Orbits on a nilmanifold under the action of a polynomial sequence of translations

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

September 25, 2006

Abstract

It is known that the closure \( \overline{\text{Orb}_g(x)} \) of the orbit \( \text{Orb}_g(x) \) of a point \( x \) of a compact nilmanifold \( X \) under a polynomial sequence \( g \) of translations of \( X \) is a disjoint finite union of subnilmanifolds of \( X \). Assume that \( g(0) = 1 \_G \) and let \( A \) be the group generated by the elements of \( g \); we show in this paper that for almost all points \( x \in X \), \( \overline{\text{Orb}_g(x)} \) are congruent (that is, are translates of each other), with connected components of \( \overline{\text{Orb}_g(x)} \) equal to (some of) the connected components of \( \overline{\text{Orb}_A(x)} \).

1. Nilmanifolds, subnilmanifolds, polynomial sequences and orbits

Let \( X \) be a compact nilmanifold, that is, a compact homogeneous space of a (not necessarily connected) nilpotent Lie group \( G \). Then \( X \) is isomorphic to (and will be identified with) \( G/\Gamma \), where \( \Gamma \) is a closed uniform subgroup of \( G \), with \( G \) acting on \( X \) by left translations. We will denote by \( \pi \) the factorization mapping \( G \longrightarrow X \), and by \( 1_X \) the point \( \pi(1_G) \), so that \( \pi(a) = a1_X, a \in G \).

We will list, without proofs, some elementary facts about nilmanifolds; for more details see [M], [L1], [L2] and [L3].

1.1. If \( X \) is not connected, it consists of finitely many isomorphic components, which may be treated independently; throughout the paper we will assume for simplicity that \( X \) is connected. The connectedness of \( X \) does not imply that \( G \) is connected; let \( G^o \) be the identity component of \( G \) and let \( \Gamma^o = \Gamma \cap G^o \). Then \( X = G^o/\Gamma^o \), so that \( X \) is a homogeneous space of the connected group \( G^o \). If \( X \) is interpreted this way, the elements of \( G \setminus G^o \) act on \( X \) not as translations but as unipotent affine transformations. (Example: the nilmanifold \( X = \left( \begin{array}{cc} 1 & \mathbb{R} \\ 0 & 1 \end{array} \right) / \left( \begin{array}{cc} 1 & \mathbb{Z} \\ 0 & 1 \end{array} \right) \) is isomorphic to the torus \( \mathbb{R}^2/\mathbb{Z}^2 \), on which the element \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) of the group \( G = \left( \begin{array}{cc} 1 & \mathbb{R} \\ 0 & 1 \end{array} \right) \) acts as the transformation \( (x, y) \mapsto (x + \alpha, y + x) \).

Conversely, if \( X \) is a nilmanifold corresponding to a group \( G \) and \( A \) is a nilpotent Lie group of unipotent affine transformations of \( X \), then the semidirect product \( \tilde{G} = G \rtimes A \)

Supported by NSF grant DMS-0345350.
is a nilpotent Lie group that contains both $G$ and $A$, and $X$ is a homogeneous space of $\tilde{G}$ on which it acts by translations.

1.2. If the subgroup $\Gamma$ is not discrete, then the connected component $\Gamma^0$ of $\Gamma$ is a normal subgroup of $G$, and we may pass from $G$ to $G/\Gamma^0$ without changing $X$ (see [L1]). Thus, we may and will assume that $\Gamma$ is a discrete subgroup of $G$.

1.3. After replacing the group $G^o$ by its universal cover, we may and will assume that $G^o$ is simply connected. One may then introduce Malcev coordinates on $G^o$, that is, a system of one-parameter subgroups $e_i(t)$, $t \in \mathbb{R}$, $i = 1, \ldots, d$, such that the elements $e_1^t, \ldots, e_d^t$ generate $\Gamma$ and any element $a$ of $G^o$ is uniquely representable in the form $a = e_1^{t_1} \cdots e_d^{t_d}$, $t_1, \ldots, t_d \in \mathbb{R}$. The “coordinate” mapping $\eta(t_1, \ldots, t_d) = a$ is a homeomorphism $\mathbb{R}^d \to G$, with $\eta(\mathbb{Z}^d) = \Gamma$. In coordinates, the multiplication in $G^o$ is given by a polynomial mapping $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

Let us say that a mapping $\varphi: \mathbb{R}^k \times \mathbb{Z}^l \to G^o$ is polynomial if it is polynomial in coordinates, that is, if $\eta^{-1} \circ \varphi: \mathbb{R}^k \times \mathbb{Z}^l \to \mathbb{R}^d$ is a polynomial mapping. Since the change-of-Malcev-coordinates mapping is an invertible bi-polynomial transformation of $\mathbb{R}^d$, this definition does not depend on the choice of Malcev coordinates on $G^o$.

1.4. A subnilmanifold $Y$ of $X$ is a closed subset of $X$ of the form $Y = Hx$, where $H$ is a closed subgroup of $G$ and $x \in X$. Since $\pi(G^o) = X$, after replacing $H$ by $H \cap G^o$ one may assume that $H \subseteq G^o$. A subnilmanifold $Y$ is a nilmanifold, since $Y \cong H/(a(\Gamma a^{-1}) \cap H)$ where $a$ is any element of $G$ with $\pi(a) = x$.

1.5. Given a closed subgroup $H$ of $G^o$ and a point $x \in X$, the set $Hx$ may not be closed and so, may not be a subnilmanifold of $X$; $Hx$ is closed iff $(a(\Gamma a^{-1}) \cap H$ is a uniform subgroup of $H$, where $a$ is any element of $\pi^{-1}(x)$. In particular, $H1_X = \pi(H)$ is closed iff $H \cap \Gamma$ is uniform in $H$; we will say that $H$ is rational in this case. There are only countably many rational closed subgroups in $G$.

We say that an element $a$ of $G$ is rational if $a^n \in \Gamma$ for some $n \in \mathbb{N}$. A closed subgroup $H$ of $G$ is rational iff rational elements are dense in $H$ ([L3]).

We say that a point $x = \pi(a) \in X$ is rational if $x = \pi(a)$ where $a$ is rational in $G$. A subnilmanifold $Y$ of $X$ is rational if it contains at least one rational point of $X$, and in this case rational points are dense in $Y$. $X$ has countably many rational sub-nilmanifolds. For any point $x \in X$ there are only countably many distinct sub-nilmanifolds in $X$ that contain $x$. (See [L3].)

1.6. Let $H$ be a closed connected subgroup of $G^o$ and let $\tau: \mathbb{R}^r \to H$ be Malcev coordinates on $H$. Then the mapping $\eta^{-1} \circ \tau: \mathbb{R}^r \to \mathbb{R}^d$ is polynomial, and thus in coordinates $H$ is the image of a polynomial mapping. Let us say that a subset $S$ of $G^o$ is polynomial if $\eta^{-1}(S)$ is an algebraic subset of $\mathbb{R}^d$, that is, is defined by one or several polynomial equations; this definition does not depend on the choice of Malcev coordinates on $G^o$. Any closed connected subgroup $H$ of $G^o$ is a polynomial subset of $G^o$; indeed, Malcev coordinates on $G^o$ can be constructed so that they extend Malcev coordinates on $H$, and in these coordinates $\eta^{-1}(H)$ is even a linear (coordinate) subspace of $\mathbb{R}^d$. Since a translation by an element $a \in G^o$ is an invertible bi-polynomial transformation of $\mathbb{R}^d$, the set $aH$ is
polynomial in $G^o$ as well.

Let us say that a set $P \subseteq X$ is *polynomial* in $X$ if $P = \pi(S)$ where $S$ is a polynomial subset in $G^o$. Note that a polynomial subset of $X$ does not have to be closed in $X$. (It may even be dense in $X$, as a line with an irrational slope in the 2-dimensional torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$.)

Let us say that a subset of $\mathbb{R}^d$, $G^o$ or $X$ is *countably polynomial* if it is a countable (or finite) union of polynomial subsets. Note that any proper countably polynomial subset is of zero (Lebesgue) measure and of first category in the corresponding space.

1.7. Let $A$ be a closed (possibly, discrete) subgroup of $G$. For $x \in X$, we will denote by $\text{Orb}_A(x)$ the orbit of $x$ under the action of $A$, $\text{Orb}_A(x) = Ax$, and by $\overline{\text{Orb}}_A(x)$ the closure of $\text{Orb}_A(x)$. By abuse of language, we will also refer to $\overline{\text{Orb}}_A(x)$ as the orbit of $x$ under the action of $A$. It is shown in [L1] that for any $x \in X$, $\overline{\text{Orb}}_A(x)$ is a (connected or disconnected) subnilmanifold of $X$. (See also [Le] and [Sh].) For any $x \in X$, the action of $A$ on $\overline{\text{Orb}}_A(x)$ is minimal, that is, $\overline{\text{Orb}}_A(y) = \overline{\text{Orb}}_A(x)$ for any $y \in \overline{\text{Orb}}_A(x)$. It follows that $X = \bigcup_{x \in X} \overline{\text{Orb}}_A(x)$ is a partition of $X$. In particular, if $\overline{\text{Orb}}_A(x) = X$ for a point $x \in X$, then $\overline{\text{Orb}}_A(y) = X$ for all $y \in X$.

The orbits of distinct points may not be translates of each other, and may even have different dimensions, as the following examples demonstrate:

1.8. Examples.

1. Let $G = \left\{ \left( \begin{array}{ccc} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{array} \right) \right\}, \ x_1, x_2, x_3 \in \mathbb{R}$ and $\Gamma = \left\{ \left( \begin{array}{ccc} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{array} \right), \ x_1, x_2, x_3 \in \mathbb{Z} \right\}$; then $X = G/\Gamma$ is identified with the 2-dimensional torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$ with coordinates $x_2, x_3 \in \mathbb{T}$. Let $a = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \in G$, then the action of $a$ on $X$ is given by $a(x_2, x_3) = (x_2, x_3 + ax_2) \mod 1,$ $(x_2, x_3) \in X$. (Equivalently, without even mentioning nilpotent groups, $X = T^2$ and $a$ is the unipotent transformation of $X$ defined by this formula.) Let $A = \{a^n\}_{n \in \mathbb{Z}}$. Then for $x = (x_2, x_3) \in X$, $\overline{\text{Orb}}_A(x) = \{ (x_2, u), \ u \in \mathbb{T} \} \simeq \mathbb{T}$ if $x_1$ is irrational, and is the finite set $\{ (x_2, nx_1), \ n \in \mathbb{N} \}$ if $x_1$ is rational.

2. Now let $G = \left\{ \left( \begin{array}{ccc} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{array} \right) \right\}$ and $\Gamma = \left\{ \left( \begin{array}{ccc} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{array} \right), \ x_1, x_2, x_3 \in \mathbb{Z} \right\}$; $X = G/\Gamma$ is then the 3-dimensional *Heisenberg manifold*. Let $a = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \in G$ where $\alpha$ is an irrational number; then the action of $a$ on $X$ is given by $ax = \left( \begin{array}{ccc} 1 & x_1 + \alpha x_3 \alpha x_2 \\ 0 & 1 \end{array} \right) \mod \Gamma, x = \left( \begin{array}{ccc} 1 & x_1 \alpha x_2 \\ 0 & 0 \end{array} \right) \in X$. Let $A = \{a^n\}_{n \in \mathbb{Z}}$ and $x = \left( \begin{array}{ccc} 1 & x_1 \alpha x_2 \\ 0 & 0 \end{array} \right) \in X$. If $\alpha$ and $\alpha x_2$ are rationally independent, that is, if $x_2 \not\in \mathbb{Q} + \frac{1}{\alpha} \mathbb{Q},$ the orbit of $x = \left( \begin{array}{ccc} 1 & x_1 \alpha x_2 \\ 0 & 0 \end{array} \right)$ is the 2-dimensional torus $\overline{\text{Orb}}_A(x) = \left\{ \left( \begin{array}{ccc} 1 & u \alpha x_2 \\ 0 & 0 \end{array} \right), \ u \in \mathbb{T} \right\}$. Otherwise $\overline{\text{Orb}}_A(x)$ is a 1-dimensional torus or the union of several 1-dimensional tori; for example, if $x_2 = 0$ or $x_2 = \frac{1}{\alpha},$ then $\overline{\text{Orb}}_A(x) = \left\{ \left( \begin{array}{ccc} 1 & u \alpha x_2 \\ 0 & 0 \end{array} \right), \ u \in \mathbb{T} \right\}$.
we will call “generic” below); (ii) any “non-generic” orbit is a proper subnilmanifold of the “generic” one; and (iii) the points having a “non-generic” orbit are all contained in a countable union of proper subnilmanifolds of $X$. We will show that (i) and (ii) always hold; (iii) may fail (see example 2.4 below), and must be replaced by a weaker statement:

1.10. Theorem. Let $A$ be a closed subgroup of $G$. There exists a closed subnilmanifold $Y_A$ of $X$ such that

(a) for any $x \in X$ the orbit $\overline{\Orb_A(x)}$ is congruent to some subset of $Y_A$;

(b) there exists a proper countably polynomial subset $P \subset X$ such that for all $x \notin P$ the orbit $\overline{\Orb_A(x)}$ is congruent to $Y_A$.

This theorem will be proven in Section 2. We will refer to the “standard” orbit $Y_A$ in the formulation of the theorem as the generic orbit for $A$.

1.11. A (multiparameter) polynomial sequence in $G$ is a sequence of the form $g(n) = a_1(n) \ldots a_r(n)$, $n \in \mathbb{Z}^l$, where $a_1, \ldots, a_n \in G$ and $p_1, \ldots, p_r$ are polynomials $\mathbb{Z}^l \to \mathbb{Z}$. In the terminology introduced above, a polynomial sequence is just a polynomial mapping $\mathbb{Z}^l \to G$. For $x \in X$ we will denote by $\Orb_g(x)$ the orbit of $x$ under the action of $g$, $\Orb_g(x) = g(\mathbb{Z}^l)x = \{g(n)x, n \in \mathbb{Z}^l\}$, and by $\overline{\Orb_g(x)}$ the closure of $\Orb_g(x)$; by abuse of language, we will also refer to $\overline{\Orb_g(x)}$ as the orbit of $x$ under the action of $g$. It is shown in [L2] that $\overline{\Orb_g(x)}$ is of the form $\bigcup_{i=1}^L Hx_i$, where $H$ is a connected closed subgroup of $G$ and $x_1, \ldots, x_L \in X$, and thus is either a connected subnilmanifold of $X$ or the union of a finite collection of pairwise disjoint connected subnilmanifolds of same dimension. Let us call such a union a FU-subnilmanifold; in particular, any (connected or disconnected) subnilmanifold of $X$ is a FU-subnilmanifold.

Let us say that a FU-subnilmanifold is rational if all its connected components are rational subnilmanifolds. It is shown in [L3] that for any rational point $x$ of $X$, $\overline{\Orb_g(x)}$ is a rational FU-subnilmanifold.

1.12. The orbits under the action of a polynomial sequence do not have to partition $X$; in the following example, due to Frantzkinakis and Kra, the generic orbit is the whole space $X$, whereas nongeneric orbits are proper subnilmanifolds of $X$.

Example. Let $X$ be the 3-dimensional torus $\mathbb{T}^3$ and let $a$ and $b$ be the transformations of $X$ defined by $ax = (x_1 + \alpha, x_2 + 2x_1 + \alpha, x_3)$ and $bx = (x_1, x_2, x_3 + \alpha)$, $x = (x_1, x_2, x_3) \in X$. (As mentioned in 1.1 above, since $a$ and $b$ are commuting unipotent transformations of $X$ they can be viewed as elements of a nilpotent Lie group for which $X$ is a homogeneous space.) Define $g(n) = a^n b^{n^2}$, $n \in \mathbb{Z}$. Then, for $x = (x_1, x_2, x_3)$, one has $g(n)x = (x_1 + n\alpha, x_2 + 2nx_1 + n^2\alpha, x_3 + n^2\alpha)$. If $x_1$ is irrational, $\overline{\Orb_g(x)} = X$. If $x_1$ is rational, $\overline{\Orb_g(x)}$ is a proper subtorus or a union of several 2-dimensional subtori of $X$. For example, if $x_1 = 0$, $\overline{\Orb_g(x)} = \{(u, v, v), u, v \in \mathbb{T}\}$.

1.13. We will show that, like in the case of a linear action, under the action of a polynomial sequence $g$ almost all points of $X$ have congruent orbits:
Theorem. Let \( g \) be a polynomial sequence in \( G \).

I. There exists a closed FU-subnilmanifold \( Y_g \) of \( X \) such that

(a) for any \( x \in X \) the orbit \( \text{Orb}_g(x) \) is congruent to some subset of \( Y_g \);
(b) there exists a proper countably polynomial subset \( P \subset X \) such that for all \( x \notin P \) the orbit \( \text{Orb}_g(x) \) is congruent to \( Y_g \).

II. Assume that \( g(0) = 1_G \), let \( A \) be the subgroup of \( G \) generated by the elements of \( g \) and let \( Y_A \) be the generic orbit for \( A \). Then \( Y_g \) consists of one or several components of \( Y_A \); in particular, if \( Y_A \) is connected, \( Y_g = Y_A \).

Part I of this theorem will be proved in Section 2, Part II will be proved in Section 4. In Section 3 we study the property of “normality” of generic orbits. In Section 5 we investigate the orbit of a subnilmanifold of \( X \).

2. The generic orbits

2.1. Theorem 1.10 and (the first part of) Theorem 1.13 are corollaries of the following simple general fact:

Theorem. Let \( M \) be a set and let \( \varphi: \mathbb{R}^k \times M \to G \) be a mapping; assume that for each fixed \( m \in M \), \( \varphi \) is polynomial with respect to \( \mathbb{R}^k \), and for each \( t \in \mathbb{R}^k \) the set \( Y_t = \pi(\varphi(t,M)) \) is a rational FU-subnilmanifold of \( X \). Then there exist a FU-subnilmanifold \( Y_\varphi \) of \( X \) and a proper countably polynomial subset \( S \subset \mathbb{R}^k \) such that

(a) \( Y_t \subseteq Y_\varphi \) for all \( t \in \mathbb{R}^k \);
(b) \( Y_t = Y_\varphi \) for all \( t \notin S \).

Proof. Let \( Y_\varphi \) be the minimal FU-subnilmanifold of \( X \) such that \( Y_t \subseteq Y_\varphi \) for all \( t \in \mathbb{R}^k \). Assume that \( Z \) is a rational FU-subnilmanifold of \( X \) such that \( Z \nsubseteq Y_\varphi \); then there exists \( t_0 \in \mathbb{R}^k \) such that \( Y_{t_0} \nsubseteq Z \). Let \( Z_1, \ldots, Z_s \) be connected components of \( Z \) and let \( H_1, \ldots, H_s \) be connected closed subgroups of \( G^o \) such that \( Z_i = \pi(H_i), i = 1, \ldots, s \). There exists \( m_0 \in M \) such that \( \varphi(t_0,m_0) \notin \bigcup_{i=1}^s H_i \). Each \( H_i \) is a polynomial subset of \( G^o \), and the mapping \( t \to \varphi_t(m_0), t \in \mathbb{R}^k \), is polynomial, thus the set \( S_Z = \{ t \in \mathbb{R}^k : \varphi(t,m_0) \in \bigcup_{i=1}^s H_i \} \) is a proper polynomial subset of \( \mathbb{R}^k \). For any \( t \notin S_Z \) we have \( \varphi(t,m_0) \notin Z \) and so, \( Y_t \neq Z \). We now put \( S = \bigcup S_Z \), where \( Z \) runs over the set of rational FU-subnilmanifolds of \( X \) (which is countable by 1.5).

2.2. We will now deduce a generalization of Theorem 1.10:

Theorem. Let \( V \) be a connected subnilmanifold of \( X \), let \( K \) be a connected component of \( \pi^{-1}(V) \) and \( A \) be a closed subgroup of \( G \). There exists a closed subnilmanifold \( Y_{V,A} \) of \( X \) such that

(a) for any \( x \in V \) one has \( \text{Orb}_A(x) \subseteq aY_{V,A} \) whenever \( a \in K \), \( \pi(a) = x \);
(b) there exists a proper countably polynomial subset \( P \subset V \) such that for any \( x \in V \setminus P \) one has \( \text{Orb}_A(x) = aY_{V,A} \) whenever \( a \in K \), \( \pi(a) = x \).

We call the subnilmanifold \( Y_{V,A} \) the generic orbit for \( A \) on \( V \); in the case \( V = X \), \( Y_{V,A} \) is just the generic orbit for \( A \) and will be denoted by \( Y_A \).
Proof. We may assume that \( \dim V \geq 1 \), \( V \ni 1_X \) and \( K \) is a connected closed subgroup of \( G^0 \). Let \( \tau : \mathbb{R}^r \rightarrow V \) be a (Malcev) coordinate system on \( K \). Define a mapping \( \varphi : \mathbb{R}^r \times A \rightarrow G \) by \( \varphi(t, b) = \tau(t)^{-1}b\tau(t) \). Then \( \varphi \) is a polynomial mapping, and for each \( t \in \mathbb{R}^k \) and \( a = \tau(t) \) the set

\[
Y_t = \overline{\pi(\varphi(t, A))} = \overline{\pi(a^{-1}Aa)} = \overline{a^{-1}Aa1_X} = \overline{\text{Orb}_{a^{-1}Aa}(1_X)}
\]

is a rational subnilmanifold of \( X \). By Theorem 2.1, there exist a FU-subnilmanifold \( Y_{V,A} \subseteq X \) and a proper countably polynomial subset \( S \subseteq \mathbb{R}^k \) such that \( Y_t \subseteq Y_{V,A} \) for all \( t \in \mathbb{R}^k \) and \( Y_t = Y_{V,A} \) for all \( t \in \mathbb{R}^k \setminus S \). Since all \( Y_t \) are subnilmanifolds, \( Y_{V,A} \) is also a subnilmanifold. Finally, for any \( x \in V \), \( x = \pi(a) \) with \( a = \tau(t) \in K \), we have

\[
\text{Orb}_A(x) = \overline{Ax} = \overline{Aa1_X} = \overline{a^{-1}Aa1_X} = \overline{\tau(t)^{-1}A\tau(t)}1_X = \overline{\pi(\varphi(t, A))} = aY_t,
\]

so \( \text{Orb}_A(x) \subseteq aY_{V,A} \), and \( \text{Orb}_A(x) = aY_{V,A} \) whenever \( x \notin P = \pi(\tau(S)) \).

2.3. We generalize Theorem 1.13.I in a similar manner:

**Theorem.** Let \( V \) be a connected subnilmanifold of \( X \), let \( K \) be a connected component of \( \pi^{-1}(V) \) and let \( g : \mathbb{Z}^l \rightarrow G \) be a polynomial sequence in \( G \). There exists a closed FU-subnilmanifold \( Y_{V,g} \) of \( X \) such that

(a) for any \( x \in V \) one has \( \overline{\text{Orb}_g(x)} \subseteq aY_{V,g} \) whenever \( a \in K \), \( \pi(a) = x \);

(b) there exists a proper countably polynomial subset \( P \subset V \) such that for any \( x \in V \setminus P \) one has \( \overline{\text{Orb}_g(x)} = aY_{V,g} \) whenever \( a \in K \), \( \pi(a) = x \).

We call the FU-subnilmanifold \( Y_{V,g} \) the generic orbit for \( g \) on \( V \); in the case \( V = X \), \( Y_{V,g} \) is just the generic orbit for \( g \) and will be denoted by \( Y_g \).

**Proof.** We may assume that \( g(0) = 1_G \), \( \dim V \geq 1 \), \( V \ni 1_X \) and \( K \) is a connected closed subgroup of \( G^0 \). Let \( \tau : \mathbb{R}^r \rightarrow V \) be a (Malcev) coordinate system on \( K \). Define a mapping \( \varphi : \mathbb{R}^r \times \mathbb{Z}^l \rightarrow G \) by \( \varphi(t, n) = \tau(t)^{-1}g(n)\tau(t) \), then \( \varphi \) is a polynomial mapping. For \( t \in \mathbb{R}^k \) let \( Y_t = \overline{\pi(\varphi(t, \mathbb{Z}^l))} \). Putting \( a = \tau(t) \), we get

\[
Y_t = \overline{\pi(\varphi(t, \mathbb{Z}^l))} = \overline{\pi(a^{-1}g(\mathbb{Z}^l)a)} = \overline{a^{-1}g(\mathbb{Z}^l)a1_X} = \overline{\text{Orb}_{g,a}(1_X)},
\]

where \( g^a \) is the polynomial mapping \( g^a(n) = a^{-1}g(n)a \), \( n \in \mathbb{Z}^l \). Hence, \( Y_t \) is a rational FU-subnilmanifold of \( X \). By Theorem 2.1, there exist a FU-subnilmanifold \( Y_{V,g} \subseteq X \) and a proper countably polynomial subset \( S \subseteq \mathbb{R}^k \) such that \( Y_t \subseteq Y_{V,g} \) for all \( t \in \mathbb{R}^k \) and \( Y_t = Y_{V,g} \) for all \( t \in \mathbb{R}^k \setminus S \). For any \( x \in V \), \( x = \pi(a) \) with \( a = \tau(t) \in K \), we have

\[
\text{Orb}_g(x) = \overline{g(\mathbb{Z}^l)x} = \overline{g(\mathbb{Z}^l)a1_X} = \overline{a^{-1}g(\mathbb{Z}^l)a1_X} = \overline{a^{-1}g(\mathbb{Z}^l)\tau(t)1_X} = \overline{a(\pi(\varphi(t, \mathbb{Z}^l)))} = aY_t,
\]

so \( \text{Orb}_g(x) \subseteq aY_{V,g} \), and \( \text{Orb}_g(x) = aY_{V,g} \) whenever \( x \notin P = \pi(\tau(S)) \).

2.4. The following example shows that the set of points having non-generic orbits may not be a union of subnilmanifolds of \( X \).
Example. Let $G = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}, \ x_{i,j} \in \mathbb{R} \right\}$, $\Gamma = \left\{ \begin{pmatrix} 1 & x_{1,1} & x_{1,2} \\ 0 & 1 & x_{2,2} \\ 0 & 0 & 1 \end{pmatrix}, \ x_{i,j} \in \mathbb{Z} \right\}$ and $X = G/\Gamma$. Let $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $\alpha$ is an irrational number, and $A = \{ b^n \}_{n \in \mathbb{Z}}$.

For $a = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} \\ 0 & 1 & x_{2,2} \\ 0 & 0 & 1 \end{pmatrix} \in G$ one finds that $a^{-1} b^n a = \begin{pmatrix} 1 & n \alpha & n \alpha x_{2,4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$. So, the generic orbit for $A$ is the 3-dimensional torus $Y_A = \begin{pmatrix} 1 \nu \nu \nu \end{pmatrix}$; when the numbers $\alpha, \alpha x_{2,3}, \alpha x_{2,4}$ are rationally dependent, the point $\pi(a)$ has a nongeneric orbit, which is a 1- or a 2-dimensional subtorus of $Y_A$.

Let $Q = \left\{ \begin{pmatrix} 1 & x_{1,1} & x_{1,2} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \ x_{i,j} \in \mathbb{R} \right\} \subset G$, then every $x \in \pi(Q)$ has a 2-dimensional nongeneric orbit. $Q$ is a connected polynomial subset of $G$ with $\pi(Q)$ dense in $X$.

3. The normality of generic orbits

If $g$ is a polynomial sequence in $G$ with generic orbit $Y_g$ on $X$ and $x$ is a point of $X$ having generic orbit under the action of $g$, then $\overline{\text{Orb}_g}(x) = aY_g$ for all $a \in G^o$ with $\pi(a) = x$. This gives us some additional information about generic orbits.

3.1. Let us say that a subnilmanifold $Y$ of $X$ is normal if $Y = Hx$ where $x \in X$ and $H$ is a normal subgroup of $G^o$.

3.2. The importance of the notion of normality is manifested by the following fact:

Proposition. Let $Y$ be a connected subnilmanifold of $X$, $Y = Hx$ where $x \in X$ and $H$ is a connected closed subgroup of $G^o$. The following are equivalent:
(i) $Y$ is normal;
(ii) the sets $Hy$ are closed in $X$ for all $y \in X$;
(iii) the subnilmanifolds $aY$, $a \in G^o$, partition $X$.

Proof. If $Y$ is normal then $H$ is normal in $G^o$, so $aH = Ha$ and thus, $aY = Hax$ for all $a \in G^o$. The sets $aH$, $a \in G^o$, are closed and the sets $Hax$, $a \in G^o$, partition $X$, so we have (ii) and (iii).

Assume that the sets $Hy$ are all closed. This means that the sets $\pi(Ha)$, $a \in G^o$, are closed, and so, the sets $\pi(a^{-1}Ha)$, $a \in G^o$, are closed. Thus, for any $a \in X$, $a^{-1}Ha$ is a rational subgroup of $G^o$; since there are only countably many of such and $a^{-1}Ha$ continuously depends on $a$, $a^{-1}Ha = H$ for all $a \in G^o$.

Let now the sets $aY$, $a \in G^o$, partition $X$. We may assume that $x = 1_X$, so that $Y = \pi(H)$ and $1_X \in Y$. Then for any $\gamma \in \Gamma^o$, $\gamma Y$ contains $1_X$, thus $\gamma Y = Y$. Let $\gamma \in \Gamma^o$; then $\gamma H\Gamma^o = H\Gamma^o$ and, since $H$ is connected, $\gamma H = H\gamma'$ for some $\gamma' \in \Gamma^o$. So, $\gamma H \gamma^{-1} = H\gamma' \gamma^{-1}$, and since $\gamma H \gamma^{-1}$ is a subgroup of $G^o$, $\gamma H \gamma^{-1} = H$. It remains to apply the following lemma:
3.3. Lemma. If a subgroup $H$ of $G^o$ is normalized by $\Gamma^o$ then $H$ is normal in $G^o$.

Sketch of the proof. $G^o$ is an exponential group, which means that for any $a \in G^o$ there exists a one-parametric flow $t \to a^t$, $t \in \mathbb{R}$, passing through $a$. Let $a \in G^o$ be such that $a^{t_0} \in \Gamma$ for some nonzero $t_0 \in \mathbb{R}$. The condition “$a^t$ normalizes $H$” is polynomial with respect to $t$, so, since $a^{nt_0}$ normalizes $H$ for all $n \in \mathbb{Z}$, $a^t$ normalizes $H$ for all $t \in \mathbb{R}$. $G^o$ is generated by elements $a \in G^o$ with $a^{t_0} \in \Gamma$ for some nonzero $t_0 \in \mathbb{R}$, thus $G^o$ normalizes $H$. ■

3.4. If $Y$ is a normal sub-nilmanifold of $X$, the factor-nilmanifold $X/Y$ is defined. Indeed, assume that $1_X \in Y$ and let $H$ be the closed normal subgroup of $G^o$ such that $Y = \pi(H)$. Then $\pi^{-1}(Y) = H\Gamma^o$ is a closed uniform subgroup of $G^o$; define $Z = G^o/(H\Gamma^o)$. $Z$ is a nilmanifold, and the fibers of the natural mapping $X \longrightarrow Z$ are translates of $Y$.

3.5. We will now show:

Theorem. Let $A$ be a subgroup of $G$ and $Y_A$ be the generic orbit for $A$. The connected components of $Y_A$ are normal sub-nilmanifolds of $X$.

Proof. Let $P$ be the set, introduced in Theorem 2.2, of points whose orbits under the action of $A$ are nongeneric on $X$. Let $x \notin P$; we may assume that $x = 1_X$. Then, by Theorem 2.2, for any $\gamma \in \Gamma^o$, $\operatorname{Orb}_A(1_X) = \overline{\operatorname{Orb}_A}(\pi(\gamma)) = \gamma Y_A$. So, $\gamma Y_A = Y_A$ for all $\gamma \in \Gamma^o$. Let $H$ be the closed subgroup of $G^o$ such that $Y_A = \pi(H)$ and let $H^o$ be the identity component of $H$. Let $\gamma \in \Gamma^o$, then $\gamma H\Gamma^o = H\Gamma^o$, and $\gamma H^o = H^o c' \gamma'$ for some $c \in H$ and $\gamma' \in \Gamma^o$. So $\gamma H^o \gamma^{-1} = H^o c' \gamma' \gamma^{-1}$, and since $\gamma H^o \gamma^{-1}$ is a subgroup of $G^o$, $\gamma H^o \gamma^{-1} = H^o$. Hence, $H^o$ is normalized by $\Gamma^o$; by Lemma 3.3, $H^o$ is normal in $G^o$. ■

3.6. Similarly, we have

Theorem. If $Y_g$ is the generic orbit for a polynomial sequence $g$ in $G$ then the connected components of $Y_g$ are normal sub-nilmanifolds of $X$.

Proof. Let $P$ be the set, introduced in Theorem 2.3, of points whose orbits under the action of $g$ are nongeneric on $X$. Let $x \notin P$; we may assume that $x = 1_X$. Then, by Theorem 2.3, for any $\gamma \in \Gamma^o$, $\overline{\operatorname{Orb}_g}(1_X) = \overline{\operatorname{Orb}_g}(\pi(\gamma)) = \gamma Y_g$. So, $\gamma Y_g = Y_g$ for all $\gamma \in \Gamma^o$. Let $H$ be the connected closed subgroup of $G^o$ such that $Y_g = \bigcup_{i=1}^n Hx_i$. Let $\gamma \in \Gamma^o$, then $\gamma H = Hc\gamma'$ for some $c \in G^o$ and $\gamma' \in \Gamma^o$. So $\gamma H \gamma^{-1} = Hc\gamma' \gamma^{-1}$, and since $\gamma H \gamma^{-1}$ is a subgroup of $G^o$, $\gamma H \gamma^{-1} = H$. Hence, $H$ is normalized by $\Gamma^o$; by Lemma 3.3, $H$ is normal in $G^o$. ■

3.7. Let us informally describe the picture we have got. Let $A$ be a subgroup of $G$. If $\overline{\operatorname{Orb}_A}(x) = X$ for some point $x \in X$ then $\overline{\operatorname{Orb}_A}(y) = X$ for all $y \in X$. Otherwise, the generic orbit $Y_A$ for $A$ is a proper sub-nilmanifold of $X$. Let $Y$ be a connected component of $Y_A$, then $Y$ is normal in $X$ and thus the factor-nilmanifold $Z = X/Y$ is defined; the fibers of the projection mapping $\eta:X \longrightarrow Z$ are translates of $Y$. $A$ acts on $Z$ in a finite way; after passing to a subgroup of finite index in $A$ we may assume that the action of $A$ on $Z$ is trivial, and $A$ acts only on the fibers of $\eta$. For almost every $z \in Z$ the action of $A$ on $\eta^{-1}(z)$ is minimal, that is, the sub-nilmanifold $\eta^{-1}(z)$ is the orbit of all its points. If a
fiber $V = \eta^{-1}(z)$ is not the orbit of its points then the generic orbit $Y_{V,A}$ of points of $V$ is a proper subnilmanifold of $V$, $V$ is partitioned by translates of its connected component, etc.

For the action on $X$ of a polynomial sequence $g$ the picture is similar. A difference is that orbits of distinct points do not partition $X$; they may contain one another, or have a nontrivial intersection. That is, assuming $g(0) = 1_G$, even if a translate $V = aY_g$ of the generic orbit $Y_g$ for $g$ is the orbit of some point, $V = \text{Orb}_g(x)$, it does not have to be true for all other points of $V$; however, in this case $V = \text{Orb}_g(y)$ for almost all $y \in V$.

4. Relation between linear and polynomial generic orbits

Let $g: \mathbb{Z}^l \to G$ be a polynomial sequence in $G$ with $g(0) = 1_G$ and let $A$ be the subgroup of $G$ generated by the elements of $g$. Let $Y_g \subseteq X$ be the generic orbit for $g$ and $Y_A \subseteq X$ be the generic orbit for $A$. We will investigate the relation between $Y_g$ and $Y_A$.

Clearly, $Y_g \subseteq Y_A$.

4.1. Let us first assume that $Y_g$ is connected. Let $x \in X$ be a point of $X$ that has generic orbit under the action of $g$; let $x = \pi(a)$, $a \in \mathcal{G}^0$, so that $\text{Orb}_g(x) = aY_g$. For any $y \in aY_g$ by Theorem 2.3(a) and Theorem 3.6 we have $\text{Orb}_g(y) \subseteq aY_g$, so $g(n)y \in aY_g$ for all $n \in \mathbb{Z}^l$.

It follows that $A$ preserves $\text{Orb}_g(x)$ and hence, $\text{Orb}_A(x) \subseteq \text{Orb}_g(x)$. Since this is true for almost all points of $X$, we have $Y_A \subseteq Y_g$.

4.2. We obtain the following result:

**Proposition.** Let $g: \mathbb{Z}^l \to G$ be a polynomial sequence in $G$ with $g(0) = 1_G$, let $A$ be the subgroup of $G$ generated by the elements of $g$, let $Y_g \subseteq X$ be the generic orbit for $g$ and $Y_A \subseteq X$ be the generic orbit for $A$. If $Y_g$ is connected, then $Y_g = Y_A$.

4.3. The case where $Y_g$ is not connected can be reduced to the previous one. It is proven in [L2] that, given a point $x \in X$, there exists a subgroup $\omega$ of finite index in $\mathbb{Z}^l$ such that, for the restriction $g_\omega$ of $g$ on $\omega$ the orbit $\text{Orb}_{g_\omega}(x)$ is connected. It follows that for some subgroup $\omega \subseteq \mathbb{Z}^l$ of finite index the generic orbit $Y_{g_\omega}$ for $g_\omega$ is connected. (Indeed, since $\mathbb{Z}^l$ has only countably many subgroups, there exists a subgroup $\omega$ of finite index for which the set of $x$ with connected orbits under the action of $g_\omega$ has positive measure.) $Y_{g_\omega}$ is then a connected component of $Y_g$.

Let $A_\omega$ be the group generated by the elements of $g_\omega$; by Proposition 4.2, the generic orbit for $A_\omega$ is $Y_{g_\omega}$. It is easy to see that $A_\omega$ has finite index in $A$, thus $Y_{g_\omega}$ coincides with one of the connected components of $Y_g$. Hence, the connected components of $Y_g$ have the same dimension as components of $Y_A$, and so, coincide with them. This proves Theorem 1.13.II:

4.4. **Theorem.** Let $g: \mathbb{Z}^l \to G$ be a polynomial sequence in $G$ with $g(0) = 1_G$, let $A$ be the subgroup of $G$ generated by the elements of $g$, let $Y_g \subseteq X$ be the generic orbit for $g$ and $Y_A \subseteq X$ be the generic orbit for $A$. Then $Y_g$ is a union of connected components of $Y_A$.
4.5. Remark. If \( V \) is a connected subnilmanifold of \( X \), the generic orbit \( Y_{V,g} \) for \( g \) on \( V \) may not be a union of connected components of the generic orbit \( Y_{V,A} \) for \( A \) on \( V \). This can already be seen on the trivial example where \( V \) is a single point.

4.6. Corollary. The connected components of \( Y_g \) are all congruent.

4.7. An open question. I cannot answer the following question: are the connected components of any nongeneric orbit for \( g \) also congruent to each other?

4.8. Let \( L = [G^o, G^o] \setminus X \); we will call \( L \) the maximal factor-torus of \( X \).

Let \( g \) be a polynomial sequence in \( G \). It is proven in [L2] that if \( \text{Orb}_g(u) = L \) for a point \( u \in L \) then \( \text{Orb}_g(x) = X \) for any \( x \in X \).

For “linear” actions on \( X \) a stronger statement holds. Now let \( N = [G, G] \setminus X \). \( N \) is a factor-torus of \( L \), and dealing with \( N \) is easier than with \( L \) since \( G \) acts on the torus \( N \) by conventional, “abelian” shifts, whereas on \( L \) it may act by “sqew-shifts”, that is, by unipotent affine transformations (see Example 1.81 above). Let \( A \) be a subgroup of \( G \);

assuming that \( G \) is generated by \( G^o \) and \( A \), one has \( \text{Orb}_A(x) = X \) for all \( x \in X \) whenever \( \text{Orb}_A(v) = N \) for some \( v \in N \). Example 1.12 shows that an analogous statement does not hold for a polynomial action; we, however, get the following:

4.9. Corollary. Let \( g \) be a polynomial sequence in \( G \) and assume that \( G \) is generated by \( G^o \) and elements of \( g \). Let \( N = [G, G] \setminus X \), and assume that \( \text{Orb}_g(v) = N \) for some \( v \in N \). Then the generic orbit for \( g \) is equal to \( X \).

Proof. We may assume that \( g(0) = 1_G \). Let \( A \) be the group generated by the elements of \( g \). Then \( \text{Orb}_A(v) = N \), so the generic orbit for \( A \) is \( X \), and by Proposition 4.2, \( X \) is the generic orbit for \( g \). ■

5. Orbits of a subnilmanifold

Let \( V \) be a connected subnilmanifold of \( X \); we will assume for simplicity that \( V \ni 1_X \) and so, \( V = \pi(K) \) where \( K \) is a connected closed subgroup of \( G^o \). For a subgroup \( A \) of \( G \) or a polynomial sequence \( g: \mathbb{Z}^l \rightarrow G \) we may now investigate (the closures of) the orbits \( \overline{\text{Orb}}_A(V) = AV \) and \( \overline{\text{Orb}}_g(V) = g(\mathbb{Z}^l)V \) of \( V \) under the action of \( A \) and \( g \) respectively. It is shown in [L3] that \( \overline{\text{Orb}}_A(V) \) is a subnilmanifold and \( \overline{\text{Orb}}_g(V) \) is a FU-subnilmanifold of \( X \); in this section we will study a relation between these orbits of \( V \) and the generic orbits for \( A \) and \( g \) on \( V \).

5.1. We first extend the notion of normality of a subnilmanifold introduced in 3.1. Let \( Y \) be a subnilmanifold of \( X \), \( Y = Hx \) where \( x \in X \) and \( H \) is a closed subgroup of \( G^o \). Let us say that \( Y \) is normal with respect to \( V \) if \( K \) normalizes \( H \).

5.2. Proposition. Let \( H \) be a closed subgroup of \( G^o \) and let \( x \in V \). If the subnilmanifold \( Y = Hx \) of \( X \) is normal with respect to \( V \), then

(i) the sets \( Hx, y \in V \), are all closed;
(ii) the set \( W = HV \) is a subnilmanifold of \( X \) with \( \dim W = \dim V + \dim Y - \dim(V \cap Y) \),
and the sets \( aY, a \in K \), partition \( W \);
(iii) the subnilmanifolds \( aY \cap V, a \in K \), of \( V \) are all congruent and partition \( V \).

**Proof.** We may assume that \( x = 1_X \) and so, \( Y = \pi(H) \). Since \( K \) normalizes \( H \), the set \( HK = KH \) is a closed subgroup of \( G^\circ \). \( \Gamma \cap K \) is uniform in \( K \) and \( \Gamma \cap H \) is uniform in \( H \), thus \( \Gamma \cap (KH) \) is uniform in \( KH \). Thus, \( W = \pi(HK) = HV \) is a subnilmanifold of \( X \).

\( H \) is a normal subgroup of \( KH \), thus \( Y \) is a normal subnilmanifold of \( W \). Hence, the sets \( H_y, y \in V \), are equal to the sets \( aY, a \in K \), are closed and partition \( W \). \( H \cap K \) is a normal subgroup of \( K \), thus \( Y \cap V \) is a normal subnilmanifold of \( V \), so the sets \( aY \cap V = aY \cap aV = a(Y \cap V) \) partition \( V \). The factor-nilmanifold \( W/Y \) is isomorphic to the factor-nilmanifold \( V/(V \cap Y) \), so \( \dim W = \dim Y + \dim(V/(V \cap Y)) = \dim Y + \dim V - \dim(V \cap Y) \).

5.3. Let us denote the subnilmanifold \( W = HV \), appearing in Proposition 5.2, by \( YA \).

5.4. We now have:

**Theorem.** Let \( A \) be a subgroup of \( G \). The connected components of the generic orbit \( Y_{V,A} \) of \( A \) on \( V \) are normal with respect to \( V \).

5.5. We need an extension of Lemma 3.3:

**Lemma.** Let \( H \) and \( K \) be subgroups of \( G^\circ \), let \( \Lambda \) be a uniform subgroup of \( K \) and assume that \( \Lambda \) normalizes \( H \). Then \( K \) normalizes \( H \).

The proof of this lemma is similar to the proof of Lemma 3.3.

5.6. **Proof of Theorem 5.4.** Let \( \Lambda = \Gamma \cap K \), this is a uniform subgroup of \( K \). Let \( P \) be the set, introduced in Theorem 2.2, of points of \( V \) whose orbits under the action of \( A \) are nongeneric on \( V \). Let \( x \in V \setminus P \); we may assume that \( x = 1_X \). Then, by Theorem 2.2, for any \( \lambda \in \Lambda \), \( \text{Orb}_A(1_X) = \text{Orb}_A(\pi(\lambda)) = \lambda Y_{V,A} \). So, \( \lambda Y_{V,A} = Y_{V,A} \) for all \( \lambda \in \Lambda \). Let \( H \) be the closed subgroup of \( G^\circ \) such that \( Y_{V,A} = \pi(H) \) and let \( H^0 \) be the identity component of \( H \). For \( \lambda \in \Lambda \) we have \( \lambda H^0 = H^0 \gamma \), and \( \lambda H^0 = H^0 c^\gamma \) for some \( c \in H \) and \( \gamma \in \Gamma^0 \). So \( \lambda H^0 \gamma^{-1} = H^0 c^\gamma \lambda^{-1} \), and since \( \lambda H^0 \gamma^{-1} \) is a subgroup of \( G^\circ \), \( \lambda H^0 \gamma^{-1} = H^0 \). Hence, \( H^0 \) is normalized by \( \Lambda \); by Lemma 5.5, \( H^0 \) is normalized by \( K \).

5.7. As a corollary, we get

**Theorem.** Let \( A \) be a subgroup of \( G \) and \( Y \) be a connected component of the generic orbit \( Y_{V,A} \) of \( A \) on \( V \). The connected components of the orbit \( \text{Orb}_A(V) \) of \( V \) under the action of \( A \) are translates of \( YV \).

**Proof.** If \( Y_{V,A} = Y \) is connected, it is normal with respect to \( V \), thus \( YV \) is defined and is a closed subnilmanifold of \( X \). For every point \( x \in V \), \( x = \pi(a) \) with \( a \in K \), we have \( \text{Orb}_A(x) \subseteq aY \subseteq YV \), thus \( \text{Orb}_A(V) \subseteq YV \). For almost every point \( x \in V \), \( x = \pi(a) \) with \( a \in K \), we have \( \text{Orb}_A(x) = aY \), thus \( \bigcup_{a \in K} \text{Orb}_A(x) \) is dense in \( YV \), and so, \( \text{Orb}_A(V) = YV \).

If \( Y_{V,A} \) is not connected and \( Y \) is its connected component, we can find in \( A \) a subgroup \( B \) of finite index such that \( Y_{V,B} = Y \). Thus, \( \text{Orb}_B(V) = YV \). Now, \( A = \bigcup_{i=1}^s b_i B \) for
some $b_1, \ldots, b_s \in A$, and thus $\overline{\text{Orb}}_A(V) = \bigcup_{i=1}^s b_i Y V$. ■

5.8. Similarly, we have

**Theorem.** Let $g$ be a polynomials sequence in $G$. The connected components of the generic orbit $Y_{V,g}$ of $g$ on $V$ are normal with respect to $V$.

**Proof.** Let $g : \mathbb{Z}^l \to G$; after passing to a subgroup of finite index in $\mathbb{Z}^l$ we may assume that $Y_{V,g}$ is connected. Next, we may assume that $g(0) = 1_G$. Let $P$ be the set, introduced in Theorem 2.3, of points whose orbits under the action of $g$ are nongeneric on $X$. Let $x \notin P$; we may assume that $x = 1_X$. Let $\Lambda = \Gamma \cap K$, this is a uniform subgroup of $K$. By Theorem 2.3, for any $\lambda \in \Lambda$, $\overline{\text{Orb}}_g(1_X) = \overline{\text{Orb}}_g(\pi(\lambda)) = \lambda Y_{V,g}$. So, $\lambda Y_{V,g} = Y_{V,g}$ for all $\lambda \in \Lambda$. Let $H$ be the connected closed subgroup of $G^o$ such that $Y_{V,g} = \pi(H)$. Let $\lambda \in \Lambda$, then $\lambda H = H c_\gamma$ for some $\gamma \in \Gamma^o$. So $\gamma H \lambda^{-1} = H \gamma \lambda^{-1}$, and since $\lambda H \lambda^{-1}$ is a subgroup of $G^o$, $\lambda H \lambda^{-1} = H$. Hence, $H$ is normalized by $\Lambda$; by Lemma 5.5, $H$ is normalized by $K$. ■

5.9. And as a corollary we obtain

**Theorem.** Let $g$ be a polynomial sequence in $G$. Every connected component of the orbit $\overline{\text{Orb}}_g(V)$ of $V$ under the action of $g$ is a translate of $Y V$, where $Y$ is a connected component of the generic orbit $Y_{V,g}$ of $g$ on $V$.

**Proof.** Again, by passing to a subgroup of finite index in $\mathbb{Z}^l$ the problem is reduced to the case $Y_{V,g} = Y$ is connected. $Y$ is normal with respect to $V$, thus $Y V$ is a closed sub-nilmanifold of $X$. For every point $x \in V$, $x = \pi(a)$ with $a \in K$, we have $\overline{\text{Orb}}_g(x) \subseteq a Y \subseteq Y V$, thus $\overline{\text{Orb}}_g(V) \subseteq Y V$. For almost every point $x \in V$, $x = \pi(a)$ with $a \in K$, we have $\overline{\text{Orb}}_g(x) = a Y$, thus $\bigcup_{a \in K} \overline{\text{Orb}}_g(x)$ is dense in $Y V$, and so, $\overline{\text{Orb}}_g(V) = Y V$. ■

**Acknowledgment.** I thank the anonymous referee for comments and suggestions.

**Bibliography**


