$ instead of $\rho$. So, we will assume that the measure $\rho$ is supported by a connected subnilmanifold $V$ of $X$ and $\rho(W) = 0$ for any proper subnilmanifold $W$ of $V$.

First, let $V = X$:

**Lemma 4.5.** Let $\rho$ be a finite Borel measure on $G$ such that for $\tilde{\rho} = \pi_*(\rho)$ one has $\tilde{\rho}(W) = 0$ for any proper subnilmanifold $W$ of $X$. Then for any $f \in C(X)$ the sequence $\varphi(n) = \int_X f(\pi(a^n))d\rho(a)$, $n \in \mathbb{Z}$, converges to $\int_X f d\mu_X$ in uniform density.

**Proof.** We may assume that $\int_X f d\mu_X = 0$; we then have to show that $\varphi$ is a null-sequence. Let $(\Phi_N)$ be a Følner sequence in $\mathbb{Z}$. By the dominated convergence theorem we have

$$
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_G f(\pi(a^n))d\rho(a) \int_G \tilde{f}(\pi(b^n))d\rho(b)
$$

$$
= \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{G \times G} f(\pi(a^n))\tilde{f}(\pi(b^n))d(\rho \times \rho)(a,b)
$$

$$
= \int_{G \times G} \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi \times 2(a^n, b^n)) d\rho \times 2(a,b)
$$

$$
= \int_{G \times G} F(a,b) d\rho \times 2(a,b),
$$

where $\pi \times 2 = \pi \times \pi$, $\rho \times 2 = \rho \times \rho$, and $F(a,b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi \times 2(a^n, b^n))$, $a,b \in G$. For $a,b \in G$, if the sequence $u_n = \pi \times 2(a^n, b^n)$, $n \in \mathbb{Z}$, is well distributed in $X \times X$ then $F(a,b) = \int_{X \times X} f \otimes \tilde{f} d\mu_{X \times X} = \int_X f d\mu \int_X \tilde{f} d\mu = 0$. So, $F(a,b) \neq 0$ only if the sequence $(u_n)$ is not well distributed in $X \times X$, which only happens if the point
\[ \pi^2(a, b) \text{ is contained in a proper subnilmanifold } D \text{ of } X \times X \text{ with } 1_{X \times X} \in D. \] So,

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 \leq \sum_{D \in \mathcal{D}} \int_{(\pi^2)^{-1}(D)} |F(a, b)| \, d\rho^2(a, b),
\]

where \( \mathcal{D} \) is the (countable) set of proper subnilmanifolds of \( X \times X \) containing \( 1_{X \times X} \). Let \( D \in \mathcal{D} \); then either for any \( x \in X \) the fiber \( W'_x = \{ y \in X : (x, y) \in D \} \) of \( D \) over \( x \) is a proper subnilmanifold of \( X \), or for any \( y \in X \) the fiber \( W'_y = \{ x \in X : (x, y) \in D \} \) of \( D \) over \( y \) is a proper subnilmanifold of \( X \), (or both). Since, by our assumption, \( \tilde{\rho}(W) = 0 \) for any proper subnilmanifold \( W \) of \( X \), in either case \( \tilde{\rho}^2(D) = 0 \), so \( \rho^2((\pi^2)^{-1}(D)) = 0 \). Hence, \( \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = 0 \), which means that \( \varphi \in \mathcal{Z} \). \( \square \)

Thus, in this case, \( \varphi \) is a constant plus a null-sequence, that is, \( \varphi \in \mathcal{M}^0 \).

Let now \( V \) be of the form \( V = cY \), where \( Y \) is a (proper) connected subnilmanifold of \( X \) with \( 1_X \in Y \) and \( c \in G^c \). We may and will assume that the orbit \( \{c^nY \mid n \in \mathbb{Z} \} \) of \( Y \) is dense in \( X \). Let \( Z \) be the normal closure of \( Y \) in \( X \). In this situation the following generalization of Lemma 4.5 does the job:

**Lemma 4.6.** Let \( Z \) be a normal subnilmanifold of \( X \) and let \( c \in G \) be such that \( \{c^nZ \mid n \in \mathbb{Z} \} \) is dense in \( X \). Let \( \rho \) be a finite Borel measure on \( G \) such that for \( \tilde{\rho} = \pi_\ast(\rho) \) one has \( \text{supp}(\tilde{\rho}) \subset cZ \) and \( \tilde{\rho}(cW) = 0 \) for any proper normal subnilmanifold \( W \) of \( Z \). Let \( \varphi(n) = \int_G f(\pi(c^n)) \, d\rho(a) \), \( n \in \mathbb{Z} \), let \( \hat{X} = X/Z \), and let \( \hat{f} = E(f|\hat{X}) \). Then \( \varphi - \hat{f} \) is a null-sequence.

**Proof.** After replacing \( f \) by \( f - \hat{f} \) we will assume that \( E(f|\hat{X}) = 0 \); we then have to prove that \( \varphi \) is a null-sequence. Let \( L = \pi^{-1}(Z) \). Let \( (\Phi_N) \) be a Følner sequence in \( Z \); as in Lemma 4.5, we obtain

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \int_{G \times G} F(a, b) \, d\rho^2(a, b) = \int_{(cL) \times (cL)} F(a, b) \, d\rho^2(a, b),
\]

where \( F(a, b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi^2((ca^n, cb^n))) \), \( a, b \in L \). Let us “shift” \( \rho \) to the origin, by replacing it by \( c^{-1}_a \rho(a) \), \( a \in G \), so that now \( \text{supp}(\rho) \subset L \), \( \text{supp}(\rho) \subset Z \), and

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\varphi(n)|^2 = \int_{L \times L} F(a, b) \, d\rho^2(a, b),
\]

where \( F(a, b) = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f \otimes \tilde{f}(\pi^2((ca^n, cb^n))) \), \( a, b \in L \).

For \( a, b \in L \), the sequence \( u_n = \pi^2((ca^n, cb^n)) \), \( n \in \mathbb{Z} \), is contained in \( X \times \hat{X} = \{(x, y) : \delta(x) = \delta(y)\} \), where \( \delta \) is the factor mapping \( X \to \hat{X} \). If this sequence is well distributed in \( X \times \hat{X} \), then \( F(a, b) = \int_{X \times \hat{X}} f \otimes \tilde{f} \, d\mu_{X \times \hat{X}} = \int_{\hat{X}} E(f|\hat{X}) E(f|\hat{X}) \, d\mu_{\hat{X}} = 0 \). So, \( F(a, b) \neq 0 \) only if the sequence \( (u_n) \) is not well distributed in \( X \times \hat{X} \), which only happens if the image \( (\tilde{u}_n) \) of \( (u_n) \) is not well distributed in the nil-maximal factor-torus \( T \) of \( X \times \hat{X} \). Using additive notation on \( T \) we have \( \tilde{u}_n = n\tilde{a} + n\tilde{b}, n \in \mathbb{Z} \), where \( T \) contains the direct sum \( S \oplus S \) of two copies of a torus \( S \), \( \tilde{a} \in S \oplus \{0\} \), \( \tilde{b} \in \{0\} \oplus S \), and

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the sequence \((n\epsilon)\) is dense in the factor-torus \(T/(S \oplus S)\). The sequence \((\tilde{u}_n)\) is not dense in \(T\) only if the point \((\tilde{a}, \tilde{b})\) is contained in a proper subtorus \(R\) of \(S \oplus S\), and either for each \(x \in S\) the fiber \(\{\tilde{y} \in S : (\tilde{x}, \tilde{y}) \in R\}\) is a proper subtorus of \(S\), or for each \(\tilde{y} \in S\) the fiber \(\{\tilde{x} \in S : (\tilde{x}, \tilde{y}) \in R\}\) is a proper subtorus of \(S\) (or both). Without loss of generality, assume that the first possibility holds. Then, returning back to \(X \times \hat{X}\), we obtain that the sequence \((u_n)\) is not well distributed in this space only if the point \(\pi \times_2(a, b)\) is contained in a sub-nilmanifold \(D\) (the preimage of the torus \(R\)) in \(Z \times Z\) with \(D \ni 1 \times \hat{X}\), such that for every \(x \in Z\) the fiber \(W_x = \{y \in Z : (x, y) \in D\}\) is a proper normal sub-nilmanifold of \(Z\). Since, by our assumption, \(\tilde{p}(W_x) = 0\) for all \(x\), we have \(\tilde{p} \times_2(D) = 0\), so \(\rho \times_2((\pi \times_2)^{-1}(D)) = 0\). The function \(F(a, b)\) may only be nonzero on the union of a countable collection of the sub-nilmanifolds \(D\) like this, so \(\int_{L \times L} F(a, b) d\rho \times_2(a, b) = 0\). Hence, \(\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} |\phi(n)|^2 = 0\), which means that \(\phi \in \mathcal{Z}\).

Since, in the notation of Lemma 4.6, the sequence \(\hat{f}(\pi(c^n)) = \hat{f}(c^n 1_X)\), \(n \in \mathbb{Z}\), is a basic nilsequence, \(\phi\) is a sum of a nilsequence and a null-sequence, so \(\phi \in \mathcal{M}^\circ\) in this case as well.

The proof of Proposition 4.2 in the case \(d \geq 2\) is not much harder than in the case \(d = 1\), and we will only sketch it. For each \(\omega \in \Omega\), instead of a single element \(a_\omega \in G_\omega\) we now have \(d\) commuting elements \(a_{\omega, 1}, \ldots, a_{\omega, d} \in G_\omega\). After passing to a single nilmanifold \(X = G/\Gamma\), we obtain \(d\) mappings \(\tau_i : \Omega \to G\), \(\omega \mapsto a_{\omega, i}\), \(i = 1, \ldots, d\), and so, the mapping \(\tau = (\tau_1, \ldots, \tau_d) : \Omega \to C^d\). We define a measure \(\rho\) on \(C^d\) by \(\tau_\ast(\theta \nu)\); then Proposition 4.2 follows from the following modification of Proposition 4.3:

**Proposition 4.7.** Let \(X = G/\Gamma\) be a nilmanifold, let \(\rho\) be a finite Borel measure on \(C^d\), and let \(f \in C(X)\). Then the sequence \(\phi(n_1, \ldots, n_d) = \int_{C^d} f(\pi(a_1^{n_1} \ldots a_d^{n_d})) d\rho(a_1, \ldots, a_d), (n_1, \ldots, n_d) \in \mathbb{Z}^d\), is in \(\mathcal{M}^\circ\). (If \(X\) is an \(r\)-step nilmanifold, then \(\phi \in \mathcal{M}_r^\circ\).)

The proof of this proposition is the same as of Proposition 4.3, with \(a^n\) replaced by \(a_1^{n_1} \ldots a_d^{n_d}\), and the mapping \(G \to X\), \(a \mapsto \pi(a)\), replaced by the mapping \(G^d \to X\), \((a_1, \ldots, a_d) \mapsto \pi(a_1 \ldots a_d)\).

Uniting Proposition 4.2 with Lemma 4.1, we obtain:

**Theorem 4.8.** Let \((\Omega, \nu)\) be a measure space and let \(\Omega \to \mathcal{M}, \omega \mapsto \omega_\ast\), be an integrable mapping. Then the sequence \(\phi(n) = \int_{\Omega} \phi_\omega(n) d\nu\) is in \(\mathcal{M}\) as well. If, for some \(r\), \(\phi_\omega \in \mathcal{M}_r\) for all \(\omega\), then \(\phi \in \mathcal{M}_r\).

5. Multiple polynomial correlation sequences and nilsequences

Now let \((W, \mathcal{B}, \mu)\) be a probability measure space and let \(T\) be an ergodic invertible measure preserving transformation of \(W\). Let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \to \mathbb{Z}\). Let \(A_1, \ldots, A_k \in \mathcal{B}\) and let \(\phi(n) = \mu(T^{p_1(n)} A_1 \cap \ldots \cap T^{p_k(n)} A_k)\), \(n \in \mathbb{Z}^d\), or, more generally, let \(f_1, \ldots, f_k \in L^\infty(W)\) and \(\phi(n) = \int_W T^{p_1(n)} f_1 \ldots T^{p_k(n)} f_k d\mu\), \(n \in \mathbb{Z}^d\). Then, given \(\epsilon > 0\), there exist an \(r\)-step nilsystem \((X, a), X = G/\Gamma, a \in G\), and functions \(\hat{f}_1, \ldots, \hat{f}_k \in L^\infty(X)\) such that, for \(\phi(n) = \int_X a^{p_1(n)} \hat{f}_1 \cdots a^{p_k(n)} \hat{f}_k d\mu_X, n \in \mathbb{Z}^d\), the set \(\{n \in \mathbb{Z}^d : |\phi(n) - \phi(n)| > \epsilon\}\) is dense in \(X\).
\(\varepsilon\) has zero uniform density. Moreover, there is a universal integer \(r\) that works for all systems \((W, B, \mu, T)\), functions \(h_i\), and \(\varepsilon\), and depends only on the polynomials \(p_i\); for the minimal such \(r\), the integer \(c = r - 1\) is called the complexity of the system \(\{p_1, \ldots, p_k\}\) (see [L5]).

(Here is a sketch of the proof, for completeness; for more details see [HK1] and [BHK]. By [L3], there exists \(c \in \mathbb{N}\), which only depends on the polynomials \(p_i\), such that, if \((V, \nu, S)\) is an ergodic probability measure preserving system and \(Z_c(V)\) is the \(c\)-th Host-Kra-Ziegler factor of \(V\) and \(h_1, \ldots, h_k \in L^\infty(V)\) are such that \(E(h_i | Z_c(V)) = 0\) for some \(i\), then for any Følner sequence \((\Phi_N)\) in \(\mathbb{Z}^d\) one has \(\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_V S_{p_1(n)} h_1 \cdots S_{p_k(n)} h_k d\nu = 0\).

Applying this to the ergodic components of the system \((W \times W, \mu \times \mu, T \times T)\) and the functions \(h_i = f_i \otimes \tilde{f}_i, i = 1, \ldots, k\), we obtain that for any Følner sequence \((\Phi_N)\) in \(\mathbb{Z}^d\),

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \left| \int_{W \times W} T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k d\mu \right|^2
= \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{W \times W} T_{p_1(n)} f_1(x) \cdot T_{p_1(n)} \tilde{f}_1(y) \cdots T_{p_k(n)} f_k(x) \cdot T_{p_k(n)} \tilde{f}_k(y) d(\mu(x) \times \mu(y)) = 0
\]

whenever, for some \(i\), the function \(f_i \otimes \tilde{f}_i\) has zero conditional expectation with respect to almost all ergodic components of \(Z_c(W \times W)\). This is so if \(E(f_i | Z_{c+1}(W)) = 0\), and we obtain that the sequence \(\int_W T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k d\mu\) tends to zero in uniform density whenever \(E(f_i | Z_r(W)) = 0\) for some \(i\), where \(r = c + 1\). It follows that for any \(f_1, \ldots, f_k \in L^\infty(W)\) the sequence

\[
\int_W T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k d\mu - \int_{Z_r(W)} T_{p_1(n)} E(f_1 | Z_r(W)) \cdots T_{p_k(n)} E(f_k | Z_r(W)) d\mu_{Z_r(W)}
\]

tends to zero in uniform density. Now, \(Z_r(W)\) has the structure of the inverse limit of a sequence of \(r\)-step nilmanifolds on which \(T\) acts as a translation; given \(\varepsilon > 0\), we can therefore find an \(r\)-step nilmanifold factor \(X\) of \(W\) such that \(\| E(f_i | Z_r(W)) - E(f_i | X) \|_{L^\infty(W)} < \varepsilon / \prod_{j=1}^k \| f_j \|_{L^\infty(W)}\) for all \(i\). Putting \(\tilde{f}_i = E(f_i | X), i = 1, \ldots, k\), and denoting the translation induced by \(T\) on \(X\) by \(a\), we then have

\[
\left| \int_{Z_r(W)} T_{p_1(n)} E(f_1 | Z_r(W)) \cdots T_{p_k(n)} E(f_k | Z_r(W)) d\mu_{Z_r(W)} - \int_X a_{p_1(n)} \tilde{f}_1 \cdots a_{p_k(n)} \tilde{f}_k d\mu_X \right| < \varepsilon
\]

for all \(n\), which implies the assertion.)

So, there exists \(\lambda \in \mathbb{Z}\) such that \(\| \psi - (\psi + \lambda) \| < \varepsilon\). After replacing \(\tilde{f}_i\) by \(L^1\)-close continuous functions, we may assume that \(\tilde{f}_1, \ldots, \tilde{f}_k \in C(X)\), and still \(\| \varphi - (\psi + \lambda) \| < \varepsilon\).

Applying Theorem 3.3 to the nilmanifold \(X^k = G^k / \Gamma^k\), the diagonal subnilmanifold \(Y = \{(x, \ldots, x), \ x \in X\} \subseteq X^k\), the polynomial sequence \(g(n) = (a_{p_1(n)} \cdots, a_{p_k(n)}), n \in \mathbb{Z}^d\), in \(G^k\), and the function \(f(x_1, \ldots, x_k) = \tilde{f}_1(x_1) \cdots \tilde{f}_k(x_k) \in C(X^k)\), we obtain that \(\psi \in \mathcal{M}^\circ(r, s)\), so also \(\psi + \lambda \in \mathcal{M}^\circ(r, s)\). Since \(\varepsilon\) is arbitrary and, by Lemma 2.3, \(\mathcal{M}_{r,s}\) is the closure of \(\mathcal{M}_{r,s}^\circ\), we obtain:
Proposition 5.1. Let \((W, B, \mu, T)\) be an ergodic invertible probability measure preserving system, let \(f_1, \ldots, f_k \in L^\infty(W)\), and let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \to \mathbb{Z}\). Then the sequence \(\varphi(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k \, d\mu, \, n \in \mathbb{Z}^d\), is in \(\mathcal{M}\). If the complexity of the system \(\{p_1, \ldots, p_k\}\) is \(c\) and \(\deg p_i \leq s\) for all \(i\), then \(\varphi_n \in \mathcal{M}_{c+1,s}\).

Let now \((W, B, \mu, T)\) be a non-ergodic (or, rather, not necessarily ergodic) system. Let \(\mu = \int_\Omega \mu_\omega \, d\nu(\omega)\) be the ergodic decomposition of \(\mu\). For each \(\omega \in \Omega\), let \(\varphi_\omega(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k \, d\mu_\omega, \, n \in \mathbb{Z}^d\); then \(\omega \mapsto \varphi_\omega\) is a measurable mapping \(\Omega \to l^\infty\), and \(\varphi(n) = \int_\Omega \varphi_\omega(n) \, d\nu(\omega), \, n \in \mathbb{Z}^d\). By Proposition 5.1, for each \(\omega \in \Omega\) we have \(\varphi_\omega \in \mathcal{M}_{c+1,s} \subseteq \mathcal{M}_l\), where \(l = 2(c+1)s\). By Theorem 4.8 we obtain:

Theorem 5.2. Let \((W, B, \mu, T)\) be an invertible probability measure preserving system, let \(f_1, \ldots, f_k \in L^\infty(W)\), and let \(p_1, \ldots, p_k\) be polynomials \(\mathbb{Z}^d \to \mathbb{Z}\). Then the sequence \(\varphi(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k \, d\mu, \, n \in \mathbb{Z}^d\), is in \(\mathcal{M}\). If the complexity of the system \(\{p_1, \ldots, p_k\}\) is \(c\) and \(\deg p_i \leq s\) for all \(i\), then \(\varphi_n \in \mathcal{M}_l\), where \(l = 2(c+1)s\).

Since \(\mathcal{M} \subseteq \mathcal{P} + \mathbb{Z}\), where \(\mathcal{P}\) is the closure in \(l^\infty\) of the algebra of bounded generalized polynomials (see the last paragraph of Section 2), we get as a corollary:

Corollary 5.3. Up to a null-sequence, the sequence \(\varphi\) is uniformly approximable by generalized polynomials.

Bibliography


