Topological multiple recurrence for polynomial configurations in nilpotent groups

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Abstract

We establish a general multiple recurrence theorem for an action of a nilpotent group by homeomorphisms of a compact space. This theorem can be viewed as a nilpotent version of our recent polynomial Hales-Jewett theorem ([BL2]) and contains nilpotent extensions of many known "abelian" results as special cases.

0. Introduction

0.1. The celebrated van der Waerden theorem on arithmetic progressions, published in 1927 ([vdW]) states that if the set of integers is partitioned into finitely many classes then at least one of the classes contains arbitrarily long arithmetic progressions. A few years later Grünwald (=Gallai) obtained the following multidimensional extension of van der Waerden's theorem (see [R], p. 123).

0.2. Theorem. *Let $d \in \mathbb{N}$. For any finite coloring of $\mathbb{Z}^d$ and any finite set $E \subset \mathbb{Z}^d$ there exist $v \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that the set $v + nE = \{v + nz \mid z \in E\}$ is monochromatic.*

In [FW] Furstenberg and Weiss offered a new approach, based on methods of topological dynamics, to results of this type. A dynamical version of the Gallai theorem proved in [FW] (from which Theorem 0.2 can be easily derived) reads as follows:

0.3. Theorem. *Let (X, ρ) be a compact metric space and let $g_1, \ldots, g_k$ be commuting self-homeomorphisms of X. Then for any $\varepsilon > 0$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $\rho(g_i^n x, x) < \varepsilon$ for all $i = 1, \ldots, k$.*

0.4. More recently, a polynomial extension of Theorem 0.3 was proved in [BL1]:

Theorem. *Let (X, ρ) be a compact metric space, let $g_1, \ldots, g_l$ be commuting self-homeomorphisms of X and let $p_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq l$, be polynomials $\mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $p_{i,j}(0) = 0$. Then for any $\varepsilon > 0$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $\rho(g_1^{p_{i,1}(n)} \cdots g_l^{p_{i,l}(n)} x, x) < \varepsilon$ for all $i = 1, \ldots, k$.*

0.5. Corollary. *Let $d, k \in \mathbb{N}$ and let $P : \mathbb{Z}^k \longrightarrow \mathbb{Z}^d$ be a polynomial mapping satisfying $P(0) = 0$. Then for any finite coloring of $\mathbb{Z}^k$ and any finite set $E \subset \mathbb{Z}^k$ there exist $v \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that the set $v + P(nE)$ is monochromatic.*

It was S. Yuzvinsky who conjectured in 80's that Theorem 0.3 might be still true if one replaces the assumption of commutativity of the homeomorphisms $g_1, \dots, g_k$ by the condition that they generate a nilpotent group. Yuzvinsky's conjecture was confirmed in [L1], where the following "nilpotent" extension of Theorem 0.4 was proved.

0.6. Theorem. *Let self-homeomorphisms $g_1, \dots, g_l$ of a compact metric space (X, ρ) generate a nilpotent group and let $p_{i,j}$, $1 \le i \le k$, $1 \le j \le l$, be polynomials $\mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $p_{i,j}(0) = 0$. Then for any $\varepsilon > 0$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $\rho(g_1^{p_{i,1}(n)} \cdots g_l^{p_{i,l}(n)} x, x) < \varepsilon$ for all $i = 1, \dots, k$.*

0.7. Here is a combinatorial corollary of Theorem 0.6:

Corollary. *Let G be a nilpotent group, let $g_1, \dots, g_l \in G$ and let $p_{i,j}$, $1 \le i \le k$, $1 \le j \le l$, be polynomials $\mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $p_{i,j}(0) = 0$. For any finite coloring of G there exist $h \in G$ and $n \in \mathbb{N}$ such that the elements $hg_1^{p_{1,1}(n)} \cdots g_l^{p_{1,l}(n)}, \dots, hg_1^{p_{k,1}(n)} \cdots g_l^{p_{k,l}(n)}$ of G are all of the same color.*

0.8. While Theorem 0.6 provides a satisfactory result pertaining to finitely many homeomorphisms (or, equivalently, to partition theorems involving finitely generated nilpotent groups), it is desirable to have an extension of Theorem 0.3 which would deal with infinitely many homeomorphisms (and would have as combinatorial corollaries Ramsey-theoretical results about infinitely generated (semi)groups). One such extension, the (abelian) IP-van der Waerden theorem is contained in the paper of Furstenberg and Weiss alluded to above. To formulate it we need to recall the notion of IP-system, introduced in [FW]. Denote by $\mathcal{F}$ the set of finite subsets of $\mathbb{N}$. *An IP-system* in a commutative semigroup G (which should be viewed as a generalized sub-semigroup of G) is a mapping from $\mathcal{F}$ into G, $\alpha \mapsto g_\alpha$, $\alpha \in \mathcal{F}$, which satisfies $g_{\alpha \cup \beta} = g_\alpha g_\beta$ whenever $\alpha \cap \beta = \emptyset$. In particular, if $\{g_i\}_{i \in \mathbb{N}}$ is a sequence of elements of G, the IP-system generated by $\{g_i\}_{i \in \mathbb{N}}$ is the set of all products of the form $g_\alpha = \prod_{i \in \alpha} g_i$, $\alpha \in \mathcal{F}$. It is easy to see that any IP-system in G can be obtained in this way.

0.9. Theorem. ([FW]) *Let $\{g_\alpha^{(1)}\}_{\alpha \in \mathcal{F}}, \dots, \{g_\alpha^{(k)}\}_{\alpha \in \mathcal{F}}$ be IP-systems in an abelian group of self-homeomorphisms of a compact metric space (X, ρ). For any $\varepsilon > 0$ there exist $x \in X$ and a nonempty $\alpha \in \mathcal{F}$ such that $\rho(g_\alpha^{(i)} x, x) < \varepsilon$ for all $i = 1, \dots, k$.*

0.10. An equivalent combinatorial form of Theorem 0.9 reads as follows:

Theorem. *Let G be an abelian group, and let $\{g_\alpha^{(1)}\}_{\alpha \in \mathcal{F}}, \dots, \{g_\alpha^{(k)}\}_{\alpha \in \mathcal{F}}$ be IP-systems in G. For any finite coloring of G there exist $h \in G$ and a nonempty $\alpha \in \mathcal{F}$ such that the elements $hg_\alpha^{(1)}, \dots, hg_\alpha^{(k)}$ all have the same color.*

0.11. The following corollary of Theorem 0.10, which is a special case of the Geometric Ramsey Theorem, due to Graham, Leeb and Rothschild ([GLR]), deals with infinitely generated abelian groups of the form $\bigoplus K$, where K is (the additive group of) a finite field.

Theorem. *Let V be an infinite dimensional vector space over a finite field. Then for* *any finite coloring of V there are arbitrarily large monochromatic finite dimensional affine subspaces.*

For a derivation of this theorem from Theorem 0.9 see [B2].

0.12. Our goal in this paper is to establish a *nil-IP-multiple recurrence theorem* which would extend all the *abelian* results mentioned above to a nilpotent setup. We postpone the formulation of our main result (Theorem 0.24 below) and formulate first some of its corollaries. We start with nilpotent versions of Theorems 0.9 and 0.10:

0.13. Theorem. *Let G be a nilpotent group of self-homeomorphisms of a compact metric space (X,ρ) and let $g_j^{(i)}\in G$, $i=1,\ldots,k$, $j\in\mathbb{N}$. For any $\varepsilon>0$ there exist $x\in X$ and a nonempty finite set $\alpha\subset\mathbb{N}$ such that $\rho\bigl(\prod_{j\in\alpha}g_j^{(i)}x,x\bigr)<\varepsilon$ for all $i=1,\ldots,k$.*

0.14. Theorem. *Let G be a nilpotent group and let $g_j^{(i)}\in G$, $i=1,\ldots,k$, $j\in\mathbb{N}$. For any finite coloring of G there exist $h\in G$ and a nonempty finite set $\alpha\subset\mathbb{N}$ such that the elements $h\prod_{j\in\alpha}g_j^{(i)}$, $i=1,\ldots,k$, all have the same color.*

0.15. We will now discuss a nilpotent extension of Theorem 0.11. Let G be a nilpotent group with uniformly bounded torsion: for some $d\in\mathbb{N}$, $g^d=\mathbf{1}_G$ for all $g\in G$. Let a finite coloring of G be given. If G is "large" then, in accordance with the principles of Ramsey theory, one should be able to find in one color arbitrarily large "highly organized" configurations. In the case of our group G, which has uniformly bounded torsion, it is natural to look for monochromatic cosets of arbitrarily large subgroups. An even better result would be not only to get monochromatic cosets of arbitrarily large subgroups, but to have these subgroups to be as "noncommutative" as G is. We bring here two results of this type.

0.16. Theorem. *Let $q\in\mathbb{N}$ and let G be the (multiplicative) group of $(q+1)\times(q+1)$ upper triangular matrices with unit diagonal over an infinite field of finite characteristic. For any finite coloring of G and any $c\in\mathbb{N}$ there exists a subgroup H of G of nilpotency class q and of cardinality $\geq c$ such that for some $h\in G$ the coset hH is monochromatic. Moreover, one may require that not only H, but also all q nontrivial terms of its lower central series have cardinality $\geq c$.*

0.17. Let p be a prime integer and let q be an integer with $q<p$. Let us say that a group G is *a free q-step nilpotent group with torsion p* if G is defined by a generating set S and the following relations: $g^p=\mathbf{1}_G$ for all $g\in S$, and $\bigl[\ldots[[g_1,g_2],g_3],\ldots,g_{q+1}\bigr]=\mathbf{1}_G$ for all $g_1,\ldots,g_{q+1}\in G$. (Note that all elements of G have torsion p: $g^p=\mathbf{1}_G$ for all $g\in G$.) Free nilpotent groups may be viewed as nilpotent analogue of free abelian groups with torsion p (which are of the form $\bigoplus\mathbb{Z}_p$). The following fact demonstrates that free nilpotent groups with torsion have nice Ramsey-theoretical properties.

Theorem. *Let G be an infinite free q-step nilpotent group with torsion p. For any finite coloring of G and any $c\in\mathbb{N}$ there exists a free q-step nilpotent subgroup $H\subset G$ of cardinality $|H|\geq c$, such that for some $h\in G$ the coset hH is monochromatic.*

0.18. We will also be able to obtain a nilpotent version of a classical partition result of Hilbert which we will presently formulate. The following theorem, arguably the first non-trivial theorem of Ramsey-theoretical nature, is contained in [H] and reads as follows.

Theorem. ([H]) *For any finite coloring of $\mathbb{N}$ and for any $k \in \mathbb{N}$ there is a k-element set $\{n_1, \ldots, n_k\} \subset \mathbb{N}$ such that one can find in one color infinitely many translates of the set of finite sums* $\mathrm{FS}(\{n_j\}_{j=1}^k) = \left\{\sum_{j=1}^k \epsilon_j n_j \mid \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\right\}$.

(Hilbert needed this theorem in order to prove his irreducibility theorem, stating that if a polynomial $p(x,y) \in \mathbb{Z}[x,y]$ is irreducible then for some $x_0 \in \mathbb{N}$ the polynomial $p(x_0, y) \in \mathbb{Z}[y]$ is also irreducible. It is rather curious that although Hilbert's original proof of this theorem occupied more than 2 pages, a stronger result containing it as quite a special case can be proved in few lines by simply iterating a version of the Poincaré recurrence theorem (see [B1] Proposition 2.5 and Remark 2.6.))

0.19. Given a finite set $D = \{h_1, \ldots, h_k\}$ in a (non-abelian) group G, let $Q(D)$ denote the set of the products of $h_1, \ldots, h_k$ in all possible orders: $Q(D) = \left\{\prod_{j=1}^k h_{\sigma(j)} \mid \sigma \in S_k\right\}$. The following open question, dealing with a strong noncommutative generalization of Hilbert's theorem, very likely has a negative answer for general groups.

Question. Let $\{g_\alpha\}_{\alpha\in\mathcal{F}}$ be an IP-set in a group G. Is it true that for any finite coloring of G and any $k \in \mathbb{N}$ there exist a k-element set $D \subseteq \{g_\alpha\}_{\alpha\in\mathcal{F}}$ and $h \in G$ such that the set $hQ(D)$ is monochromatic?

0.20. For nilpotent groups the answer to Question 0.19 is positive:

Theorem. *Let G be an infinite nilpotent group, let $k \in \mathbb{N}$ and let $g_j^{(i)} \in G$, $1 \le i \le k$, $j \in \mathbb{N}$. For any r-coloring of G there exist a finite nonempty set $\alpha \subset \mathbb{N}$ and infinitely many $h \in G$ such that for $h_i = \prod_{j\in\alpha} g_j^{(i)}$, $i = 1, \ldots, k$, the products $hh_{i_1}h_{i_2}\ldots h_{i_l}$ with $0 \le l \le k$ and distinct $i_1, i_2, \ldots, i_l$ are all of the same color.*

0.21. In the abelian case one gets the proofs of the "linear" results, Theorem 0.9 and its corollaries, in a self-contained way. The situation is different in nilpotent case: the only known to us way of deriving the "linear facts", Theorems 0.13 – 0.20, is to obtain them as special cases of more general "polynomial" statements.

First, we formulate the general abelian polynomial IP-multiple recurrence theorem which extends both Theorem 0.4 and Theorem 0.9. Let, again, $\mathcal{F}$ be the set of all finite subsets of $\mathbb{N}$. A mapping P from $\mathcal{F}$ into a commutative (semi)group G is *an IP-polynomial of degree 0* if P is constant, and, inductively, is *an IP-polynomial of degree* $\le d$ if for any $\beta \in \mathcal{F}$ there exists an IP-polynomial $D_\beta P\colon \mathcal{F}(\mathbb{N}\setminus\beta) \longrightarrow G$ of degree $\le d-1$ (where $\mathcal{F}(\mathbb{N}\setminus\beta)$ is the set of finite subsets of $\mathbb{N}\setminus\beta$) such that $P(\alpha\cup\beta) = P(\alpha) + (D_\beta P)(\alpha)$ for every $\alpha \in \mathcal{F}$ with $\alpha\cap\beta = \emptyset$. (One easily checks that IP-systems introduced in 0.8 are just IP-polynomials of degree 1 satisfying $P(\emptyset) = \mathbf{1}_G$.)

0.22. Theorem. ([BL2]) *Let G be an abelian group of self-homeomorphisms of a compact metric space (X,ρ) and let $P_1, \ldots, P_k$ be IP-polynomials $\mathcal{F} \longrightarrow G$ satisfying $P_1(\emptyset) = \ldots = P_k(\emptyset) = \mathbf{1}_G$. For any $\varepsilon > 0$ there exist $x \in X$ and a nonempty $\alpha \in \mathcal{F}$ such that*

$\rho\big(P(\alpha)x, x\big) < \varepsilon$ *for all* $i = 1, \ldots, k$.

0.23. If G is an abelian group, it is proven in [BL2], Theorem 8.3, that a mapping $P\colon \mathcal{F} \longrightarrow G$ is an IP-polynomial of degree $\leq d$ with $P(\emptyset) = \mathbf{1}_G$ if and only if there exists a family $\{g_{(j_1,\ldots,j_d)}\}_{(j_1,\ldots,j_d)\in\mathbb{N}^d}$ of elements of G such that for any $\alpha \in \mathcal{F}$ one has $P(\alpha) = \prod_{(j_1,\ldots,j_d)\in\alpha^d} g_{(j_1,\ldots,j_d)}$. It is this characterization of commutative IP-polynomials which makes sense in the nilpotent setup as well. Namely, let G be a nilpotent group. We will call a mapping $P\colon \mathcal{F} \longrightarrow G$ *an IP-polynomial* if for some $d \in \mathbb{N}$ there exist a family $\{g_{(j_1,\ldots,j_d)}\}_{(j_1,\ldots,j_d)\in\mathbb{N}^d}$ of elements of G and a linear order $\prec$ on $\mathbb{N}^d$ such that for any $\alpha \in \mathcal{F}$ one has $P(\alpha) = \prod^{\prec}_{(j_1,\ldots,j_d)\in\alpha^d} g_{(j_1,\ldots,j_d)}$. (The entries in the product $\prod^{\prec}$ are multiplied in accordance with $\prec$.)

0.24. We may now formulate our main result, which generalizes both Theorem 0.6 and Theorem 0.13:

Theorem. *Let G be a nilpotent group of self-homeomorphisms of a compact metric space (X, ρ) and let $P_1, \ldots, P_k\colon \mathcal{F} \longrightarrow G$ be polynomial mappings satisfying $P_1(\emptyset) = \ldots = P_k(\emptyset) = \mathbf{1}_G$. For any $\varepsilon > 0$, there exist $x \in X$ and a nonempty $\alpha \in \mathcal{F}$ such that $\rho\big(P_i(\alpha)x, x\big) < \varepsilon$ for all $i = 1, \ldots, k$.*

0.25. The structure of the paper is as follows. In Sections 1 and 2 we introduce the necessary definitions and establish some facts about polynomial mappings. Sections 3 and 4 are devoted to the proof of our main theorem (Theorem 4.1). In Section 5 we consider various corollaries of the main theorem. Finally, in Section 6 we make concluding remarks and formulate some natural conjectures.

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1. Polynomial mappings $\mathcal{F}(S) \longrightarrow G$

1.1. Given a set T, $\mathcal{F}(T)$ will denote the set of all finite subsets of T, $\mathcal{F}^{=d}(T)$ the set of all subsets of T of cardinality d, $\mathcal{F}^{\leq d}(T)$ the set of all subsets of T of cardinality $\leq d$. In particular, $\mathcal{F}^{=0}(T) = \{\emptyset\}$.

Let $\{g_t\}_{t\in T}$ be a collection of elements of a group G indexed by a finite set $T = \{t_1, \ldots, t_n\}$, let $\prec$ be a linear order on T (or on some superset of T). Let $i_1, i_2, \ldots, i_n \in \{1, \ldots, n\}$ satisfy $i_1 \prec i_2 \prec \ldots \prec i_n$; we define $\prod^{\prec}_{t\in T} g_t = g_{t_{i_1}} g_{t_{i_2}} \cdots g_{t_{i_n}}$. If T is empty, we put $\prod^{\prec}_{t\in T} g_t = \mathbf{1}_G$. When the order $\prec$ is fixed or does not matter (for example, when G is abelian), we will sometimes write $\prod_{t\in T} g_t$ instead of $\prod^{\prec}_{t\in T} g_t$.

1.2. Let G be a group. *The commutator* $[x, y]$ of elements $x, y \in G$ is $x^{-1}y^{-1}xy$; so $xy = yx[x, y]$. For two subsets $A, B \subseteq G$, their commutator $[A, B]$ is the subgroup generated by the set $\big\{[x, y] \bigm| x \in A,\ y \in B\big\}$. *The lower central series of* G is the sequence $G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$ of normal subgroups of G defined inductively: $G = G_1$, $G_{k+1} = [G, G_k]$,

$k = 1, 2, \ldots$. It is well known that $[G_k, G_l] \subseteq G_{k+l}$ for any $k, l \in \mathbb{N}$.

A group G is called *nilpotent* if its lower central series is finite: *G is nilpotent of class q*, or *q-step nilpotent*, if $G_{q+1} = \{\mathbf{1}_G\}$ and $G_q \neq \{\mathbf{1}_G\}$. In particular, abelian groups are nilpotent of class 1.

1.3. Let S be a nonempty set, let G be a nilpotent group. *A monomial of degree d on S with values in G* is a pair $(u, \prec)$ consisting of a mapping $u: S^d \longrightarrow G$ and a linear order $\prec$ on S^d. When it can not lead to confusion, we will omit $\prec$ and write u instead of $(u, \prec)$. A monomial $(u, \prec)$ induces *a monomial mapping* $P_u: \mathcal{F}(S) \longrightarrow G$ by the rule $P_u(\alpha) = \prod_{s \in \alpha^d}^{\prec} u(s)$, $\alpha \in \mathcal{F}(S)$.

1.4. Examples. Constant mappings $\mathcal{F}(S) \longrightarrow G$ are monomial of degree 0: they are induced by monomials $S^0 = \{\emptyset\} \longrightarrow G$.

Given a sequence $\{g_i\}_{j \in \mathbb{N}}$ of elements of G, the mapping $P: \mathcal{F}(\mathbb{N}) \longrightarrow G$ defined by $P(\alpha) = \prod_{j \in \alpha} g_j$ is monomial of degree 1.

Let $g \in G$; put $P(\alpha) = g^{|\alpha|^2}$ for $\alpha \in \mathcal{F}(S)$ (where $|\alpha|$ denotes the cardinality of α). Then $P: \mathcal{F}(S) \longrightarrow G$ is a monomial mapping of degree 2: it is induced by the constant monomial which equals g on S^2.

1.5. Note that the monomial mapping P_u induced by a monomial $(u, \prec)$ of degree d is also induced by a monomial $(u', \prec')$ of degree $d+1$ which can be constructed as follows. Fix any $s_0 \in S$, put

$$u'(s_1, \ldots, s_d, s_{d+1}) = \begin{cases} u(s_1, \ldots, s_d) \text{ if } s_{d+1} = s_0 \\ \mathbf{1}_G \text{ otherwise.} \end{cases}$$

Then define an order $\prec'$ on $S^d \times \{s_0\}$ by $\big(s_1 \times \{s_0\}\big) \prec' \big(s_2 \times \{s_0\}\big)$ if $s_1 \prec s_2$, and lift $\prec'$ to any linear order on S^{d+1}.

1.6. Note also that the composition of a monomial mapping and a group homomorphism is a monomial mapping as well: if $P_u: \mathcal{F}(S) \longrightarrow G$ is a monomial mapping induced by a monomial $(u, \prec)$ and $\varphi: G \longrightarrow G'$ is a homomorphism of nilpotent groups, then $\varphi \circ P_u$ is induced by the monomial $(\varphi \circ u, \prec)$.

1.7. *The level* of a monomial $(u, \prec)$ of degree d is the positive integer l satisfying $u(S^d) \subseteq G_l \setminus G_{l+1}$; we will denote the level of $(u, \prec)$ by $l(u)$. If G is nilpotent of class q, then $1 \leq l(u) \leq q$ for nontrivial monomials u; we define the level of the trivial monomial, $u(s) = \mathbf{1}_G$ for all $s \in S$, to be $q+1$. *The weight $w(u)$ of* $(u, \prec)$ is the pair $(l(u), d)$. The set W of weights of monomials, that is, the set of pairs (l, d) with $l, d \in \mathbb{Z}$, $1 \leq l \leq q$, $d \geq 0$, is well-ordered by the rule: $(l_1, d_1) \leq (l_2, d_2)$ if either $l_1 > l_2$, or $l_1 = l_2$ and $d_1 \leq d_2$.

1.8. *A polynomial mapping* $P: \mathcal{F}(S) \longrightarrow G$ is the product of finitely many monomial mappings: $P(\alpha) = P_{u_1}(\alpha) \ldots P_{u_m}(\alpha)$, $\alpha \in \mathcal{F}(S)$, where $P_{u_1}, \ldots, P_{u_m}$ are monomial mappings, corresponding to monomials $(u_1, \prec_1), \ldots, (u_m, \prec_m)$. *The weight $w(P)$* of a polynomial mapping P is the minimum, taken over the set of all representations of P as the product $P = P_{u_1} \ldots P_{u_m}$ of monomial mappings, of the maximum of the weights $w(u_i)$, $i = 1, \ldots, m$, of monomials participating in this representation. If $w(P) = (l, d)$, we will call l *the level of* P and denote it by $l(P)$, and we will call d *the degree of* P. A polynomial mapping of level l takes values in the subgroup G_l of G.

1.9. Given a set S and a nilpotent group G, polynomial mappings $\mathcal{F}(S) \longrightarrow G$ form a group with respect to the element-wise multiplication. Indeed, it is clear from definition that the product PQ, $(PQ)(\alpha) = P(\alpha)Q(\alpha)$ of polynomial mappings P and Q is polynomial as well. The inverse P_u^{-1}, $P_u^{-1}(\alpha) = P_u(\alpha)^{-1}$, of the monomial mapping P_u induced by a monomial $(u, \prec)$ of degree d, is also a monomial mapping: it is induced by the monomial $(u^{-1}, \succ)$, where $u^{-1}(s) = u(s)^{-1}$ and $\succ$ is the order on S^d which is inverse to $\prec$.

1.10. Lemma. *Let* $P, Q\colon \mathcal{F}(S) \longrightarrow G$ *be polynomial mappings. Then*
(i) $w(P^{-1}) = w(P)$*;*
(ii) $w(PQ) \le \max\big(w(P), w(Q)\big)$*;*
(iii) *if* $w(Q) < w(P)$*, then* $w(PQ) = w(QP) = w(P)$.

Proof. Assertions (i) and (ii) follow from the definition. To prove (iii) note that the assumption $w(PQ) < w(P)$ leads to a contradiction, since it implies that

$$w(P) = w(PQQ^{-1}) \le \max\big(w(PQ), w(Q^{-1})\big) < w(P).$$

■

1.11. Corollary. *Given a weight* (l, d)*, polynomial mappings* $P\colon \mathcal{F}(S) \longrightarrow G$ *with* $w(P) \le (l, d)$ *form a group.*

1.12. The following proposition describes the basic properties of monomial mappings: it tells us that if G is a nilpotent group then certain elementary operations with monomial mappings taking values in G are "nilpotent": they are trivial modulo polynomial mappings of higher levels.

Proposition. *Let* S *be a set and* G *be a nilpotent group.*
(i) *Let* $(u, \prec)$ *and* $(u, \prec')$ *be two monomials of weight* (l, d)*, given by the same mapping* $u\colon S^d \longrightarrow G$ *and different linear orders* $\prec, \prec'$ *on* S^d*, and let* P *and* P' *be the corresponding monomial mappings. Then* $P = P'Q$ *where* Q *is a polynomial mapping* $\mathcal{F}(S) \longrightarrow G$ *with* $l(Q) > l$.
(ii) *Let* $(u_1, \prec_1)$ *and* $(u_2, \prec_2)$ *be two monomials on* S *with values in* G*, and let* P_1 *and* P_2 *be the corresponding monomial mappings. Then* $P_1P_2 = P_2P_1Q$*, where* Q *is a polynomial mapping* $\mathcal{F}(S) \longrightarrow G$ *with* $l(Q) > \max\big(l(P_1), l(P_2)\big)$.
(iii) *Let* $u_1, u_2\colon \mathcal{F}(S^d) \longrightarrow G$ *be two mappings, let* $\prec$ *be a linear order on* S^d*, and let* P_1*,* P_2 *and* P *be the monomial mappings induced by the monomials* $(u_1, \prec)$*,* $(u_2, \prec)$ *and* $(u_1u_2, \prec)$ *respectively. Then* $P = P_1P_2Q$*, where* Q *is a polynomial mapping* $\mathcal{F}(S) \longrightarrow G$ *with* $l(Q) > \max\big(l(P_1), l(P_2)\big)$.

1.13. The formal proof of Proposition 1.12 is cumbersome, but its idea is simple: interchanging two products $\prod_{s\in A} g_s$ and $\prod_{t\in B} h_t$ of elements of G creates commutator expressions indexed by products of several copies of A and B:

$$\prod_{s\in A} g_s \prod_{t\in B} h_t = \prod_{t\in B} h_t \prod_{s\in A} g_s \prod_{(s,t)\in A\times B} [g_s, h_t] \prod_{\substack{(s_1,s_2,t)\in A^2\times B \\ s_1\prec s_2}} \big[[g_{s_1}, h_t], g_{s_2}\big] \prod \cdots$$

and leads to appearance of monomial mappings of higher levels.

To clarify the idea of the proof we first give the proof in the case where G has nilpotency class 2 (that is, $G_2 = [G,G]$ is contained in the center of G).

(i) Let $(u,\prec)$ and $(u,\prec')$ be two monomials of weight $(1,d)$ and let P and P' be the corresponding monomial mappings. Then for $\alpha \in \mathcal{F}(S)$ we have

$$P'(\alpha) = \prod_{s\in\alpha^d}^{\prec'} u(s) = \prod_{s\in\alpha^d}^{\prec} u(s) \prod_{\substack{s,t\in\alpha^d\\ s\prec' t\\ t\prec s}} \big[u(s),u(t)\big] = P(\alpha)Q(\alpha),$$

where Q is the monomial mapping induced by the monomial

$$(s,t) \mapsto \begin{cases} \big[u(s),u(t)\big] \text{ if } s\prec' t \text{ and } t\prec s\\ \mathbf{1}_G \text{ otherwise}\end{cases}$$

which is of degree $2d$ and of level ≥ 2. (The order on S^{2d} does not matter since the range of this monomial lies in the abelian group G_2.)

(ii) Let $(u_1,\prec_1)$ and $(u_2,\prec_2)$ be monomials, $u_1\colon S^{d_1}\longrightarrow G$, $u_2\colon S^{d_2}\longrightarrow G$, and P_1 and P_2 be the corresponding monomial mappings. Then for $\alpha\in\mathcal{F}(S)$

$$P_1(\alpha)P_2(\alpha) = \prod_{s\in\alpha^{d_1}}^{\prec_1} u_1(s) \prod_{s\in\alpha^{d_2}}^{\prec_2} u_2(s) = \prod_{s\in\alpha^{d_1}}^{\prec_2} u_2(s) \prod_{s\in\alpha^{d_2}}^{\prec_1} u_1(s) \prod_{\substack{s\in\alpha^{d_1}\\ t\in\alpha^{d_2}}} \big[u_1(s),u_2(t)\big]$$
$$= P_2(\alpha)P_1(\alpha)Q(\alpha),$$

where Q is the monomial mapping induced by the monomial $(s,t)\mapsto \big[u_1(s),u_2(t)\big]$ of degree d_1+d_2 and level ≥ 2.

(iii) Let P_1, P_2 and P be the monomial mappings induced respectively by monomials $(u_1,\prec)$, $(u_2,\prec)$ and $(u_1u_2,\prec)$, where $u_1,u_2\colon S^d\longrightarrow G$. Then for $\alpha\in\mathcal{F}(S)$

$$P(\alpha) = \prod_{s\in\alpha^d}^{\prec} u_1(s)u_2(s) = \prod_{s\in\alpha^d}^{\prec} u_1(s) \prod_{s\in\alpha^d}^{\prec} u_2(s) \prod_{\substack{s,t\in\alpha^d\\ t\prec s}} \big[u_2(t),u_1(s)\big] = P_1(\alpha)P_2(\alpha)Q(\alpha),$$

where Q is the monomial mapping induced by the monomial

$$(s,t) \mapsto \begin{cases} \big[u_2(t),u_1(s)\big] \text{ if } t\prec s\\ \mathbf{1}_G \text{ otherwise.}\end{cases}$$

1.14. Proof of Proposition 1.12. We confine ourselves to the proof of statement (i); the proofs of (ii) and (iii) are similar. Let G be a nilpotent group of class q.

We introduce first some notation. Given a set B, denote by $C(B)$ the set of words in the alphabet $B\cup\{[\}\cup\{,\}\cup\{]\}$ defined inductively: $B\subset C(B)$, and if $c_1,c_2\in C(B)$ then $[c_1,c_2]\in C(B)$. $C(B)$ is "the set of commutators with entries from B". For example, if $b_1,b_2\in B$, then $[b_1,b_2]\in C(B)$ and $\big[b_1,[b_2,b_1]\big]\in C(B)$. Also notice that

$C\big(C(B)\big) = C(B)$. Let *the depth* $d(c)$ of $c \in C(B)$ be defined by $d(b) = 1$ for $b \in B$ and $d([c_1, c_2]) = d(c_1) + d(c_2)$ for $c_1, c_2 \in C(B)$. (Examples: if $b_1, b_2 \in B$, then $d\big([b_1, b_2]\big) = 2$, $d\big(\big[b_1, [b_2, b_1]\big]\big) = 3$.)

Now, let $u: B \longrightarrow G$ be a mapping. We can lift u onto $C(B)$ by putting $u\big([c_1, c_2]\big) = \big[u(c_1), u(c_2)\big] \in G$, $c_1, c_2 \in C(B)$. (Note that if $d(c) > q$, then $u(c) = \mathbf{1}_G$.) Let $D \subseteq C(B)$ and let $\prec$ be a linear order on D; then for any $\alpha \in \mathcal{F}(B)$ we put $u_\alpha(D, \prec) = \prod^{\prec}_{c \in D \cap C(\alpha)} u(c)$. Let $D_1, D_2 \subseteq B$ and let $\prec_1$ and $\prec_2$ be linear orders on D_1 and D_2 respectively. Then we will write $u(D_1, \prec_1) = u(D_2, \prec_2)$ if $u_\alpha(D_1, \prec_1) = u_\alpha(D_2, \prec_2)$ for all $\alpha \in \mathcal{F}(B)$. (Example: let $B = \{b_1, b_2\}$, $D_1 = \{b_1, b_2\}$, $b_1 \prec_1 b_2$, and $D_2 = \big\{b_1, b_2, [b_1, b_2]\big\}$, $b_2 \prec_2 b_1 \prec_2 [b_1, b_2]$. Then $u(D_1, \prec_1) = u(D_2, \prec_2)$ for any u.)

1.15. Lemma. *For any linear orders $\prec_1, \prec_2$ on a set B there exist $D \subseteq C(B)$ with $B \subseteq D$, and a linear order $\prec$ on D such that $\prec|_B = \prec_2$, $b \prec c$ for any $b \in B$, $c \in D \setminus B$, and $u(B, \prec_1) = u(D, \prec)$ for any $u: B \longrightarrow G$.*

Proof. The idea of the proof is to "place" the elements of B in accordance with $\prec_1$ and then "move" them to the left in accordance with $\prec_2$; when $b \in B$ passes a commutator $c \in C(B)$, we replace c, b by $b, c, [c, b]$. To put this more formally, we define

$$D = B \cup \Big\{ \big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big] \;\Big|\; k \in \mathbb{N},\ b, b_1, b_2, \ldots, b_k \in B,\\ b \prec_1 b_1,\ b \prec_1 b_2, \ldots,\ b \prec_1 b_k, \quad b_1 \prec_2 b, \quad b_1 \prec_2 b_2 \prec_2 \ldots \prec_2 b_k \Big\},$$

and define a linear order $\prec$ on D as follows:

$$b \prec c \text{ for any } b \in B,\ c \in C(B) \setminus B; \qquad \text{for } b, b' \in B,\ b \prec b' \text{ iff } b \prec_2 b';$$

$$\big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big] \prec \big[\ldots\big[\big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big], b_{k+1}\big], \ldots, b_l\big];$$

$$\big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big] \prec \big[\ldots\big[[b', b'_1], b'_2\big], \ldots, b'_l\big] \text{ iff } b \prec b';$$

$$\big[\ldots\big[\big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big], b_{k+1}\big], \ldots, b_l\big] \prec \big[\ldots\big[\big[\ldots\big[[b, b_1], b_2\big], \ldots, b_k\big], b'_{k+1}\big], \ldots, b'_m\big]\\ \text{iff } b'_{k+1} \prec_2 b_{k+1}.$$

■

Let $R = C(\{*\})$. (R is the set of "commutator patterns"; for example, $* \in R$, $\big[*, [*, *]\big] \in R$.) For $c \in C(B)$ we will say that c *has type* r, $r \in R$, if after replacing all B-entries of c by $*$, c transforms into r. (Example: for $b_1, b_2 \in B$, b_1 has type $*$ and $\big[[b_1, b_2], b_1\big]$ has type $\big[[*, *], *\big]$.) On the other hand, every $r \in R$ defines a mapping $B^{d(r)} \longrightarrow C(B)$ which can be described as follows: $*(b) = b$ and

$$[r_1, r_2](b_1, b_{r_1}, b_{r_1+1}, \ldots, b_{d(r_1)+d(r_2)}) = \big[r_1(b_1, \ldots, b_{d(r_1)}), r_2(b_{d(r_1+1}, \ldots, b_{d(r_1)+d(r_2)})\big]$$

($*$-s are consecutively replaced in r by $b_1, \ldots, b_{d(r)}$). Let $R_q = \big\{r \in R \;\big|\; d(r) \le q\big\}$. R_q is a finite set, let $R_q = \{r_0, r_1, \ldots, r_k\}$ with $r_0 = *$ and $d(r_{i-1}) \le d(r_i)$ for all $i = 1, \ldots, k$.

Now all the preparatory work has been done, and we pass to the proof of Proposition 1.12(i). Let u be a mapping $S^d \longrightarrow G$, let $\prec, \prec'$ be linear orders on S^d and let P, P' be the monomial mappings $\mathcal{F}(S^d) \longrightarrow G$ induced by the monomials $(u, \prec)$ and $(u, \prec')$

respectively. First, applying Lemma 1.15 to $B_0 = S^d$, find $D_1 \subseteq C(B_0)$, $B_0 \subseteq D_1$, with a linear order $\prec_1'$ on D_1 such that $\prec_1' |_{B_0} = \prec'$, $s \prec_1' c$ for any $s \in B_0$, $c \in D_1 \setminus B_0$, and $u(B_0, \prec) = u(D_1, \prec_1')$. Let $B_1 = D_1 \setminus B_0$. Then for any $\alpha \in \mathcal{F}(S^d)$ we have

$$P(\alpha) = u_\alpha(B_0, \prec) = u_\alpha(D_1, \prec_1') = u_\alpha(B_0, \prec')u_\alpha(B_1, \prec_1') = P'(\alpha)u_\alpha(B_1, \prec_1').$$

Now we will "separate" commutators which have type r_1. Let $S_1 = \{c \in B_1 \mid c \text{ has type } r_1\}$. Introduce any linear order $\prec_1$ on B_1 which satisfies $c_1 \prec_1 c_2$ for any $c_1 \in S_1$, $c_2 \in B_1 \setminus S_1$. Applying Lemma 1.15 to B_1, find $D_2 \subseteq C(B_1) \subseteq C(S^d)$, $B_1 \subseteq D_2$, and a linear order $\prec_2'$ on D_2 such that $\prec_2' |_{B_1} = \prec_1$, $b \prec_2' c$ for any $b \in B_1$, $c \in D_2 \setminus B_1$, and $u(B_1, \prec_1') = u(D_1, \prec_2')$. Define a monomial $u_1 \colon (S^d)^{d(r_1)} \longrightarrow G$ by

$$u_1(s_1, \dots, s_{d(r_1)}) = \begin{cases} r_1\big(u(s_1), \dots, u(s_{d(r_1)})\big) \text{ if } r_1(s_1, \dots, s_{d(r_1)}) \in S_1 \\ \mathbf{1}_G \text{ otherwise} \end{cases}$$

and by the order $\prec_1$, and let P_1 be the monomial mappings induced by u_1. Let $B_2 = D_2 \setminus B_1$. Then for any $\alpha \in \mathcal{F}(S^d)$ we have

$$u_\alpha(B_1, \prec_1') = u_\alpha(D_2, \prec_2') = u_\alpha(S_1, \prec_1)u_\alpha(B_2, \prec_2') = P_1(\alpha)u_\alpha(B_2, \prec_2'),$$

and hence, $P(\alpha) = P'(\alpha)P_1(\alpha)u_\alpha(B_2, \prec_2')$. Note also that, since $d(c) > 1$ for all $c \in S_1$, $l(P_1) > l(u)$.

After repeating this procedure $k-1$ more times, that is, after consecutively separating commutators having types $r_1, \dots, r_k$, we arrive at the representation $P(\alpha) = P'(\alpha)P_1(\alpha)\dots P_k(\alpha)u_\alpha(B_{k+1}, \prec_{k+1}')$, $\alpha \in \mathcal{F}(S^d)$, where B_{k+1} consists of commutators of depth $> q$. Hence, the last term of this product vanishes, and we get $P = P'P_1 \dots P_k$. ■

1.16. Corollary. *Let P_1, P_2 be polynomial mappings $\mathcal{F}(S) \longrightarrow G$. Then $P_1P_2 = P_2P_1Q$, where Q is a polynomial mapping of level $l(Q) > \max\big(l(P_1), l(P_2)\big)$.*

1.17. Corollary. *The group of polynomial mappings $\mathcal{F}(S) \longrightarrow G$ is nilpotent (and has the same nilpotency class as G).*

1.18. Corollary. *Every polynomial mapping $P\mathcal{F}(S) \longrightarrow G$ can be represented in the form $P = P_uQ$, where P_u is a monomial mapping induced by a monomial u of weight $w(u) = w(P)$ and Q is a polynomial mapping with $l(Q) > l(P)$.*

Proof. Let $w(P) = (l, d)$ and let $P_{u_1} \dots P_{u_m}$ be the "minimal" representation of P. That is, let $P_{u_1}, \dots, P_{u_m}$ be the monomial mappings corresponding to monomials $(u_1, \prec_1), \dots, (u_m, \prec_m)$ with $w(u_i) \leq (l, d)$, $i = 1, \dots, m$. Let $(u_{i_1}, \prec_{i_1}), \dots, (u_{i_t}, \prec_{i_t})$ be the monomials whose level is l. By 1.5, we may assume that all these monomials are of the same degree d. Choose a linear order $\prec$ on S^d, and using Proposition 1.12 (i), replace all $\prec_{i_1}, \dots, \prec_{i_t}$ by $\prec$ using the identity $P_{u_{i_j}} = P_jQ_j$, $j = 1, \dots, t$, where P_j is the monomial mapping induced by the monomial $(u_{i_j}, \prec)$ and Q_j is a polynomial mapping of level $> l$. Using Proposition 1.12 (ii), write $P = P_1 \dots P_tQ$ with $l(Q) > l$. Now, by Proposition 1.12 (iii) and (ii), $P_1 \dots P_t = P_uQ'$, where P_u is the monomial mapping induced by the monomial $(u_{i_1} \dots u_{i_t}, \prec)$. ■

2. Triangular monomials

A monomial u carries superfluous information in comparison with the corresponding monomial mapping P. Indeed, in every product $P(\alpha)=\prod_{s\in\alpha^d}u(s)$ an entry $u(s_1,\dots,s_d)$ appears together with $u(s_{\sigma(1)},\dots,s_{\sigma(d)})$ for all permutations σ of $(1,\dots,d)$. We will now introduce a more compact "encoding" of monomial mappings. As before, let S be a set and G be a nilpotent group of class q.

2.1. *A triangular monomial of degree* d is the pair $(v,\prec)$ where v is a mapping $\mathcal{F}^{=d}(S)\longrightarrow G$ and $\prec$ is a linear order on $\mathcal{F}^{=d}(S)$. A triangular monomial $(v,\prec)$ *induces* a mapping $P_v\colon\mathcal{F}(S)\longrightarrow G$ by the rule

$$P_v(\alpha)=\prod_{t\in\mathcal{F}^{=d}(\alpha)}^{\prec}v(t).$$

It is clear that P_v is a monomial mapping. Indeed, let $<$ be a linear order on S. Then $\mathcal{F}^{=d}(S)$ can be embedded into S^d by $\{s_1,\dots,s_d\}\longrightarrow(s_1,\dots,s_d)$ under the assumption $s_1<s_2<\dots<s_d$. (This embedding is the source of the term "triangular".) Now, put $u(s)=v(s)$ for $s\in\mathcal{F}^{=d}(S)$ and $u(s)=\mathbf{1}_G$ for $s\in S^d\setminus\mathcal{F}^{=d}(S)$, and lift the order $\prec$ from $\mathcal{F}(S^d)$ to a linear order on S^d. Then the obtained monomial $(u,\prec)$ induces the mapping P_v.

On the other hand, any monomial mapping can be represented as a product of monomial mappings induced by triangular monomials. Indeed, let $(u,\prec)$ be a monomial of degree d and let P_u be the corresponding monomial mapping. For $s\in S^d$, let t_s be the set of entries of s (for example, $t_{(1,2,2,1)}=\{1,2\}$). Let, for each $i=d,d-1,\dots,0$, $\prec_i$ be a linear order on $\mathcal{F}^{=i}(S)$. Introduce a new linear order $\prec'$ on S^d in the following way:
(i) if $|t_{s_1}|>|t_{s_2}|$ then $s_1\prec' s_2$;
(ii) if $|t_{s_1}|=|t_{s_2}|=i$ and $t_{s_1}\prec_i t_{s_2}$, then $s_1\prec' s_2$;
(iii) if $t_{s_1}=t_{s_2}$, then $s_1\prec' s_2$ iff $s_1\prec s_2$. Let P'_u be the monomial mapping induced by the monomial $(u,\prec')$. Then for $\alpha\in\mathcal{F}(S)$,

$$\begin{aligned}P'_u(\alpha)&=\prod_{s\in\alpha^d}^{\prec'}u(s)=\prod_{\substack{s\in\alpha^d\\|t_s|=d}}^{\prec'}u(s)\prod_{\substack{s\in\alpha^d\\|t_s|=d-1}}^{\prec'}u(s)\dots\prod_{\substack{s\in\alpha^d\\|t_s|=0}}^{\prec'}u(s)\\&=\prod_{t\in\mathcal{F}^{=d}(\alpha)}^{\prec_d}\prod_{s:t_s=t}^{\prec'}u(s)\prod_{t\in\mathcal{F}^{=d-1}(\alpha)}^{\prec_{d-1}}\prod_{s:t_s=t}^{\prec'}u(s)\dots\prod_{t\in\mathcal{F}^{=0}(\alpha)}^{\prec_0}\prod_{s:t_s=t}^{\prec'}u(s).\end{aligned}$$

For $t\in\mathcal{F}^{=i}(S)$ put $v_i(t)=\prod_{s:t_s=t}^{\prec'}u(s)$, $i=d,d-1,\dots,0$. Then the triangular monomials $(v_i,\prec_i)$, $i=d,d-1,\dots,0$, induce monomial mappings P_i such that $P'_u=P_dP_{d-1}\dots P_0$. By Proposition 1.12, $P_u=P'_uQ$ where Q is a polynomial mapping with $l(Q)>l(u)$. We arrive at the following fact:

2.2. Proposition. *Every polynomial mapping P, $w(P)=(l,d)$, is representable in the form $P=P_dP_{d-1}\dots P_0Q$, where for each $i=d,d-1,\dots,0$, P_i is either the monomial mapping induced by a triangular monomial of weight (l,i) or is trivial, and Q is a polynomial mapping of level $>l$.*

Proof. By Corollary 1.18 P can be represented in the form $P = P_u P'$, where P_u is the monomial mapping induced by a monomial of weight (l,d), and P' is a polynomial mapping of weight $< (l,d)$. Write $P_u = P_v P''$, where P_v is the monomial mapping induced by a triangular monomial of degree d, and $w(P'') < (l,d)$. Then $P = P_v P'' P'$, with $w(P''P') < (l,d)$, and we may apply induction on the weight of P. Moreover, $w(P_v) = (l,d)$, since we would have $w(P) < (l,d)$ otherwise. ∎

2.3. The representation of a polynomial mapping P in the form $P = P_d P_{d-1} \dots P_0 Q$, where for each $i = d, d-1, \dots, 0$, P_i is the monomial mapping induced by a triangular monomial v_i of degree i and Q is a polynomial mapping of a higher level, is still not unique. The reason for this is the freedom in choosing an order on $\mathcal{F}^{=i}(S)$: if we change the order on $\mathcal{F}^{=i}(S)$ corresponding to some of v_i, it will affect the mapping Q. However, this representation is uniquely defined if we deal with an abelian group, because in this case Q is trivial:

Proposition. *Let S be a set, H be an abelian group and $P\colon \mathcal{F}(S) \longrightarrow H$ be a polynomial mapping of weight (l,d). Then P is uniquely representable in the form $P = P_d P_{d-1} \dots P_0$, where P_i is the monomial mapping induced by a triangular monomial of degree i, $i = d, d-1, \dots, 0$.*

Proof. The uniqueness of this representation follows by induction on i from the formula

$$P(\alpha) = v_i(\alpha) \prod_{\beta \subset \alpha} \bigl(P_{i-1}(\beta) \dots P_0(\beta)\bigr) \tag{2.1}$$

for $\alpha \in \mathcal{F}^{=i}(S)$. ∎

2.4. It follows that, in the case of an abelian group G, any polynomial mapping of degree d from $\mathcal{F}(S)$ to G is defined by its values at subsets of S of cardinality $\leq d$:

Corollary. *Let S be a set and H be an abelian group. If polynomial mappings $P, P'\colon \mathcal{F}(S) \longrightarrow H$ coincide on $\mathcal{F}^{\leq d}(S)$, then $P = P'$.*

Proof. Write $P = P_d \dots P_0$, $P' = P'_d P'_{d-1} \dots P'_0$, where for each $i = d, d-1, \dots, 0$, P_i, P'_i are the monomial mappings induced by, respectively, triangular monomials v_i, v'_i of degree i. Now, it follows from (2.1) by induction on i that $v_i = v'_i$ for all $i = 0, 1, \dots, d$. ∎

2.5. Let us return to the case of a general (nonabelian) nilpotent group. We have defined the weight $w(P)$ of a polynomial mapping P as the minimal possible weight of the "senior" monomial in a representation of P as a product of monomial mappings. If $w(P) = (l,d)$, we have $P(\mathcal{F}(S)) \subseteq G_l$. But we may not be sure that, in fact, $P(\mathcal{F}(S))$ is not contained in G_{l+1}. (Compare with conventional polynomials: for $p(x) = x^2 + x - x^2 + 1$ the degree of p is less than 2, though its senior term has degree 2.) We will now show that the representation of P described in Proposition 2.2 gives the "correct" weight of P. We fix a set S and a nilpotent group G of class q and consider polynomial mappings $\mathcal{F}(S) \longrightarrow G$.

Lemma. *If P is a nontrivial polynomial mapping of level $l \leq q$, then $P(\mathcal{F}(S)) \subseteq G_l \backslash G_{l+1}$.*

Proof. Write $P = P_{v_d}P_{v_{d-1}}\dots P_{v_0}Q$, where P_{v_i}, $i = d, d-1, \dots, 0$, is the monomial mapping induced by a triangular monomial v_i of weight (l,i), and Q is a polynomial mapping of level $> l$. We may assume that v_d has level l. Let $\varphi\colon G_l \longrightarrow G_l/G_{l+1}$ be the mapping of factorization. Assume that $P(\mathcal{F}(S)) \subseteq G_{l+1}$. Then we have $\mathbf{1}_{G_l/G_{l+1}} = P_{\varphi\circ v_d}P_{\varphi\circ v_{d-1}}\dots P_{\varphi\circ v_0}$ for the monomial mappings $P_{\varphi\circ v_i} = \varphi\circ P_{v_i}$, $i = d, d-1, \dots, 0$, induced by the triangular monomials $\varphi\circ v_i$, taking values in the abelian group G_l/G_{l+1}. Since $\varphi\circ v_d$ is nontrivial, this is impossible by Proposition 2.3. ∎

2.6. Corollary. *Let $P = P_{v_d}P'$, where P_{v_d} is the monomial mapping induced by a triangular monomial v_d of weight (l,d), and P' is a polynomial mapping of weight $< (l,d)$. Then $w(P) = (l,d)$.*

Proof. We have $w(P) \le (l,d)$ by definition. Assume that $w(P) < (l,d)$. Then $w(P_{v_d}) = w(PP'^{-1}) < (l,d)$ as well. Write $P_{v_d} = P_{v_{d-1}}P_{v_{d-2}}\dots P_{v_0}Q$, where P_{v_i}, $i = d-1, d-2, \dots, 0$, is the monomial mapping induced by a triangular monomial v_i of weight (l,i), and Q is a polynomial mapping of level $> l$. Let $\varphi\colon G_l \longrightarrow G_l/G_{l+1}$ be the mapping of factorization. Then we have $P_{\varphi\circ v_d} = P_{\varphi\circ v_{d-1}}P_{\varphi\circ v_{d-2}}\dots P_{\varphi\circ v_0}$, which is impossible by Proposition 2.3 since $\varphi\circ v_d$ is nontrivial. ∎

3. The principal part of a polynomial mapping, systems and PET-induction

In this section, we fix a set S and a nilpotent group G of class q.

3.1. Let $P\colon\mathcal{F}(S) \longrightarrow G$ be a polynomial mapping of weight (l,d). Represent P in the form $P = P_vQ$, where P_v is the monomial mapping induced by a triangular monomial v, $w(v) = (l,d)$, and Q is a polynomial mapping of weight $< (l,d)$. Let $\varphi\colon G_l \longrightarrow G_l/G_{l+1}$ be the mapping of factorization. We call the mapping $\varphi\circ v\colon S^d \longrightarrow G_l/G_{l+1}$ *the principal part of* P and denote it by $M(P)$. We will say that polynomial mappings P and P' are *equivalent* and write $P \sim P'$ if $w(P) = w(P')$ and their principal parts coincide: $M(P) = M(P')$. We define *the weight* of an equivalence class of polynomial mappings as the weight of any of its members.

3.2. Proposition. *Let $P, P'\colon\mathcal{F}(S) \longrightarrow G$ be polynomial mappings. Then $P \sim P'$ if and only if $w(P^{-1}P') < w(P)$.*

(For comparison: if p and p' are conventional polynomials, then p and p' have equal senior terms if and only if $\deg(p - p') < \deg(p)$.)

Proof. Let $P \sim P'$. Write $P = P_vQ$, $P' = P_{v'}Q'$, where P_v and $P_{v'}$ are the monomial mappings induced by triangular monomials v and v' of weight (l,d) and Q, Q' are polynomial mappings of weights $< (l,d)$. Then by Proposition 1.12, $P^{-1}P' = P_{v^{-1}v'}Q''$, where $P_{v^{-1}v'}$ is the monomial mapping induced by the monomial $v^{-1}v'$ and Q'' has weight $< (l,d)$. Since $\varphi\circ(v^{-1}v') = (\varphi\circ v)^{-1}(\varphi\circ v') = \mathbf{1}_{G_l/G_{l+1}}$, the range of $v^{-1}v'$ lies in G_{l+1} and so, $v^{-1}v'$ has level $\ge l+1$.

Now, let $w(P^{-1}P') < w(P) = (l,d)$. By Lemma 1.10(iii), $w(P') = (l,d)$ as well. As before, represent $P = P_vQ$ and $P' = P_{v'}Q'$, $w(v) = w(v') = (l,d)$ and $w(Q) < (l,d)$, $w(Q') < (l,d)$. Then $P^{-1}P' = P_{v^{-1}v'}Q''$, where $w(Q'') < (l,d)$ and $v^{-1}v'$ is a monomial mapping of degree d. Since we are given that $w(P^{-1}P') < (l, d)$, it follows from Corollary 2.6 that $l(v^{-1}v') > l$. Hence, $v^{-1}v'$ is trivial modulo G_{l+1}. ■

3.3. Proposition. (i) *If $P, Q: \mathcal{F}(S) \longrightarrow G$ are polynomial mappings with $w(Q) < w(P)$, then $PQ \sim P$.*
(ii) *If $P_1, P_2, Q: \mathcal{F}(S) \longrightarrow G$ are polynomial mappings such that $P_1 \sim P_2$ and $P_1 \not\sim Q$, then $Q^{-1}P_1 \sim Q^{-1}P_2$.*
(iii) *For any polynomial mappings $P, Q: \mathcal{F}(S) \longrightarrow G$ one has $Q^{-1}PQ \sim P$.*

(For comparison: (i) if p, q are conventional polynomials with $\deg(q) < \deg(p)$, then the senior terms of $p + q$ and p coincide; (ii) if the senior terms of polynomials p_1 and p_2 are equal but differ from the senior term of a polynomial q, then the senior terms of the polynomials $p_1 + q$ and $p_2 + q$ are equal.)

Proof. (i) is obvious: multiplying by Q does not affect the principal part of P. Under the assumptions of (ii), if $w(Q) \neq w(P_1)$, then (ii) follows from (i). If $w(Q) = w(P_1) = w(P_2)$, we have $M(Q^{-1}P_1) = M(Q)^{-1}M(P_1) = M(Q)^{-1}M(P_2) = M(Q^{-1}P_2)$, since this mapping is nontrivial.

To prove (iii), write $Q^{-1}PQ = PQ^{-1}QQ' = PQ'$, where $w(Q') < w(P)$ by Corollary 1.16, and use (i). ■

3.4. Let $\gamma \in \mathcal{F}(S)$, let P be a mapping $\mathcal{F}(S) \longrightarrow G$. Define $U_\gamma P: \mathcal{F}(S \setminus \gamma) \longrightarrow G$ by $U_\gamma P(\alpha) = P(\alpha \cup \gamma)$.

Proposition. *Let P be a polynomial mapping and $\gamma \in \mathcal{F}(S)$. Then $U_\gamma P$ is a polynomial mapping and $U_\gamma P \sim P|_{\mathcal{F}(S \setminus \gamma)}$.*

(For comparison: if p is a conventional polynomial, then $p(x)$ and $p(x + c)$ have equal senior terms.)

Proof. We may assume that P is the monomial mapping induced by a triangular monomial $(v, \prec)$ of weight (l, d). Moreover, we may assume that the order $\prec$ on $\mathcal{F}^{=d}(S)$ is such that (i) $|s_1 \cap \gamma| < |s_2 \cap \gamma|$ implies $s_1 \prec s_2$, and (ii) elements of $\mathcal{F}^{=d}(S)$ whose intersections with γ are equal "arise in succession", that is, if $s_1 \cap \gamma = s_2 \cap \gamma$ and $s_1 \prec s_3 \prec s_2$, then $s_3 \cap \gamma = s_1 \cap \gamma$. Then for any $\alpha \in \mathcal{F}(S \setminus \gamma)$,

$$
\begin{aligned}
&U_\gamma P(\alpha) = P(\alpha \cup \gamma) \\
&= \prod_{t \in \mathcal{F}^{=d}(\alpha)}^{\prec} v(t) \Big(\prod_{r \in \mathcal{F}^{=1}(\gamma)}^{\prec} \prod_{t \in \mathcal{F}^{=d-1}(\alpha)}^{\prec} v(r \cup t) \prod_{r \in \mathcal{F}^{=2}(\gamma)}^{\prec} \prod_{t \in \mathcal{F}^{=d-2}(\alpha)}^{\prec} v(r \cup t) \ldots \prod_{r \in \mathcal{F}^{=d}(\gamma)}^{\prec} v(r) \Big).
\end{aligned}
$$

We have $\prod_{t \in \mathcal{F}^{=d}(\alpha)}^{\prec} v(\alpha) = P(\alpha)$, and the expression in the large parentheses is a polynomial mapping of weight $< (l, d)$. ■

3.5. Corollary. *$w(P^{-1}U_\gamma P) < w(P)$.*

3.6. Remark. Given $\gamma \in \mathcal{F}(S)$, define on the set of polynomial mappings $P\colon \mathcal{F}(S) \longrightarrow G$ an operator of "differentiation" D_γ by $D_\gamma P = P^{-1}U_\gamma P$. It follows from Corollary 3.5 that every polynomial mapping P cancels out after applying to P several differential operators of the form D_γ, $\gamma \in \mathcal{F}(S)$: there exist $k \in \mathbb{N}$ such that for any pairwise disjoint $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathcal{F}(S)$, $D_{\gamma_k} D_{\gamma_{k-1}} \ldots D_{\gamma_1} P \equiv \mathbf{1}_G$. In fact, polynomial mappings are characterized by this property.

3.7. Given a set S and a nilpotent group G, we call a nonempty finite set of polynomial mappings $\mathcal{F}(S) \longrightarrow G$ *a system.*

Denote by W the set of weights of polynomials $\mathcal{F}(S) \longrightarrow G$, that is, the set of pairs (l, d) with $l, d \in \mathbb{Z}$, $1 \leq l \leq q$, $d \geq 0$. Let $\mathcal{A}$ be a system; *the weight vector $\omega(\mathcal{A})$ of $\mathcal{A}$* is a function $W \longrightarrow \{0, 1, 2, \ldots\}$ defined by $\omega(\mathcal{A})(w) =$ the number of equivalence classes of polynomial mappings $\mathcal{F}(S) \longrightarrow G$ of weight w having its representatives in $\mathcal{A}$. Since $\mathcal{A}$ is finite, $\omega(\mathcal{A})$ has a finite support. We order the weight vector lexicographically: $\omega(\mathcal{A}) < \omega(\mathcal{A}')$ if for some $w \in W$ one has $\omega(\mathcal{A})(w) < \omega(\mathcal{A}')(w)$ and $\omega(\mathcal{A})(w') = \omega(\mathcal{A}')(w')$ for all $w' > w$. We say that *a system $\mathcal{A}$ precedes a system $\mathcal{A}'$* if $\omega(\mathcal{A}) < \omega(\mathcal{A}')$.

3.8. We will prove our main result, Theorem 4.1 below, by utilizing the so-called *PET-induction*, the induction on the well ordered set of weight vectors. Our application of the PET-induction is based on the following lemma:

Lemma. *Let S be a set, let G be a nilpotent group and let $\mathcal{A}$ be a system of polynomial mappings $\mathcal{F}(S) \longrightarrow G$.*
(i) *If $\gamma \in \mathcal{F}(S)$ and a system $\mathcal{A}'$ of polynomial mappings $\mathcal{F}(S \setminus \gamma) \longrightarrow G$ is such that each element of $\mathcal{A}'$ is equivalent to $P|_{\mathcal{F}(S\setminus\gamma)}$ for some $P \in \mathcal{A}$, then $\omega(A') \leq \omega(A)$.*
(ii) *If a system $\mathcal{A}'$ consists of polynomial mappings of the form $Q^{-1}PQ$ where $P \in \mathcal{A}$ and Q is a polynomial mapping $\mathcal{F}(S) \longrightarrow G$, then $\omega(\mathcal{A}') \leq \omega(\mathcal{A})$.*
(iii) *If $\mathcal{A}'$ is formed by polynomial mappings of the form PQ and QP where $P \in \mathcal{A}$ and Q is a polynomial mapping $\mathcal{F}(S) \longrightarrow G$ with $w(Q) < w(P)$, then $\omega(\mathcal{A}') \leq \omega(\mathcal{A})$.*
(iv) *Let $Q \in \mathcal{A}$ be a nontrivial polynomial mapping with $w(Q) \leq w(P)$ for all $P \in \mathcal{A}$. If $\mathcal{A}'$ is a system of polynomial mappings of the form $Q^{-1}P$ and PQ^{-1}, then $\omega(\mathcal{A}') < \omega(\mathcal{A})$.*

Proof. (i) is clear from the definition. (ii) and (iii) easily follow from Proposition 3.3 (iii) and (i) respectively. In (iv), the equivalence classes in $\mathcal{A}$ change when we pass to $\mathcal{A}'$, but the equivalence of elements is preserved and their weights remain the same by Proposition 3.3 (ii) and Proposition 3.2. The only exception is the equivalence class containing Q: it splits into equivalence classes having smaller weights. ∎

4. The multiple recurrence theorem

4.1. Our main result is the following theorem:

Theorem. *Let G be a nilpotent group of self-homeomorphisms of a compact metric space (X, ρ). For any weight $w \in W$, any $k \in \mathbb{N}$ and any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that if S is a set of cardinality $\geq N$ and $\mathcal{A}$ is a system of k polynomial mappings $\mathcal{F}(S) \longrightarrow G$ satisfying $w(P) \leq w$ and $P(\emptyset) = \mathrm{Id}_X$, $P \in \mathcal{A}$, then there exist a point $x \in X$ and a nonempty*

$\alpha \in \mathcal{F}(S)$ *such that* $\rho\big(P(\alpha)x, x\big) < \varepsilon$ *for all* $P \in \mathcal{A}$.

Proof. We may assume that X is *minimal* with respect to the action of G, that is, that X does not contain proper nonempty closed G-invariant subsets. Note that, under the assumption of minimality of X, we can strengthen the theorem: we can claim that the set of points $x \in X$ satisfying the requirement of the theorem is dense in X. Indeed, let $\varepsilon > 0$ be given and let $U \subseteq X$ be an open set. Since X is minimal under the action of G, the G-invariant closed subset $X \setminus \bigcup_{g\in G} g^{-1}(U)$ is empty. Thus one can choose $g_1, \ldots, g_n \in G$ such that $\bigcup_{i=1}^{n} g_i^{-1}(U) = X$. Let $\delta > 0$ be such that $\rho(x_1, x_2) < \delta$ implies $\rho(g_i x_1, g_i x_2) < \varepsilon$ for all $i = 1, \ldots, n$. Now, given a system $\mathcal{A}$, let $x \in X$ and $n \in \mathcal{F}(S)$ satisfy the conclusion of the theorem for the system $\bigcup_{i=1}^{n} g_i \mathcal{A} g_i^{-1}$ (which has the same weight as $\mathcal{A}$) and δ, that is, for all $P \in \mathcal{A}$ and all $1 \leq i \leq n$, let $\rho(g_i^{-1}P(n)g_i x, x) < \delta$. Then $\rho(P(n)g_i x, g_i x) < \varepsilon$ for all $P \in \mathcal{A}$ and $i = 1, \ldots, n$, and one of the points $g_1 x, \ldots, g_n x$ lies in U.

We will prove the theorem by PET-induction, the induction on the weight vector of the system. The statement of the theorem is trivial for the system $\mathcal{A} = \{I\}$, where I is the trivial mapping; this gives the basis of the PET-induction. Assume that we are given w, k and ε, that $\mathcal{A}$ is a k-element system of polynomial mappings with $w(P) \leq w$ and $P(\emptyset) = I$ for all $P \in \mathcal{A}$, and that the theorem holds for all systems preceding $\mathcal{A}$. We may also assume that $\mathcal{A}$ does not contain constant mappings.

Let $Q \in \mathcal{A}$ be an element of the minimal weight in $\mathcal{A}$. By Lemma 3.8, the system $\mathcal{A}_1 = \big\{PQ^{-1} \bigm| P \in \mathcal{A}\big\}$ precedes $\mathcal{A}$. The PET-induction hypothesis implies that there is $N_1 \in \mathbb{N}$ such that whenever $|S| \geq N_1$, there exist $y_0 \in X$ and a nonempty $\alpha_1 \in \mathcal{F}(S)$ satisfying $\rho\big(P(\alpha_1)Q^{-1}(\alpha_1)y_0, y_0\big) < \varepsilon/2$ for all $P \in \mathcal{A}$. Assuming $|S| > N_1$, choose a subset $S_1 \subseteq S$ with $|S_1| = N_1$, and find such $y_0 \in X$ and $\alpha_1 \in \mathcal{F}(S_1)$. Put $x_0 = y_0$ and $x_1 = Q(\alpha_1)^{-1}y_0$, then $\rho\big(P(\alpha_1)x_1, x_0\big) < \varepsilon/2$ for all $P \in \mathcal{A}$.

Now, let δ_1, $0 < \delta_1 < \varepsilon/4$, be such that $\rho(x, x_1) < \delta_1$ implies $\rho\big(P(\alpha_1)x, x_0\big) < \varepsilon/2$ for all $P \in \mathcal{A}$. By Lemma 3.8 and Proposition 3.4, the system

$$\mathcal{A}_2 = \big\{PQ^{-1}, P(\alpha_1)^{-1}(U_{\alpha_1}P)Q^{-1} \bigm| P \in \mathcal{A}\big\}$$

precedes $\mathcal{A}$. Thus, by induction hypothesis there is $N_2 \in \mathbb{N}$ such that if $|S \setminus S_1| \geq N_2$, then there are $y_1 \in X$ and a nonempty $\alpha_2 \in \mathcal{F}(S \setminus S_1)$ such that $\rho\big(R(\alpha_2)y_1, y_1\big) < \varepsilon/4$. Furthermore, since we assume X to be minimal under the action of G, y_1 can be found in the δ_1-neighborhood U of x_1. Choose $S_2 \subseteq S \setminus S_1$ with $|S_2| = N_2$, find $y_1 \in U$ and $\alpha_2 \in \mathcal{F}(S_2)$, and put $x_2 = Q(\alpha_2)^{-1}y_1$. Then $\rho\big(P(\alpha_2)x_2, y_1\big) < \varepsilon/4$ and so, $\rho\big(P(\alpha_2)x_2, x_1\big) < \varepsilon/2$ for all $P \in \mathcal{A}$. Also, $\rho\big(P(\alpha_1)^{-1}P(\alpha_1 \cup \alpha_2)x_2, x_1\big) < \delta_1$, and hence, by the choice of δ_1, $\rho\big(P(\alpha_1 \cup \alpha_2)x_2, x_0\big) < \varepsilon/2$ for all $P \in \mathcal{A}$.

Continuing this process, we find $N_1, N_2, \ldots \in \mathbb{N}$, disjoint $S_1, S_2, \ldots \subseteq S$ with $|S_j| = N_j$, $x_0, x_1, x_2, \ldots \in X$ and a nonempty $\alpha_1 \in \mathcal{F}(S_1), \alpha_2 \in \mathcal{F}(S_2), \ldots$ such that for any $0 \leq l < m$,

$$\rho\big(P(\alpha_{l+1} \cup \ldots \cup \alpha_m)x_m, x_l\big) < \varepsilon/2$$

for all $P \in \mathcal{A}$. Let K be the cardinality of a finite $\frac{\varepsilon}{2}$-net in X. Then there exist $0 \leq l < m \leq K$ for which $\rho(x_l, x_m) < \varepsilon/2$. For $x = x_m$ and $\alpha = \alpha_{l+1} \cup \ldots \cup \alpha_m$ we will have $\rho(P(\alpha)x, x) < \varepsilon$, and for all this to be done it is enough to have $|S| \geq N_1 + \ldots + N_K$. ■

4.2. In order to derive a "coloring" version of Theorem 4.1, fix $r \in \mathbb{N}$ and consider the set Ω of all r-colorings of a nilpotent group G, that is, the set of all mappings from G to a fixed r-element set. Without loss of generality we may assume that G is countable, $G = \{g_1, g_2, \ldots\}$. A metric ρ on Ω is introduced by $\rho(\chi_1, \chi_2) = 1/k$, where k is the minimal integer for which $\chi_1(g_k) \neq \chi_2(g_k)$; this turns Ω into a compact metric space. G acts on Ω by $(g\chi)(h) = \chi(hg)$.

Given an r-coloring χ of G, denote by X the closure of its orbit $G\chi$ in Ω. Let S be a set and let $P_1, \ldots, P_k \colon \mathcal{F}(S) \longrightarrow G$ be polynomial mappings satisfying $P_i(\emptyset) = \mathbf{1}_G$, $i = 1, \ldots, k$. Applying Theorem 4.1 to X (under the assumption that S is large enough) find a coloring $\chi' \in X$ and a nonempty set $\alpha \in \mathcal{F}(S)$ such that the colorings $P_1(\alpha)\chi', \ldots, P_k(\alpha)\chi'$ are all close to χ':

$$\chi'(\mathbf{1}_G) = P_i(\alpha)\chi'(\mathbf{1}_G) = \chi'\big(P_i(\alpha)\big), \ i = 1, \ldots, k.$$

Find $h \in G$ for which $h\chi$ is close enough to χ': $h\chi\big(P_i(\alpha)\big) = \chi'\big(P_i(\alpha)\big)$, $i = 1, \ldots, k$. Then $\chi\big(P_i(\alpha)h\big)$, $i = 1, \ldots, k$, do all coincide.

4.3. We have obtained the following theorem:

Theorem. *Let G be a nilpotent group. For any $w \in W$ and any $k, r \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if S is a set of cardinality $\geq N$ and $P_1, \ldots, P_k$ are polynomial mappings $\mathcal{F}(S) \longrightarrow G$ which satisfy $w(P_i) \leq w$ and $P_i(\emptyset) = \mathbf{1}_G$, $i = 1, \ldots, k$, then for any r-coloring of G there exist a nonempty $\alpha \in \mathcal{F}(S)$ and $h \in G$ such that the elements $P_1(\alpha)h, \ldots, P_k(\alpha)h$ have the same color.*

4.4. Of course, in the formulation of Theorem 4.4 the element h can be placed on the left of P_i:

Theorem. *Let G be a nilpotent group. For any $w \in W$ and any $k, r \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if S is a set of cardinality $\geq N$ and $P_1, \ldots, P_k$ are polynomial mappings $\mathcal{F}(S) \longrightarrow G$ which satisfy $w(P_i) \leq w$ and $P_i(\emptyset) = \mathbf{1}_G$, $i = 1, \ldots, k$, then for any r-coloring of G there exist a nonempty $\alpha \in \mathcal{F}(S)$ and $h \in G$ such that the elements $hP_1(\alpha), \ldots, hP_k(\alpha)$ have the same color.*

Proof. Let χ be a finite coloring of G. Put $P_i' = P_i^{-1}$, $i = 1, \ldots, k$, and consider the coloring χ' of G defined by $\chi'(g) = \chi(g^{-1})$. Find $h' \in G$ and $n \in \mathcal{F}(S)$ such that $P_1'(n)h', \ldots, P_k'(n)h'$ have the same color with respect to χ'. Then for $h = h'^{-1}$, $hP_1(n), \ldots, hP_k(n)$ have the same color with respect to χ. ∎

4.5. Also notice that if G is infinite, then $h\chi$ is close to χ' for infinitely many $h \in G$. This implies that, in the case of an infinite G, one can find a nonempty $\alpha \in \mathcal{F}(S)$ and infinitely many $h \in G$ for which $hP_i(\alpha)$, $i = 1, \ldots, k$, have the same color:

Theorem. *Let G be an infinite nilpotent group. For any $w \in W$ and any $k, r \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if S is a set of cardinality $\geq N$ and $P_1, \ldots, P_k$ are polynomial mappings $\mathcal{F}(S) \longrightarrow G$ which satisfy $w(P_i) \leq w$ and $P_i(\emptyset) = \mathbf{1}_G$, $i = 1, \ldots, k$, then for any r-coloring of G there exist a nonempty $\alpha \in \mathcal{F}(S)$ and infinitely many $h \in G$ such that the elements $hP_1(\alpha), \ldots, hP_k(\alpha)$ have the same color.*

5. Applications

We will now derive from our main combinatorial results, Theorems 4.4 and 4.5, a few combinatorial corollaries (some of which were mentioned in the introduction).

5.1. The following is the "linear" case of Theorem 4.4:

Theorem. *Let G be a nilpotent group. For any $k, r \in \mathbb{N}$ and linear orders $\prec_1, \dots, \prec_k$ on $\mathbb{N}$ there is $N \in \mathbb{N}$ such that for any r-coloring, $G = \bigcup_{m=1}^r C_m$, of G and any k collections $g^{(i)} = \{g_j^{(i)}\}_{j=1}^N$, $i = 1, \dots, k$, of N elements from G, there exist $m \in \{1, \dots, r\}$, a nonempty set $\alpha \subseteq \{1, \dots, N\}$ and $h \in G$ such that $h \prod_{j\in\alpha}^{\prec_1} g_j^{(1)}, \dots, h \prod_{j\in\alpha}^{\prec_k} g_j^{(k)} \in C_m$. If G is inifnite, there exist $m \in \{1, \dots, r\}$ and a nonempty $\alpha \subseteq \{1, \dots, N\}$ for which the set $\Big\{h \mid h \prod_{j\in\alpha}^{\prec_1} g_j^{(1)}, \dots, h \prod_{j\in\alpha}^{\prec_k} g_j^{(k)} \in C_m\Big\}$ is infinite.*

5.2. Theorem 5.1 is a special case of the following statement:

Theorem. *Let G be a nilpotent group, let F be the free group generated by a (finite) set $\{z_1, \dots, z_t\}$, let $E \subset F$ be finite, let $\prec_1, \dots, \prec_t$ be linear orders on $\mathbb{N}$ and let $r \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for any r-coloring, $G = \bigcup_{m=1}^r C_m$, of G and any $g_j^{(i)} \in G$, $1 \le i \le t$, $1 \le j \le N$, there exist $m \in \{1, \dots, r\}$, a nonempty set $\alpha \subseteq \{1, \dots, N\}$ and $h \in G$ such that for the homomorhism $\varphi: F \longrightarrow G$ defined by $\varphi(z_i) = \prod_{j\in\alpha}^{\prec_i} g_j^{(i)}$, $i = 1, \dots, t$, one has $h\varphi(E) \subseteq C_m$. If G is infinite then there exist $m \in \{1, \dots, r\}$ and a nonempty $\alpha \subseteq \{1, \dots, N\}$ for which the set $\big\{h \in G \bigm| h\varphi(E) \subseteq C_m\big\}$ is infinite.*

5.3. For example, taking E to be $\{z_1 z_2^2 z_1^{-3}, z_2^{-1} z_1^2 z_2\}$, one can find N such that for any r-coloring of G and any $g_1^{(1)}, \dots, g_N^{(1)}, g_1^{(2)}, \dots, g_N^{(2)}$ there exist $1 \le j_1 < \dots < j_l \le N$, $1 \le m \le r$ and $h \in G$ such that for $h_1 = g_{j_1}^{(1)} \dots g_{j_l}^{(1)}$ and $h_2 = g_{j_1}^{(2)} \dots g_{j_l}^{(2)}$, the products $hh_1 h_2^2 h_1^{-3}$ and $hh_2^{-1} h_1^2 h_2$ have the same color.

5.4. Proof of Theorem 5.2. Let F be the free group generated by $\{z_1, \dots, z_m\}$, let E be a finite subset of F and let χ be an r-coloring of G. Let N satisfy the conclusion of Theorem 4.4 for $w = (1, 1)$, $k = |E|$ and the given r. Put $S = \{1, \dots, N\}$. Given $g_j^{(i)} \in G$, $1 \le i \le m$, $1 \le j \le N$, for each $i = 1, \dots, m$ define a monomial $u_i: S \longrightarrow G$ by $u_i(j) = g_j^{(i)}$, and let P_i be the monomial mapping induced by u_i. Then every element $z \in F$ defines a polynomial mapping $P_z: S \longrightarrow G$ in the following way: for $z = \prod_{t=1}^l z_{i_t}^{\epsilon_t}$, $\epsilon_t = \pm 1$, let $P_z = \prod_{t=1}^l P_{i_t}^{\epsilon_t}$. Now Theorem 4.4, applied to the system $\mathcal{A} = \{P_z,\ z \in E\}$, gives the desired result. If G is infinite, Theorem 4.5 is applicable. ■

5.5. From Theorem 5.2 one derives Theorem 0.20, the nilpotent generalization of Hilbert's theorem:

Theorem. *Let G be an infinite nilpotent group. For any $k, r \in \mathbb{N}$ there exist $N \in \mathbb{N}$ such that for any $g_j^{(i)} \in G$, $1 \le i \le k$, $1 \le j \le N$, and any r-coloring of G there exist a nonempty $\alpha \subseteq \{1, \dots, N\}$ and infinitely many $h \in G$ such that for $h_i = \prod_{j\in\alpha} g_j^{(i)}$, $i = 1, \dots, k$, the products $hh_{i_1} h_{i_2} \dots h_{i_l}$ with $0 \le l \le k$ and distinct $i_1, i_2, \dots, i_l$, are all of* *the same color.*

Indeed, the theorem follows if in the notation of Theorem 5.2 we put $t = k$, let each of $\prec_1, \ldots, \prec_t$ be the natural order on $\mathbb{N}$ and let E be the set of all possible products of distinct $z_{i_1}, \ldots, z_{i_l}$ with $l \leq k$.

5.6. We now turn to Theorems 0.16 – 0.17. Let G be a nilpotent group with bounded torsion: $g^d = \mathbf{1}_G$ for all $g \in G$. Then any finitely generated subgroup H of G is finite. (If H is generated by a finite set $\{h_1, \ldots, h_t\}$ and G has nilpotency class q, then $|H| < (t + t^2 + \ldots + t^q)^d$.) Moreover, there is a finite set E of words in the alphabet $\{z_1, \ldots, z_t\}$ such that whenever H is a group generated by $h_1, \ldots, h_t \in G$, and φ is the homomorphism from the free group F generated by $\{z_1, \ldots, z_t\}$ into G which maps z_l to h_l, $l = 1, \ldots, t$, one has $\varphi(E) = H$. Therefore, by Theorem 5.2 for any r there exists N such that, given an r-coloring of G and t N-element sequences $g^{(i)} = \{g_j^{(i)}\}_{j=1}^N$, $i = 1, \ldots, t$, in G, one can find a nonempty set $\alpha \subseteq \{1, \ldots, N\}$ such that the group H generated by $h_1 = \prod_{j\in\alpha} g_j^{(1)}, \ldots, h_t = \prod_{j\in\alpha} g_j^{(t)}$ has a monochromatic coset. It only remains to choose the elements $g_j^{(i)}$ which would guarantee that the rest of the requirements of Theorems 0.16 and 0.17 are satisfied.

5.7. For a group H let H_q be the q-th term of the lower central series of H.

Theorem. *Let $q \in \mathbb{N}$ and let G be the multiplicative group of $(q + 1) \times (q + 1)$ upper triangular matrices with unit diagonal over an infinite field F of finite characteristic. For any finite coloring of G and any $c \in \mathbb{N}$ there exists a subgroup H of G with $|H_q| \geq c$ such that the cosets hH of H are monochromatic for infinitely many $h \in G$.*

Proof. Let $t \in \mathbb{N}$ satisfy $\binom{t}{q} \geq c$ and let $N \in \mathbb{N}$ be large enough so that the conclusion of 5.6 is valid. For $\emptyset \neq \alpha \subseteq \{1, \ldots, N\}$, $1 \leq i_1 < \ldots < i_q \leq t$ and $x_j^{(i)} \in G$, $i = 1, \ldots, q$, $j = 1, \ldots, N$, let $R_{\alpha,i_1,\ldots,i_q}(x_1^{(1)}, \ldots, x_N^{(t)})$ be the upper-right corner entry of the commutator expression $\big[\ldots\big[\big[\prod_{j\in\alpha} x_j^{(i_1)}, \prod_{j\in\alpha} x_j^{(i_2)}\big], \prod_{j\in\alpha} x_j^{(i_3)}\big], \ldots, \prod_{j\in\alpha} x_j^{(i_q)}\big]$. For $\emptyset \neq \alpha \subseteq \{1, \ldots, N\}$ and $1 \leq i_1, \ldots, i_q \leq t$, $R_{\alpha,i_1,\ldots,i_q}$ are distinct nonzero polynomials over F in the entries of the matrices $x_j^{(i)}$. Order the pairs $(i, j) \in \mathbb{N}^2$ lexicographically. Having $g_j^{(i)} \in G$ with $(i, j) < (l, n)$ already chosen, find $g_n^{(l)} \in G$ so that the polynomials $R_{\alpha,i_1,\ldots,i_q}(g_1^{(1)}, \ldots, g_n^{(l)}, x_{n+1}^{(l)}, \ldots, x_N^{(t)})$ (to $x_j^{(i)}$ with $(i, j) \leq (l, n)$ the value $g_j^{(i)}$ is assigned) for $\emptyset \neq \alpha \subseteq \{1, \ldots, N\}$ and $1 \leq i_1, \ldots, i_q \leq t$, are all distinct. Then for any nonempty $\alpha \subseteq \{1, \ldots, N\}$ and $h_i = \prod_{j\in\alpha} g_j^{(i)}$, $i = 1, \ldots, t$, the elements $\big[\ldots[[h_{i_1}, h_{i_2}], h_{i_3}], \ldots, h_{i_q}\big]$ are distinct for different collections $\{i_1, \ldots, i_q\}$ with $1 \leq i_1 < \ldots < i_q \leq t$. This guarantees that for the group H generated by $h_1, \ldots, h_t$ one has $|H_q| \geq \binom{t}{q} \geq c$. By 5.6, for any finite coloring of G there exists a nonempty $\alpha \subseteq \{1, \ldots, N\}$ such that the group H generated by the corresponding $h_1, \ldots, h_t$ has monochromatic cosets. ■

5.8. In fact, the field F in Theorem 5.7 need not be infinite; it suffices for F to be large enough:

Theorem. *For any $r, q, c \in \mathbb{N}$ and a prime integer p there exists $K \in \mathbb{N}$ such that if F*

is a field of characteristic p and of cardinality $\geq K$, then for any r-coloring of the group G of $(q+1)\times(q+1)$ upper triangular matrices over F with unit diagonal there exist a subgroup H of G with $|H_q|\geq c$ and $h\in G$ such that the coset hH is monochromatic.

Indeed, let $\tilde{G}$ be the q-step nilpotent group defined by an infinite set S of generators and relations $g^p=\mathbf{1}_G$, $g\in S$. Then for any F of characteristic p, the group G of $q\times q$ upper triangular matrices over F with unit diagonal is a factor of $\tilde{G}$. Therefore, any r-coloring of G induces an r-coloring of $\tilde{G}$. If we now take N large enough to satisfy 5.6 for $\tilde{G}$, the result follows by an argument analogous to that employed in Theorem 5.7.

5.9. Let p be a prime integer and let q be a positive integer $<p$.

Theorem. *For any $c,r\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that for any r-coloring of the free q-step nilpotent group G with torsion p and with k generators there exists a free q-step nilpotent subgroup $H\subset G$ with torsion p having c generators such that a coset hH of H is monochromatic.*

Proof. $G/[G,G]$ is a k-dimensional vector space over $\mathbb{Z}_p$. We leave without proof the following fact: if (the images of) $h_1,\ldots,h_t\in G$ are linearly independent in $G/[G,G]$, then the group generated by $h_1,\ldots,h_t$ is free of nilpotency class q with torsion p. Now, take $t\geq c$, $N\in\mathbb{N}$ large enough to satisfy 5.6 (for the free q-step nilpotent group with torsion p and infinitely many generators) and choose $g_j^{(i)}$, $i=1,\ldots,q$, $j=1,\ldots,N$, so that the elements $g_j^{(i)}[G,G]$ are linearly independent in $G/[G,G]$. Then find α and define $h_1,\ldots,h_t$ and H as in 5.6. ■

6. Concluding remarks

6.1. The nil-IP-multiple recurrence results proved in this paper naturally extend to the nilpotent setup all known to us results pertaining to the multiple recurrence for actions of abelian groups by homeomorphisms of compact spaces. Taking into account that analogous statements are in general no longer true if the homeomorphisms involved generate a solvable group (see, for example, [F], p. 40), it is perhaps of interest to inquire about the general framework for multiple recurrence and to discuss some new potential directions of research.

6.2. The most natural question which has to be raised is the following: what about the validity of the corresponding *measurable* nil-IP-multiple recurrence statements? This question leads us to the following conjecture, which is a measurable counterpart of our Theorem 4.1:

6.3. Conjecture. *Let G be a nilpotent group of measure preserving transformations of a probability measure space $(X,\mathcal{B},\mu)$, let S be an infinite set and let $P_1,\ldots,P_k\colon\mathcal{F}(S)\longrightarrow G$ be polynomial mappings. Then for any $A\in\mathcal{B}$ with $\mu(A)>0$ there exists a nonempty $\alpha\in\mathcal{F}(S)$ such that $\mu\big(A\cap P_1(\alpha)A\cap\ldots\cap P_k(\alpha)A\big)>0$.*

Conjecture 6.3, if true, will give a simultaneous extension of the results recently obtained in [L2] and [BM1]. The results in [L2] and [BM1] deal with finitely generated nilpotent groups and abelian IP-systems respectively; the proof of the above conjecture in full generality will almost certainly demand introduction of new ideas and methods.

6.4. Like with some other proofs in the theory of measurable multiple recurrence (see, for example, [FK1], [FK2], [BL1], [L2], [BM1]) an important auxiliary role in the proof of Conjecture 6.3 will very likely be played by partition results extending our Theorem 4.1. While Theorem 4.1 can be viewed as a nilpotent version of our recent "PHJ", the polynomial Hales-Jewett theorem ([BL2]), and moreover, gives nilpotent extensions of those corollaries of PHJ which deal with abelian groups, it still lacks certain subtlety which the full-fledged Nil-PHJ should have. To explain this point better, let us formulate first the "abelian" PHJ:

6.5. Theorem. ([BL2]) *Let G be a commutative semigroup. For any $k, d, r \in \mathbb{N}$ there exists N such that if S is a set of cardinality $\geq N$ and $P_1, \ldots, P_k$ are monomial mappings induced, respectively, by monomials $u_1, \ldots, u_k \colon S^d \longrightarrow G$, then for any r-coloring of G there exist $\beta_1, \ldots, \beta_k \in \mathcal{F}(S^d)$ and a nonempty $\alpha \in \mathcal{F}(S)$ with $\beta_i \cap \alpha^d = \emptyset$, $i = 1, \ldots, k$, such that for $h = u_1(\beta_1) \ldots u_k(\beta_k)$ the elements $hP_1(\alpha), \ldots, hP_k(\alpha)$ have the same color.*

In comparison with our main combinatorial result, Theorem 4.4, Theorem 6.5 has two additional features. First, in its formulation one deals with a semigroup, whereas G is assumed to be a group in Theorem 4.4. Second, in the PHJ we have control over "the shift parameter" h: h is chosen from an a priori given finite set. While the requirement that G is a group rather than a semigroup does not seem to be a crucial one, the second feature, namely, the a priori condition on the range of the "shifting" element h, plays a key role in the known proofs of results similar to the one conjectured in 6.3. So, the general Nil-PHJ theorem is still ahead.

6.6. We want to conclude this section by discussing a nilpotent version of another important partition result, Hindman's finite sums theorem.

Theorem. ([Hi]) *Let $r \in \mathbb{N}$. If $\mathbb{N} = \bigcup_{m=1}^{r} C_m$, then there exist $m \in \{1, \ldots, r\}$ and an infinite set $\{n_j\}_{j=1}^{\infty} \subseteq C_m$ such that* $\mathrm{FS}\big(\{n_j\}_{j=1}^{\infty}\big) \setminus \{0\} \subseteq C_m$.

Hindman's theorem, similarly to its rather special corollary, Hilbert's theorem (Theorem 0.18 above) has a version which makes sense in any semigroup. Namely, given a finite coloring of an infinite semigroup G, one can always find an infinite sequence $\{h_i\}_{i=1}^{\infty} \subseteq G$ such that all the finite products of the form $h_{i_1} \ldots h_{i_k}$, $i_1 < \ldots < i_k$, $k \in \mathbb{N}$, will be in the same color. However, a much more interesting and subtle question is whether one can obtain a noncommutative extension of Hindman's theorem, which would guarantee the existence of monochromatic products of elements of a sequence $\{h_i\}_{i=1}^{\infty}$ taken in different orders. The only known nontrivial general result of this nature says that if G is an amenable group, then for any finite coloring of G one can always find a monochromatic quadruple $\{x, y, xy, yx\}$ where, for a large class of noncommutative amenable groups, one can guarantee $xy \neq yx$ ([BM2]).

6.7. Encouraged by the nilpotent Hilbert theorem (Theorem 0.20 above), we formulate the following conjecture:

Conjecture. *Let G be an infinite nilpotent group of nilpotency class q. Then for any* *finite coloring of G there exist an infinite sequence $\{h_i\}_{i=1}^{\infty}$ and $K \in \mathbb{N}$ such that every K distinct elements of $\{h_i\}_{i=1}^{\infty}$ generate a subgroup of G of nilpotency class q, and all the products of the form $h_{i_1} \dots h_{i_k}$, for $k \in \mathbb{N}$ and distinct $i_1, \dots, i_k \in \mathbb{N}$, are in the same color.*

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