Polynomial mappings of groups

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Abstract

A mapping φ of a group G to a group F is said to be polynomial if it trivializes after several consecutive applications of operators D_h, $h \in G$, defined by $D_h\varphi(g) = \varphi(g)^{-1}\varphi(gh)$. We study polynomial mappings of groups, mainly to nilpotent groups. In particular, we prove that polynomial mappings to a nilpotent group form a group with respect to the elementwise multiplication, and that any polynomial mapping $G \longrightarrow F$ to a nilpotent group F splits into a homomorphism $G \longrightarrow G'$ to a nilpotent group G' and a polynomial mapping $G' \longrightarrow F$. We apply the obtained results to prove the existence of the compact/weak mixing decomposition of a Hilbert space under a unitary polynomial action of a finitely generated nilpotent group.

0. Introduction

0.1. In contrast with the case of abelian groups, the element-wise product $\varphi = \varphi_1\varphi_2$ of two homomorphisms $\varphi_1, \varphi_2 : G \longrightarrow F$ of general groups need not to be a homomorphism. If F is a nilpotent group, then φ is not, however, quite arbitrary. Consider the following example. Let F be the (nilpotent) group of upper triangular $\mathbb{Z}$-matrices with unit diagonal. Then for any homomorphism $\varphi : \mathbb{Z} \longrightarrow F$ the entries of $\varphi(n)$, $n \in \mathbb{Z}$, are polynomials in n. Let us say that a mapping $\varphi : \mathbb{Z} \longrightarrow F$ **is polynomial** if the entries of $\varphi(n)$ are polynomials in n. Then the set of polynomial mappings $\mathbb{Z} \longrightarrow F$ is closed with respect to the element-wise multiplication $(\varphi_1\varphi_2)(n) = \varphi_1(n)\varphi_2(n)$. It follows that the product of finitely many homomorphism $\mathbb{Z} \longrightarrow F$ is a polynomial mapping.

0.2. We show in this paper that the example in 0.1 illustrates a general phe-

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nomenon. For a mapping φ of a group G to a group F and for $h \in G$ we define the operator of "differentiation" D_h by $(D_h\varphi)(g) = \varphi(g)^{-1}\varphi(gh)$, $g \in G$. We say that $\varphi\colon G \longrightarrow F$ **is polynomial of degree** $\leq d$ if the successive application of $d+1$ operators of differentiation cancels φ: for any $h_1, h_1, \ldots, h_{d+1} \in G$, $D_{h_1} D_{h_2} \ldots D_{h_{d+1}} \varphi \equiv \mathbf{1}_F$. Under this definition, polynomial mappings of degree 0 are constants and polynomial mappings of degree ≤ 1 are affine homomorphisms. Polynomial mappings inherit some properties of the conventional polynomials: the set of polynomial mappings is invariant under translations in both F and G; given a set S of generators of G, any polynomial mapping of degree $\leq d$ is determined by its values on the set $S^{\leq d} = \bigl\{g_1 \ldots g_k \bigm| k \leq d,\ g_1, \ldots, g_k \in S\bigr\}$. However, the "sum" (that is, the element-wise product) $\varphi(g) = \varphi_1(g)\varphi_2(g)$ of two polynomial mappings need not to be polynomial: the simplest example is provided by the homomorphisms $\varphi_1(n) = f_1^n$, $\varphi_2(n) = f_2^n$ of $\mathbb{Z}$ to the free group F generated by f_1, f_2. The example in section 3.1 below demonstrates that even in the case where F is a metabelian (2-step solvable) group, the product of two homomorphisms may not be polynomial.

0.3. The situation changes if one deals with nilpotent groups: polynomial mappings of an arbitrary group to a nilpotent group form a group with respect to the element-wise multiplication (Theorem 3.2). It seems that nilpotent groups form a natural scope for polynomial mappings: we prove that, if F is nilpotent, the composition of polynomial mappings $G \longrightarrow G_1$ and $G_1 \longrightarrow F$ is polynomial (Proposition 3.22); that a polynomial mapping of an arbitrary group G to a nilpotent group F is decomposable into the composition of an epimorphism $G \longrightarrow G'$ onto a nilpotent group G' and a polynomial mapping $G' \longrightarrow F$ (Proposition 3.21); that the operations of multiplication and arising to powers in a nilpotent group are polynomial (Corollary 3.7).

0.4. We utilize the obtained results to describe the decomposition of a Hilbert space into the sum of the compact/weak mixing subspaces under the "polynomial action" of a finitely generated nilpotent group. If φ is a unitary action of a countable amenable group G on a Hilbert space $\mathcal{H}$, then one has a decomposition $\mathcal{H} = \mathcal{H}^{\mathrm{c}}(\varphi) \oplus \mathcal{H}^{\mathrm{wm}}(\varphi)$, where $\mathcal{H}^{\mathrm{c}}(\varphi)$ and $\mathcal{H}^{\mathrm{wm}}(\varphi)$ are $\varphi(G)$-invariant subspaces of $\mathcal{H}$ such that the action φ of G is compact on $\mathcal{H}^{\mathrm{c}}(\varphi)$ and is weakly mixing on $\mathcal{H}^{\mathrm{wm}}(\varphi)$. That is, for any $u \in \mathcal{H}^{\mathrm{c}}(\varphi)$ the orbit $\varphi(G)u = \bigl\{\varphi(g)u \bigm| g \in G\bigr\}$ is precompact, and for any $u, u' \in \mathcal{H}^{\mathrm{wm}}(\varphi)$ and any $\varepsilon > 0$ the set $\bigl\{g \in G \bigm|$

$|\langle\varphi(g)u,u'\rangle|>\varepsilon\}$ has zero density in G (with respect to any Følner sequence in G) ([D]). Such a decomposition may not exist if φ is not a homomorphism of G to the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$ but the element-wise product of two homomorphisms $G\longrightarrow U(\mathcal{H})$:

Example. Let $\mathcal{H}$ be a Hilbert space with the orthonormal basis $\{u_i,v_i,x_i,y_i\}_{i\in\mathbb{Z}}$ and let T and S be unitary operators on $\mathcal{H}$ whose action on the elements of the basis are defined by

$$\begin{aligned} &T: u_i\mapsto y_i,\quad v_i\mapsto x_i,\quad x_i\mapsto v_{i-1},\quad y_i\mapsto u_{i+1}\\ \text{and}\quad &S: u_i\mapsto v_i,\quad v_i\mapsto u_{i+1},\quad x_i\mapsto y_i,\quad y_i\mapsto x_{i+1}.\end{aligned}$$

Let $\varphi(n)=T^nS^n$ (that is, $\varphi(n)w=T^n(S^nw)$, $n\in\mathbb{Z}$, $w\in\mathcal{H}$); φ is the product of the homomorphisms $n\mapsto T^n$ and $n\mapsto S^n$ of $\mathbb{Z}$ to the group of unitary operators on $\mathcal{H}$. We have

$$\varphi(n):\begin{cases} u_i\mapsto u_{i+n},\quad v_i\mapsto v_i,\quad x_i\mapsto x_i,\quad y_i\mapsto y_{i+n} & \text{if } n \text{ is even},\\ u_i\mapsto x_i,\quad v_i\mapsto y_{i+n},\quad x_i\mapsto u_{i+n},\quad y_i\mapsto v_i & \text{if } n \text{ is odd},\end{cases}$$

and it is easy to see that φ is neither compact nor weakly mixing on any vector from $\mathcal{H}$.

0.5. However, if φ_1, φ_2 are homomorphisms of a (finitely generated) amenable group G to a *nilpotent group* of unitary operators on $\mathcal{H}$, then φ is a unitary **polynomial action** of G, that is, a polynomial mapping (see 1.4) of G to $U(\mathcal{H})$. In this case we have:

Theorem. *Let φ be a polynomial mapping of a finitely generated amenable group G to a nilpotent group F of unitary operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}=\mathcal{H}^{\mathrm{c}}(\varphi)\oplus\mathcal{H}^{\mathrm{wm}}(\varphi)$, where $\mathcal{H}^{\mathrm{c}}(\varphi)$ and $\mathcal{H}^{\mathrm{wm}}(\varphi)$ are $\varphi(G)$-invariant orthogonal subspaces, φ is compact on $\mathcal{H}^{\mathrm{c}}(\varphi)$ and is weakly mixing on $\mathcal{H}^{\mathrm{wm}}(\varphi)$.*

0.6. Though we are mainly interested in polynomial mappings of nilpotent groups, many facts brought in this paper have quite general character; therefore, when possible, we will consider polynomial mappings of general groups. Polynomial mappings to abelian groups are discussed in [B], Chapter 4.

0.7. In Section 1 we describe some useful properties of polynomial mappings of general groups. In Section 2 we discuss polynomial mappings to abelian groups. Section 3 is devoted to polynomial mappings to nilpotent groups. In Section 4 we prove that "the set of zeroes" $\varphi^{-1}(\mathbf{1}_F)$ of a polynomial mapping φ of a countable amenable group G to a torsion-free group F has zero density in G. In Section 5 we consider weakly mixing/compact properties of polynomial unitary and polynomial measure preserving actions of finitely generated nilpotent groups.

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1. Polynomial mappings of groups

1.1. Let φ be a mapping of a group G to a group F. For $h \in G$, we define **the h-derivative of** φ, $D_h\varphi: G \longrightarrow F$, by $D_h\varphi(g) = \varphi(g)^{-1}\varphi(gh)$.

1.2. Lemma. *For a mapping $\varphi: G \longrightarrow F$ and for any $h, h_1, h_2 \in G$, one has $D_{h_1h_2}\varphi(g) = D_{h_1}\varphi(g)D_{h_2}\varphi(gh_1) = D_{h_1}\varphi(g)D_{h_2}\varphi(g)D_{h_1}D_{h_2}\varphi(g)$ and $D_{h^{-1}}\varphi(g)$ $= D_h\varphi(gh^{-1})^{-1}$.*

Proof. Direct computation.

1.3. Lemma. *Let $\varphi: G \longrightarrow F$ satisfy $\varphi(\mathbf{1}_G) = \mathbf{1}_F$. Then for any $h \in G$, $D_h\varphi(\mathbf{1}_G) = \varphi(h)$ and $D_h\varphi(h^{-1}) = \varphi(h^{-1})^{-1}$.*

Proof. Direct computation.

1.4. Unless stated otherwise, we will assume that G and F are groups and that $S \subseteq G$ is a generating set for G. Let d be a nonnegative integer; we say that φ is **polynomial of degree $\leq d$ (relative to S)** if for any $h_1, \ldots, h_{d+1} \in S$, $D_{h_1} \ldots D_{h_{d+1}}\varphi \equiv \mathbf{1}_F$. We will call the minimal d with this property **the degree of φ (relative to S)**: $\deg\varphi = d$.

Thus, a mapping $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$, $d \geq 1$, relative to S if and only if for all $h \in S$, the mappings $D_h\varphi$ are polynomial of degrees $\leq d-1$.

1.5. Clearly, if S_1, S_2 are two generating sets for G with $S_1 \subseteq S_2$ and a mapping $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$ relative to S_2, then φ is polynomial of degree $\leq d$ relative to S_1. So, the strongest definition of polynomiality corresponds to $S = G$.

When the set S does not matter, we will omit the words "relative to S" and simply write "φ is polynomial".

1.6. Let us remark that the requirement that S generates G is not a principal restriction. Indeed, assume that $S \subseteq G$ generates a subgroup G' of G. Then G is partitioned into the union of left cosets of G', each of which is (affine) isomorphic to G'. A mapping $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$ relative to S if and only if the restriction of φ on each of these cosets is polynomial of degree $\leq d$ relative to S.

1.7. Proposition. *A polynomial mapping of degree zero is constant. A polynomial mapping of degree one is a nonconstant affine homomorphism (that is, a homomorphism multiplied by a constant).*

Proof. Let $D_h\varphi \equiv \mathbf{1}_F$ for all $h \in S$. Then for any $g \in G$ and $h \in S$, $\varphi(gh) = \varphi(g)D_h\varphi(g) = \varphi(g)$ and $\varphi(gh^{-1}) = \varphi(g)D_h\varphi(g)^{-1} = \varphi(g)$; since S generates G, this implies $\varphi \equiv \varphi(\mathbf{1}_G)$.

Now let $D_h\varphi = \text{const} = D_h\varphi(\mathbf{1}_G)$ for all $h \in G$. Then for any $g \in G$ and $h \in S$, $D_h\varphi(g) = D_h\varphi(\mathbf{1}_G) = \varphi(\mathbf{1}_G)^{-1}\varphi(h)$, and so,

$$\varphi(\mathbf{1}_G)^{-1}\varphi(gh) = \big(\varphi(\mathbf{1}_G)^{-1}\varphi(g)\big)\big(\varphi(\mathbf{1}_G)^{-1}\varphi(h)\big).$$

Since S generates G, this implies that $\varphi(\mathbf{1}_G)^{-1}\varphi$ is a homomorphism. ∎

It follows that if φ is a polynomial mapping of degree $\leq d$ relative to S, then for any $h_1, \ldots, h_d \in S$ the mapping $D_{h_1} \ldots D_{h_d}\varphi$ is constant, for some $h_1, \ldots, h_d$ this constant differs from $\mathbf{1}_F$, and then the mapping $D_{h_1} \ldots D_{h_{d-1}}\varphi$ is a nonconstant affine homomorphism.

1.8. Polynomial mappings $\mathbb{Z}^k \longrightarrow \mathbb{Z}^l$ (relative to any generating set S in $\mathbb{Z}^k$) are ordinary polynomials in k variables (with, possibly, rational coefficients: $\frac{1}{2}n(n-1)$ is a polynomial mapping $\mathbb{Z} \longrightarrow \mathbb{Z}$).

1.9. Polynomial mappings of groups share many properties with conventional polynomials; we will describe some of these properties in a series of simple propositions.

1.10. Proposition. *If $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$ and $\pi: F \longrightarrow F'$ is a homomorphism, then the composition $\pi \circ \varphi$ is also polynomial of degree $\leq d$.*

1.11. Proposition. *Let $\varphi: G \longrightarrow F$ be a mapping, let $\pi: \tilde{G} \longrightarrow G$ be a homomorphism, and let $\tilde{\varphi} = \varphi \circ \pi$. If φ is polynomial of degree d relative to $S \subseteq G$ and $\tilde{S} = \pi^{-1}(S)$ generates $\tilde{G}$, then $\tilde{\varphi}$ is polynomial of degree d relative to $\tilde{S}$. If $\tilde{\varphi}$ is polynomial of degree d relative to $\tilde{S} \subseteq \tilde{G}$ and π is onto, then φ is polynomial of degree d relative to $S = \pi(\tilde{S})$.*

Proof. For $\tilde{h} \in \tilde{S}$, $h = \pi(\tilde{h}) \in S$ and any $\tilde{g} \in \tilde{G}$ one has

$$D_{\tilde{h}}\tilde{\varphi}(\tilde{g}) = \tilde{\varphi}(\tilde{g})^{-1}\tilde{\varphi}(\tilde{g}\tilde{h}) = \varphi\big(\pi(\tilde{g})\big)^{-1}\varphi\big(\pi(\tilde{g}\tilde{h})\big) = \varphi\big(\pi(\tilde{g})\big)^{-1}\varphi\big(\pi(\tilde{g})h\big) \\ = D_h\varphi\big(\pi(\tilde{g})\big).$$

Hence, $D_{\tilde{h}}\tilde{\varphi} = D_h\varphi \circ \pi$, and the derivatives of φ and $\tilde{\varphi}$ trivialize simultaneously. ∎

1.12. Proposition. *Let $\varphi: G \longrightarrow F$ be a mapping, S be a generating set for G and $d \in \mathbb{N}$. Assume that for any $g \in G$ and any $h_1, \ldots, h_d \in S$ the restriction of φ onto the group generated by g and $h_1, \ldots, h_d$ is polynomial of degree $\leq d$ relative to $\{h_1, \ldots, h_d\}$, then φ is polynomial of degree $\leq d$ relative to S.*

Proof. For $d = 0$ the statement is clear. Let $h \in S$. For any $g \in G$ and $h_1, \ldots, h_{d-1} \in S$ the restriction of φ on the subgroup generated by $g, h, h_1, \ldots,$ h_{d-1} is polynomial of degree $\leq d$ relative to $\{h, h_1, \ldots, h_{d-1}\}$. Hence, the restriction of $D_h\varphi$ on the subgroup generated by $g, h_1, \ldots, h_{d-1}$ is polynomial of degree $\leq d-1$ relative to $\{h_1, \ldots, h_{d-1}\}$. By induction on d, $D_h\varphi$ is polynomial of degree $\leq d-1$ relative to S and so, φ is polynomial of degree $\leq d$ relative to S. ∎

1.13. Proposition. *If $\varphi: G \longrightarrow F$ is polynomial of degree d relative to $S \subseteq G$, then for any $g_0 \in G$, $f_0 \in F$, the mappings $\varphi_1(g) = f_0\varphi(g)$, $\varphi_2(g) = \varphi(g)f_0$ and $\varphi_3(g) = \varphi(g_0g)$ are polynomial of degree d relative to S, and $\varphi_4(g) = \varphi(gg_0)$ is polynomial of degree d relative to $g_0Sg_0^{-1}$.*

Proof. $D_h\varphi_1(g) = \varphi(g)^{-1}f_0^{-1}f_0\varphi(gh) = D_h\varphi(g)$,
$D_h\varphi_2(g) = f_0^{-1}\varphi(g)^{-1}\varphi(gh)f_0 = f_0^{-1}D_h\varphi(g)f_0$,
$D_h\varphi_3(g) = \varphi(g_0g)^{-1}\varphi(g_0gh) = D_h\varphi(g_0g)$,
$D_h\varphi_4(g) = \varphi(gg_0)^{-1}\varphi(ghg_0) = \varphi(gg_0)^{-1}\varphi(gg_0g_0^{-1}hg_0) = D_{g_0^{-1}hg_0}\varphi(gg_0)$. ∎

1.14. Proposition. *A mapping $\varphi: G \longrightarrow F_1 \times F_2$, $\varphi = (\varphi_1, \varphi_2)$, is polynomial of degree $\leq d$ if and only if both φ_1 and φ_2 are polynomial of degree $\leq d$.*

1.15. Proposition. *Let $S \subseteq G$ generate G and let $\varphi: G \longrightarrow F$ be a polynomial mapping of degree d relative to S. Then φ is uniquely determined by its values on the set $S^{\le d} = \{g = h_1 \dots h_l \mid 0 \le l \le d,\ h_1, \dots, h_l \in S\}$.* (We assume that the empty word represents $\mathbf{1}_G$.)

Proof. Assume that we are given $\varphi|_{S^{\le d}}$. For $h \in S$ and $g \in S^{\le d-1}$, $D_h\varphi(g) = \varphi(g)^{-1}\varphi(gh)$ with $gh \in S^{\le d}$ and so, by induction on d, $D_h\varphi$ is uniquely determined. Lemma 1.2 shows that $D_g\varphi$ is uniquely determined for any $g \in G$, and $\varphi(g) = \varphi(\mathbf{1}_G)D_g\varphi(\mathbf{1}_G)$. ■

1.16. Corollary (of the proof)**.** *Let $\varphi: G \longrightarrow F$ be a polynomial mapping of degree d relative to S and let H be a subgroup of F. If $\varphi(S^{\le d}) \subseteq H$, then $\varphi(G) \subseteq H$.*

1.17. Corollary. *Let $\varphi: \mathbb{Z} \longrightarrow F$ be a polynomial mapping of degree d relative to $\{1\}$ and let H be a subgroup of F. If $\varphi(0), \varphi(1), \dots, \varphi(d) \in H$, then $\varphi(\mathbb{Z}) \subseteq H$.*

1.18. Corollary. *Let G be generated by a finite set S and let $\varphi: G \longrightarrow F$ be a polynomial mapping relative to S. Then $\varphi(G)$ is contained in a finitely generated subgroup of F.*

Proof. $\varphi(G)$ lies in the subgroup of F generated by $\varphi(S^{\le d})$. ■

1.19. Proposition. *Let G be the free group generated by a set S and let $d \in \mathbb{N}$. Any mapping $\eta: S^{\le d} \longrightarrow F$ is extendible to a mapping $\varphi: G \longrightarrow F$ which is polynomial of degree $\le d$ relative to S.* (By Proposition 1.15, such φ is unique.)

Proof. For $h \in S$ define a mapping $\eta_h: S^{\le d-1} \longrightarrow F$ by $\eta_h(g) = \eta(g)^{-1}\eta(gh)$, $g \in S^{\le d-1}$. By induction on d, for every $h \in S$ there is a mapping $\varphi_h: G \longrightarrow F$ which is polynomial of degree $\le d-1$ and satisfies $\varphi_h|_{S^{\le d-1}} = \eta_h$. Define a mapping $\varphi: G \longrightarrow F$ in the following way. Put $\varphi(\mathbf{1}_G) = \eta(\mathbf{1}_G)$. Assume that φ has been already defined on the elements of G representable in the alphabet $S \cup S^{-1}$ by reduced words of length m, $m \ge 0$, and let $h \in S$. Then if gh is a reduced word of length $m+1$, we put $\varphi(gh) = \varphi(g)\varphi_h(g)$, and if gh^{-1} is a reduced word of length $m+1$, we put $\varphi(gh^{-1}) = \varphi(g)\varphi_h(gh^{-1})^{-1}$. It is easy to see that $\varphi|_{S^{\le d}} = \eta$ and that for any $h \in S$, $D_h\varphi = \varphi_h$. ■

1.20. Corollary. *Let G be the free group generated by a set S, let $\varphi: G \longrightarrow F$ be a polynomial mapping of degree $\le d$ relative to S and let $\pi: \tilde{F} \longrightarrow F$ be an* *epimorphism. Then there is a mapping $\tilde{\varphi}: G \longrightarrow \tilde{F}$ which is polynomial of degree $\le d$ relative to S and such that $\pi \circ \tilde{\varphi} = \varphi$.*

Proof. Define $\tilde{\varphi}$ on $S^{\le d}$ so that $\pi \circ \tilde{\varphi}|_{S^{\le d}} = \varphi|_{S^{\le d}}$ and extend it to a polynomial mapping $G \longrightarrow \tilde{F}$ of degree $\le d$. Then $\pi \circ \tilde{\varphi}$ is a polynomial mapping of degree $\le d$ coinciding with φ on $S^{\le d}$. By Proposition 1.15, $\pi \circ \tilde{\varphi} = \varphi$. ■

1.21. We now pass to the case of a torsion-free F.

Proposition. *Let F have no torsion and let $\varphi: G \longrightarrow F$ be a nonconstant polynomial mapping. Then $\varphi(G)$ is infinite.*

Proof. If $\varphi(G)$ is finite, then for any $h \in S$, $D_h\varphi(G) \subseteq \varphi(G)^{-1}\varphi(G)$ and thus is also finite. Let $\deg \varphi = d$; then, for appropriate $h_1, \ldots, h_{d-1} \in S$, $D_{h_1} \ldots D_{h_d}\varphi$ is a nonconstant affine homomorphism $G \longrightarrow F$ with finite range, which is impossible since F has no torsion. ■

1.22. Proposition. *Let F have no torsion and let $\varphi: \mathbb{Z} \longrightarrow F$ be a polynomial mapping of degree $\le d$ relative to* $\mathbb{N}$ (sic!). *Assume that for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\varphi(m) = \varphi(m+n) = \ldots = \varphi(m+dn)$. Then φ is constant.*

Proof. Assume that φ is nonconstant. Put $\varphi_1 = D_n\varphi$, $\varphi_2 = D_n\varphi_1$, …, and let k be the maximal integer for which φ_k is nonconstant. We have $k \le d-1$ and $\varphi_k(m) = \ldots = \varphi_k\big(m+(d-k)n\big) = \mathbf{1}_F$. But φ_k is an affine homomorphism, $\varphi_k(l) = f_0 f^l$ for some $f_0, f \in F$. Thus $f_0 f^m = f_0 f^{m+n}$, which implies $f^n = \mathbf{1}_F$. Since F has no torsion, $f = \mathbf{1}_F$, which leads to contradiction. ■

1.23. Lemma. *Let $\varphi: G \longrightarrow F$ be a mapping with $\varphi(\mathbf{1}_G) = \mathbf{1}_F$, let $S \subseteq G$ be a generating set for G and let $g \in G$ be such that $D_h\varphi(gg_0) = D_h\varphi(g_0)$ for all $g_0 \in G$ and $h \in S$. Then $\varphi(gg_0) = \varphi(g)\varphi(g_0)$ for all $g_0 \in G$.*

Proof. For any $h \in S$ we have $D_h\varphi(g) = D_h\varphi(\mathbf{1}_G) = \varphi(h)$. Now, assume that for some $g_0 \in G$ one has $\varphi(g_0 g) = \varphi(g)\varphi(g_0)$. Then, for any $h \in S$,

$$\varphi(gg_0h) = \varphi(gg_0)D_h\varphi(gg_0) = \varphi(g)\varphi(g_0)D_h\varphi(g_0) = \varphi(g)\varphi(g_0h)$$

and

$$\begin{aligned}\varphi(gg_0h^{-1}) &= \varphi(gg_0)D_{h^{-1}}\varphi(gg_0) = \varphi(g)\varphi(g_0)D_h\varphi(gg_0h^{-1})^{-1} \\ &= \varphi(g)\varphi(g_0)D_h\varphi(g_0h^{-1})^{-1} = \varphi(g)\varphi(g_0)D_{h^{-1}}\varphi(g_0) = \varphi(g)\varphi(g_0h).\end{aligned}$$

■

1.24. Proposition. *Let F have no torsion, let $\varphi: G \longrightarrow F$ be a polynomial mapping relative to $S \subseteq G$ and let g be an element of G of finite order. Then for any $g_1, g_2 \in G$, $\varphi(g_1 g g_2) = \varphi(g_1 g_2)$.*

Proof. We may assume that $\varphi(\mathbf{1}_G) = \mathbf{1}_F$. By induction on the degree of φ, for any $h \in S$ and any $g_2 \in G$, $D_h\varphi(gg_0) = D_h\varphi(g_2)$. By Lemma 1.23, $\varphi(gg_2) = \varphi(g)\varphi(g_2)$ for all $g_2 \in G$. In particular, $\varphi(g^n) = \varphi(g)^n$ for all $n \in \mathbb{N}$. Since g has finite order and F has no torsion, $\varphi(g) = \mathbf{1}_F$. So, $\varphi(gg_2) = \varphi(g_2)$. Applying this formula to the polynomial mapping $\varphi_1(g) = \varphi(g_1 g)$, $g_1 \in G$, we get $\varphi(g_1 g g_2) = \varphi(g_1 g_2)$. ■

1.25. One can generalize Proposition 1.24 as follows. Let us say that a subgroup H of a group F is **closed** if $f^n \in H$ with $n \neq 0$ implies $f \in H$. Clearly, a normal subgroup H of F is closed if and only if F/H has no torsion. It is also clear that if a subgroup H of F is closed, then the conjugate subgroups $f^{-1}Hf$, $f \in F$, are also closed and thus, the normal subgroup $\bigcap_{f\in F}(f^{-1}Hf)$ is closed.

1.26. Proposition. *Let H be a closed subgroup of F and let $\varphi: G \longrightarrow F$ be a polynomial mapping. Then for any $g \in G$ of finite order and any $g_1, g_2 \in G$, $\varphi(g_1 g g_2) \in \varphi(g_1 g_2)H \cap H\varphi(g_1 g_2)$.*

Proof. Replace F by $F/\bigcap_{f\in F}(f^{-1}Hf)$ and apply Proposition 1.24. ■

1.27. Question: Is the composition of two polynomial mappings polynomial? This is true for mappings to nilpotent groups (see Proposition 3.22 below); we however doubt that this is true in general.

2. Polynomial mappings to abelian groups

The results in this section are preparatory, they will be used and strengthened in Section 3. We will assume in this section that F is an abelian group written additively.

2.1. The following is obvious:

Lemma. *Polynomial mappings $G \longrightarrow F$ of degree $\leq d$ form a group under addition.*

2.2. Lemma. *If $\varphi: G \longrightarrow F$ is polynomial of degree d relative to $S \subseteq G$, then φ*

is polynomial of degree d relative to G.

Proof. For any $h \in S$, $D_h\varphi$ is polynomial of degree $\leq d-1$ relative to S, and thus, by induction on d, is polynomial of degree $\leq d-1$ relative to G. It follows from Lemma 1.2 and Lemma 2.1 that $D_h\varphi$ are polynomial of degrees $\leq d-1$ relative to G for all $h \in G$, which implies the result. ∎

2.3. Lemma. *For a mapping $\varphi: G \longrightarrow F$ and any $k \in \mathbb{N}$, $h_1, \ldots, h_k, g \in G$, one has $D_{h_1} \ldots D_{h_k}\varphi(g) = \sum_{A\subseteq\{1,\ldots,k\}}(-1)^{k-|A|}\varphi\big(g\prod_{j\in A} h_j\big)$.*

(In the product $\prod_{j\in A} h_j$ elements h_j are taken in the natural order: if $A = \{j_1, \ldots, j_l\}$ with $j_1 < \ldots < j_l$, then $\prod_{j\in A} h_j = h_{j_1} \ldots h_{j_l}$.)

Proof. Induction on k. ∎

2.4. Lemma. *A mapping $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$ if and only if*

$$\sum_{A\subseteq\{1,\ldots,d+1\}} (-1)^{d+1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big) = 0 \text{ for any } g_1, \ldots, g_{d+1} \in G.$$

(For $d = 2$, for example, this is $\varphi(g_1g_2g_3) - \varphi(g_1g_2) - \varphi(g_1g_3) - \varphi(g_2g_3) + \varphi(g_1) + \varphi(g_2) + \varphi(g_3) - \varphi(\mathbf{1}_G) = 0$.)

Proof. By Lemma 2.3, the condition

$$D_{g_2} \ldots D_{g_{d+1}}\varphi(g_1) = \text{const} = D_{g_2} \ldots D_{g_{d+1}}\varphi(\mathbf{1}_G)$$

can be rewritten as

$$\sum_{A\subseteq\{2,\ldots,d+1\}} (-1)^{d-|A|}\varphi\Big(g_1\prod_{j\in A} g_j\Big) = \sum_{A\subseteq\{2,\ldots,d+1\}} (-1)^{d-|A|}\varphi\Big(\prod_{j\in A} g_j\Big),$$

which gives the result. ∎

2.5. Let us define, for a set X and $l \in \mathbb{N}$, $\wp^{=l}X = \big\{A \subseteq X \bigm| |A| = l\big\}$ and $\wp^{\leq l}X = \big\{A \subseteq X \bigm| |A| \leq l\big\}$.

Lemma. *If $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$, then for any $k \geq d+1$ and $g_1, \ldots, g_k \in G$,*

$$\varphi\Big(\prod_{j=1}^{k} g_j\Big) = \sum_{A\in\wp^{\leq d}\{1,\ldots,k\}} (-1)^{d-|A|}\tbinom{k-|A|-1}{d-|A|}\varphi\Big(\prod_{j\in A} g_j\Big).$$

Proof. Let $k \in \mathbb{N}$, let $g_1, \dots, g_k \in G$ and let φ be polynomial of degree $d \leq k-1$. By Lemma 2.4, $\varphi\big(\prod_{j=1}^k g_j\big) = \sum\limits_{A \in \wp^{\leq k-1}\{1,\dots,k\}} (-1)^{k-1-|A|}\varphi\big(\prod_{j\in A} g_j\big)$. By induction on decreasing d, we then have

$$
\begin{aligned}
\varphi\Big(\prod_{j=1}^k g_j\Big) &= \sum_{A \in \wp^{\leq d+1}\{1,\dots,k\}} (-1)^{d+1-|A|}\tbinom{k-|A|-1}{d+1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big) \\
&= \sum_{B \in \wp^{= d+1}\{1,\dots,k\}} \varphi\Big(\prod_{j\in B} g_j\Big) + \sum_{A \in \wp^{\leq d}\{1,\dots,k\}} (-1)^{d+1-|A|}\tbinom{k-|A|-1}{d+1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big) \\
&= \sum_{B \in \wp^{= d+1}\{1,\dots,k\}} \Big(\sum_{A \in \wp^{\leq d} B} (-1)^{d-|A|}\varphi\Big(\prod_{j\in A} g_j\Big)\Big) \\
&\qquad\qquad + \sum_{A \in \wp^{\leq d}\{1,\dots,k\}} (-1)^{d+1-|A|}\tbinom{k-|A|-1}{d+1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big) \\
&= \sum_{A \in \wp^{\leq d}\{1,\dots,k\}} \Big((-1)^{d-|A|}\tbinom{k-|A|}{d+1-|A|} + (-1)^{d+1-|A|}\tbinom{k-|A|-1}{d+1-|A|}\Big)\varphi\Big(\prod_{j\in A} g_j\Big) \\
&= \sum_{A \in \wp^{\leq d}\{1,\dots,k\}} (-1)^{d-|A|}\tbinom{k-|A|-1}{d-|A|}\varphi\Big(\prod_{j\in A} g_j\Big)
\end{aligned}
$$

∎

2.6. Proposition. *Let F and E be abelian groups, let $\varphi\colon G \longrightarrow F$ be polynomial of degree $\leq d$ and $\psi\colon F \longrightarrow E$ be polynomial of degree $\leq c$. Then the composition $\psi \circ \varphi$ is polynomial of degree $\leq dc$.*

2.7. To avoid cumbersome calculations involved in the direct proof of Proposition 2.6, we want to establish first a useful criterion of polynomiality. Let us introduce more notation. Assume that X, Y are sets and $\alpha\colon (X^0 \cup X^1 \cup X^2 \cup \ldots \cup X^d) \longrightarrow Y$ is a mapping. Let $k \in \mathbb{N}$ and $\bar{x} = (x_1, \dots, x_k) \in X^k$. Then we define a mapping $\alpha^{\bar{x}}\colon \wp^{\leq d}\{1,\dots,k\} \longrightarrow Y$ in the following way: for $A \in \wp^{\leq d}\{1,\dots,k\}$, $A = \{j_1, \dots, j_r\}$ with $j_1 < \ldots < j_r$, let $\alpha^{\bar{x}}(A) = \alpha(x_{j_1}, \dots, x_{j_r})$.

We will say that a mapping $\alpha\colon (X^0 \cup X^1 \cup X^2 \cup \ldots \cup X^d) \longrightarrow Y$ **is symmetric** if $\alpha(g_1, \dots, g_r) = \alpha(g_{\sigma(1)}, \dots, g_{\sigma(r)})$ for any $r \leq d$, any $g_1, \dots, g_r \in G$ and any permutation σ of $\{1, \dots, r\}$.

2.8. Lemma. *A mapping $\varphi\colon G \longrightarrow F$ is polynomial of degree $\leq d$ if and only if there is a mapping $\alpha\colon (G^0 \cup \ldots \cup G^d) \longrightarrow F$ such that for any $k \in \mathbb{N}$ and any $\bar{g} = (g_1, \dots, g_k) \in G^k$, $\varphi\Big(\prod\limits_{j=1}^k g_j\Big) = \sum\limits_{A \in \wp^{\leq d}\{1,\dots,k\}} \alpha^{\bar{g}}(A)$. If G is an abelian group, α is symmetric.*

Proof. Let $\alpha\colon (G^0 \cup \ldots \cup G^d) \longrightarrow F$ and $\varphi\colon G \longrightarrow F$ satisfy $\varphi\big(\prod_{j=1}^{k} g_j\big) = \sum\limits_{A\in\wp^{\le d}\{1,\ldots,k\}} \alpha^{\bar g}(A)$ for all $k \in \mathbb{N}$ and all $(g_1, \ldots, g_k) \in G^k$. Then for any $\bar g = (g_1, \ldots, g_{d+1}) \in G^{d+1}$,

$$\sum_{A\subseteq\{1,\ldots,d+1\}} (-1)^{d+1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big) = \sum_{A\subseteq\{1,\ldots,d+1\}} (-1)^{d+1-|A|}\Big(\sum_{B\in\wp^{\le d}A} \alpha^{\bar g}(B)\Big)$$

$$= \sum_{B\in\wp^{\le d}\{1,\ldots,d+1\}} \Big(\sum_{l=|B|}^{d+1} \tbinom{d+1-|B|}{l-|B|}\Big)\alpha^{\bar g}(B) = \sum_{B\in\wp^{\le d}\{1,\ldots,d+1\}} (1-1)^{d+1-|B|}\alpha^{\bar g}(B) = 0.$$

By Lemma 2.4, φ is polynomial of degree $\le d$.

Now let us assume that $\varphi\colon G \longrightarrow F$ is polynomial of degree $\le d$. Define $\alpha\colon (G^0 \cup \ldots \cup G^d) \longrightarrow F$ by $\alpha(\emptyset) = \varphi(\mathbf{1}_G)$, $\alpha(g_1, \ldots, g_r) = \sum\limits_{A\subseteq\{1,\ldots,r\}} (-1)^{r-|A|}\varphi\big(\prod_{j\in A} g_j\big)$ for $1 \le r \le d$, $\bar g = (g_1, \ldots, g_r) \in G^r$. Then for any $k \le d$ and $\bar g = (g_1, \ldots, g_k) \in G^k$ we have, using induction on k,

$$\varphi\Big(\prod_{j=1}^{d} g_j\Big) = \alpha(g_1, \ldots, g_k) - \sum_{A\in\wp^{\le k-1}\{1,\ldots,k\}} (-1)^{k-|A|}\varphi\Big(\prod_{j\in A} g_j\Big)$$

$$= \alpha(g_1, \ldots, g_k) - \sum_{A\in\wp^{\le k-1}\{1,\ldots,k\}} (-1)^{k-|A|}\Big(\sum_{B\subseteq A} \alpha^{\bar g}(B)\Big)$$

$$= \alpha(g_1, \ldots, g_k) - \sum_{B\in\wp^{\le k-1}\{1,\ldots,k\}} \Big(\sum_{l=|B|}^{k-1} (-1)^{k-l}\tbinom{k-|B|}{l-|B|}\Big)\alpha^{\bar g}(B) = \sum_{B\subseteq\{1,\ldots,k\}} \alpha^{\bar g}(B).$$

And for $k \ge d+1$ and $\bar g = (g_1, \ldots, g_k) \in G^k$, using Lemma 2.5 and induction on k we have

$$\varphi\Big(\prod_{j=1}^{k} g_j\Big) = \sum_{A\in\wp^{\le k-1}\{1,\ldots,k\}} (-1)^{k-1-|A|}\varphi\Big(\prod_{j\in A} g_j\Big)$$

$$= \sum_{A\in\wp^{\le k-1}\{1,\ldots,k\}} (-1)^{k-1-|A|}\Big(\sum_{B\in\wp^{\le d}A} \alpha^{\bar g}(B)\Big)$$

$$= \sum_{B\in\wp^{\le d}\{1,\ldots,k\}} \Big(\sum_{l=|B|}^{k-1} (-1)^{k-1-l}\tbinom{k-|B|}{l-|B|}\Big)\alpha^{\bar g}(B) = \sum_{B\in\wp^{\le d}\{1,\ldots,k\}} \alpha^{\bar g}(B).$$

In case of commutative G the symmetry of α follows from its definition. ∎

2.9. Now let us note that in 2.8, instead of $\{1,\ldots,k\}$ we can use as a set of indices any linearly ordered finite set. In case of abelian G, because of symmetry of α any non-ordered finite set can be used. It follows that for an abelian G we can reformulate Lemma 2.8 in the following way. For sets X, Y, a symmetric mapping $\beta\colon (X^0 \cup \ldots \cup X^d) \longrightarrow Y$, a set Z and a mapping $\xi\colon Z \longrightarrow X$ let us define a mapping $\beta^\xi\colon \wp^{\leq d} Z \longrightarrow Y$ by $\beta^\xi\bigl(\{z_1,\ldots,z_r\}\bigr) = \beta\bigl(\xi(z_1),\ldots,\xi(z_r)\bigr)$.

Lemma. *Let G be an abelian group (written additively). A mapping $\varphi\colon G \longrightarrow F$ is polynomial of degree $\leq d$ if and only if there is a symmetric mapping $\beta\colon (G^0 \cup \ldots \cup G^d) \longrightarrow F$ such that for any finite set Z and mapping $\xi\colon Z \longrightarrow G$ one has $\varphi\bigl(\sum_{z\in Z}\xi(z)\bigr) = \sum_{B\in\wp^{\leq d}Z}\beta^\xi(B)$.*

2.10. Proof of Proposition 2.6. Let G be a group, let F and E be abelian groups, let $\varphi\colon G \longrightarrow F$ be a polynomial mapping of degree $\leq d$ and let $\psi\colon F \longrightarrow E$ be a polynomial mapping of degree $\leq c$. Let $\alpha\colon (G^0 \cup \ldots \cup G^d) \longrightarrow F$ be the mapping defining φ as in Lemma 2.8, and let $\beta\colon (F^0 \cup \ldots \cup F^c) \longrightarrow E$ be the mapping defining ψ as in Lemma 2.9. Then for any $k \in \mathbb{N}$ and $\bar g = (g_1,\ldots,g_k) \in G^k$ we have

$$\psi\circ\varphi\Bigl(\prod_{j=1}^{k} g_j\Bigr) = \psi\Bigl(\sum_{A\in\wp^{\leq d}\{1,\ldots,k\}}\alpha^{\bar g}(A)\Bigr) = \sum_{B\in\wp^{\leq c}\wp^{\leq d}\{1,\ldots,k\}}\beta^{\alpha^{\bar g}}(B)$$
$$= \sum_{C\in\wp^{\leq dc}\{1,\ldots,k\}}\Bigl(\sum_{\substack{B\in\wp^{\leq c}\wp^{\leq d}C\\ \cup B = C}}\beta^{\alpha^{\bar g}}(B)\Bigr),$$

where for a set B we put $\bigcup B = \bigcup_{A\in B} A$. Now, if we define $\gamma\colon (G^0 \cup \ldots \cup G^{dc}) \longrightarrow F$ by $\gamma(g_1,\ldots,g_r) = \sum_{\substack{B\in\wp^{\leq c}\wp^{\leq d}\{1,\ldots,r\}\\ \cup B=\{1,\ldots,r\}}}\beta^{\alpha^{\bar g}}(B)$, $r \leq dc$, $\bar g = (g_1,\ldots,g_r) \in G^r$, we will have $\psi\circ\varphi\bigl(\prod_{j=1}^{k} g_j\bigr) = \sum_{C\in\wp^{\leq dc}\{1,\ldots,k\}}\gamma^{\bar g}(C)$ for all $k \in \mathbb{N}$ and $\bar g = (g_1,\ldots,g_k) \in G^k$. By Lemma 2.8, $\psi\circ\varphi$ is polynomial of degree $\leq dc$. ∎

2.11. Let $\mathbb{Z}[G]$ be the group ring of G; we may extend φ to a homomorphism $\mathbb{Z}[G] \longrightarrow F$ by linearity. Let $J = \bigl\{g - \mathbf{1}_G \bigm| g \in G\bigr\} \subseteq \mathbb{Z}[G]$; then, by Lemma 2.4, φ is polynomial of degree $\leq d$ if and only if it is trivial on J^{d+1}. In this case φ is also trivial on J^k for all $k \geq d+1$.

2.12. Let I be the augmentation ideal in $\mathbb{Z}[G]$, that is, the kernel of the augmentation homomorphism $\alpha\colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ (which is defined by $\alpha_{|G} \equiv 1$).

Lemma. *$\varphi\colon G \longrightarrow F$ is polynomial of degree $\leq d$ if and only if the extension of φ on $\mathbb{Z}[G]$ is trivial on I^{d+1}.* (This is the definition of polynomiality given in [B].)

Proof. I is spanned by elements of the form

$$g_1(g-\mathbf{1}_G)g_2 = (g_1-\mathbf{1}_G)(g-\mathbf{1}_G)(g_2-\mathbf{1}_G) + (g-\mathbf{1}_G)(g_2-\mathbf{1}_G) \\ +(g_1-\mathbf{1}_G)(g-\mathbf{1}_G) - (g-\mathbf{1}_G) \in J + J^2 + J^3,$$

thus $I \subseteq J + J^2 + J^3$ and so, $I^{d+1} \subseteq J^{d+1} + J^{d+2} + \ldots + J^{3d+3}$. ■

2.13. Given $h \in G$, let us define **the left h-derivative** of $\varphi\colon R \longrightarrow F$ by $D_h^L\varphi(g) = \varphi(hg) - \varphi(g)$. We will say that φ **is left-polynomial of degree** $\leq d$ if for any $h_1, \ldots, h_{d+1} \in G$, $D_{h_1}^L \ldots D_{h_{d+1}}^L \varphi \equiv 0$.

Corollary. *$\varphi\colon G \longrightarrow F$ is left-polynomial of degree $\leq d$ if and only if φ is (right-)polynomial of degree $\leq d$.*

2.14. Let $G = G_{(1)} \supset G_{(2)} \supset \ldots$ be the lower central series of G: $G_{(i+1)} = [G, G_{(i)}]$, $i \in \mathbb{N}$, and let I be the augmentation ideal in $\mathbb{Z}[G]$.

Lemma. (Cf. [B], section 27.) *For any $i \in \mathbb{N}$, $G_{(i)} - \mathbf{1}_G \subseteq I^i$.*

Proof. Assume by induction that $G_{(i-1)} - \mathbf{1}_G \subseteq I^{i-1}$. Then for any $g \in G^{(i-1)}$ and $g' \in G$ we have

$$[g, g'] - \mathbf{1}_G = g^{-1}g'^{-1}gg' - \mathbf{1}_G = g^{-1}g'^{-1}(gg' - g'g) \\ = g^{-1}g'^{-1}\big((g-\mathbf{1}_G)(g'-\mathbf{1}_G) - (g'-\mathbf{1}_G)(g-\mathbf{1}_G)\big) \in I^i.$$

$G_{(i)}$ is generated by elements of the form $[g, g']$ with $g \in G_{(i-1)}$, $g' \in G$, and if $g_1, g_2 \in G$ satisfy $g_1 - \mathbf{1}_G, g_2 - \mathbf{1}_G \in I^i$, then

$$g_1g_2 - \mathbf{1}_G = g_1(g_2 - \mathbf{1}_G) + g_1 - \mathbf{1}_G \in I^i.$$

■

2.15. Proposition. *If $\varphi: G \longrightarrow F$ is a polynomial mapping of degree $\leq d$, then there is a nilpotent group G' of class $\leq d$ such that φ splits into the composition $\varphi = \varphi' \circ \pi$ of an epimorphism $\pi: G \longrightarrow G'$ and a polynomial mapping $\varphi': G' \longrightarrow F$.*

Proof. In the notation of 2.14, φ is defined on $G/G_{(d+1)}$, that is, is constant on cosets of $G_{(d+1)}$ in G. Indeed, the extension of φ to $\mathbb{Z}[G]$ is trivial on I^{d+1}. But by Lemma 2.14, $G_{(d+1)} - \mathbf{1}_G \subseteq I^{d+1}$ and so,

$$\varphi(g_0 g) - \varphi(g_0) = \varphi\big(g_0(g - \mathbf{1}_G)\big) = 0$$

for all $g \in G_{(d+1)}$, $g_0 \in G$. ■

3. Polynomial mappings to nilpotent groups

3.1. Given a group G and a non-abelian group F, the product of two polynomial mappings $G \longrightarrow F$ does not have to be polynomial. Here is an example: let F be the group generated by f, f_1, f_2 satisfying $[f_1, f_2] = \mathbf{1}_F$, $f^{-1} f_1 f = f_1^2 f_2$ and $f^{-1} f_2 f = f_1 f_2$. Then the product of the polynomial mappings $\varphi_1, \varphi_2: \mathbb{Z} \longrightarrow F$, $\varphi_1(n) = f^{-n} f_1$, $\varphi_2(n) = f^n$, is

$$\varphi(n) = f^{-n} f_1 f^n = f_1^{a_1 \lambda_1^n + a_2 \lambda_2^n} f_2^{b_1 \lambda_1^n + b_2 \lambda_2^n},$$

where λ_1, λ_2 are the eigenvalues of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and a_1, a_2, b_1, b_2 are some constants. φ maps $\mathbb{Z}$ into the abelian group generated by f_1 and f_2 and, clearly, is not polynomial. In this example, the group F is metabelian.

3.2. However, when F is nilpotent, the situation improves:

Theorem. *If F is nilpotent, polynomial mappings $G \longrightarrow F$ form a group.*

3.3. To prove Theorem 3.2 we need to introduce the notion of the lc-degree (the degree, associated with the lower central series) of a polynomial mapping to a nilpotent group. Let F be nilpotent of class c and let $F = F_{(1)} \supset F_{(2)} \supset \ldots \supset F_{(c)} \supset F_{(c+1)} = \{\mathbf{1}_F\}$ be the lower central series of F: $F_{(i+1)} = [F_{(i)}, F]$, $i = 1, \ldots, c$. Put $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_* = \mathbb{Z}_+ \cup \{-\infty\}$. A vector $\bar{d} = (d_1, \ldots, d_c) \in \mathbb{Z}_*^c$ is said to be **superadditive** if $d_i \leq d_j$ for all $i < j$, and $d_i + d_j \leq d_{i+j}$ for all i, j with $i + j \leq c$ (we follow the convention that $-\infty < t$

and $-\infty + t = -\infty$ for any $t \in \mathbb{Z}_+$). Let $\varphi: G \longrightarrow F$ be a polynomial mapping relative to $S \subseteq G$ and let $\bar{d} = (d_1, \ldots, d_c) \in \mathbb{Z}_*^c$ be a superadditive vector. We will say that φ **has lc-degree** $\leq \bar{d}$ **(relative to** S**)**, $\text{lc-deg}\,\varphi \leq \bar{d}$, if for each $i = 1, \ldots, c$ one has:

if $d_i = -\infty$, then $\varphi(G) \in F_{(i+1)}$;

if $d_i \geq 0$, then for any $h_1, \ldots, h_{d_i+1} \in S$, $D_{h_1} \ldots D_{h_{d_i+1}} \varphi(G) \subseteq F_{(i+1)}$.

It is clear that if $\text{lc-deg}\,\varphi \leq (d_1, \ldots, d_c)$, then $\deg \varphi \leq d_c$, and if $\deg \varphi \leq d$, then $\text{lc-deg}\,\varphi \leq (d, 2d, \ldots, cd)$.

3.4. We will prove the following strengthening of Theorem 3.2:

Proposition. *Let F be nilpotent, let S be a generating set for G and let $\bar{d} \in \mathbb{Z}_*^c$ be a superadditive vector. Then polynomial mappings $G \longrightarrow F$ of lc-degree $\leq \bar{d}$ relative to S form a group.*

Proof. The proof is analogous to the proof of Theorem 1.12 in [L1]. For $d \in \mathbb{Z}_*$ and $t \in \mathbb{Z}_+$, put $d \div t = d - t$ if $d \geq t$ and $d \div t = -\infty$ otherwise. For $\bar{d} = (d_1, \ldots, d_c) \in \mathbb{Z}_*^c$, put $\bar{d} \div t = (d_1 \div t, \ldots, d_c \div t)$. Notice that $(\bar{d} \div t_1) \div t_2 = \bar{d} \div (t_1 + t_2)$, and that if $\bar{d}$ is a superadditive vector then for any $t \in \mathbb{Z}_+$, $\bar{d} \div t$ is also superadditive. It is clear that if φ is a polynomial mapping of lc-degree $\leq \bar{d}$, then for any $h \in S$, $D_h \varphi$ is polynomial of lc-degree $\leq \bar{d} \div 1$. And conversely, if for any $h \in S$ the mapping $D_h \varphi$ is polynomial of lc-degree $\leq \bar{d} = (d_1, \ldots, d_c)$, then φ is polynomial of lc-degree $\leq \bar{b} = (b_1, \ldots, b_c)$, where $b_i = d_i + 1$ if $d_i \geq 0$ and $b_i = 0$ if $d_i = -\infty$.

We will show the following:

(a) If φ_1, φ_2 are polynomial mappings of lc-degree $\leq \bar{d} \div t$, then $\varphi_1 \varphi_2$ is polynomial of lc-degree $\leq \bar{d} \div t$;

(b) If φ_1, φ_2 are polynomial mappings of lc-degrees $\leq \bar{d} \div t_1$ and $\leq \bar{d} \div t_2$ respectively, then $[\varphi_1, \varphi_2] = \varphi_1^{-1} \varphi_2^{-1} \varphi_1 \varphi_2$ is polynomial of lc-degree $\leq \bar{d} \div (t_1 + t_2)$;

(c) If φ is a polynomial mapping of lc-degree $\leq \bar{d} \div t$, then φ^{-1} is polynomial of lc-degree $\leq \bar{d} \div t$.

We will prove (a), (b) and (c) simultaneously by induction on decreasing t and $t_1 + t_2$.

First of all, if t is large enough ($t > d_c$), then $\text{lc-deg}\,\varphi \leq \bar{d} \div t$ means that $\varphi \equiv \mathbf{1}_F$, which trivially implies (a) and (c); (b) is trivially satisfied when $t_1 + t_2 > 2d_c$. Now let $s \in \mathbb{Z}_+$ and assume that (a), (b) and (c) hold for $t \geq s + 1$ and $t_1 + t_2 \geq s + 1$; we will prove that they hold for $t = t_1 + t_2 = s$.

(a) Let $t = s$ and let φ_1, φ_2 be polynomial mappings of lc-degree $\leq \bar{d} \mathbin{\dot{-}} t$. Then for any $h \in S$ and $g \in G$,

$$D_h(\varphi_1\varphi_2)(g) = \varphi_2(g)^{-1}\varphi_1(g)^{-1}\varphi_1(gh)\varphi_2(gh) = \varphi_2(g)^{-1}D_h\varphi_1(g)\varphi_2(g)D_h\varphi_2(g)$$
$$= D_h\varphi_1(g)[D_h\varphi_1(g), \varphi_2(g)]D_h\varphi_2(g).$$

$D_h\varphi_1(g)$ and $D_h\varphi_2(g)$ are polynomial mappings of lc-degree $\leq \bar{d} \mathbin{\dot{-}} (t+1)$. Thus by our assumption, $[D_h\varphi_1(g), \varphi_2(g)]$ is polynomial of lc-degree $\leq \bar{d} \mathbin{\dot{-}} (t+1+t) \leq \bar{d} \mathbin{\dot{-}} (t+1)$, and $D_h(\varphi_1\varphi_2)$ is polynomial of lc-degree $\leq \bar{d} \mathbin{\dot{-}} (t+1)$. It follows that $\varphi_1\varphi_2$ is polynomial of lc-degree $\leq (b_1, \ldots, b_c)$ with $b_i = d_i - t$ if $d_i \geq t$. To prove that lc-deg$(\varphi_1\varphi_2) \leq \bar{d} \mathbin{\dot{-}} t$ it suffices to check that $\varphi_1\varphi_2(G) \subseteq F_{(i+1)}$ if $d_i < t$. But this is so since $\varphi_1(G), \varphi_2(G) \subseteq F_{(i+1)}$ in this case.

(b) Now let $t_1 + t_2 = s$, let φ_1 be a polynomial mapping of lc-degree $\leq \bar{d} \mathbin{\dot{-}} t_1$ and φ_2 be a polynomial mapping of lc-degree $\leq \bar{d} \mathbin{\dot{-}} t_2$. We use the commutator identity

$$[xy, uv] = [x,u][x,v]\big[v,[u,x]\big]\Big[[x,v]\big[v,[u,x]\big],[x,u]\Big]\Big[[x,v]\big[v,[u,x]\big][x,u],y\Big]$$
$$\cdot[y,v]\big[v,[u,y]\big][y,u]$$

to write, for any $h \in S$ and $g \in G$,

$$\begin{aligned} D_h[\varphi_1,\varphi_2](g) &= [\varphi_1(g),\varphi_2(g)]^{-1}[\varphi_1(gh),\varphi_2(gh)] \\ &= [\varphi_1(g),\varphi_2(g)]^{-1}[\varphi_1(g)D_h\varphi_1(g),\varphi_2(g)D_h\varphi_2(g)] \\ &= [\varphi_1(g),D_h\varphi_2(g)]\big[D_h\varphi_2(g),[\varphi_2(g),\varphi_1(g)]\big] \\ &\quad\cdot\Big[[\varphi_1(g),D_h\varphi_2(g)]\big[D_h\varphi_2(g),[\varphi_2(g),\varphi_1(g)]\big],[\varphi_1(g),\varphi_2(g)]\Big] \\ &\quad\cdot\Big[[\varphi_1(g),D_h\varphi_2(g)]\big[D_h\varphi_2(g),[\varphi_2(g),\varphi_1(g)]\big][\varphi_1(g),\varphi_2(g)],D_h\varphi_1(g)\Big] \\ &\quad\cdot[D_h\varphi_1(g),D_h\varphi_2(g)]\big[D_h\varphi_2(g),[\varphi_2(g),D_h\varphi_1(g)]\big][D_h\varphi_1(g),\varphi_2(g)]. \end{aligned} \tag{3.1}$$

The mappings $D_h\varphi_1(g)$ and $D_h\varphi_2(g)$ are polynomial of lc-degrees $\leq \bar{d} \mathbin{\dot{-}} (t_1+1)$ and $\leq \bar{d} \mathbin{\dot{-}} (t_2+1)$ respectively. Thus by our assumption, all commutators on the right hand part of (3.1) are polynomial mappings of lc-degree $\leq \bar{d} \mathbin{\dot{-}} (t_1+t_2+1) = \bar{d} \mathbin{\dot{-}} (s+1)$, and such is their product $D_h[\varphi_1,\varphi_2]$. Hence, $[\varphi_1,\varphi_2]$ is polynomial of lc-degree $\leq (b_1,\ldots,b_c)$ with $b_i = d_i - (t_1+t_2)$ if $d_i \geq t_1+t_2$. It is only to check that $[\varphi_1,\varphi_2](G) \subseteq F_{(i+1)}$ if $d_i < t_1+t_2$. Fix $g \in G$, and let $i_1, i_2 \in \mathbb{N}$ be such that $\varphi_1(g) \in F_{(i_1)} \setminus F_{(i_1+1)}$ and $\varphi_2(g) \in F_{(i_2)} \setminus F_{(i_2+1)}$. Then $d_{i_1} - t_1 \geq 0$ and $d_{i_2} - t_2 \geq 0$, so $d_{i_1+i_2} \geq d_{i_1} + d_{i_2} \geq t_1 + t_2 > d_i$, and thus $i_1 + i_2 > i$. But then $[\varphi_1(g),\varphi_2(g)] \in F_{(i_1+i_2)} \subseteq F_{(i+1)}$.

(c) Let φ be a polynomial mapping of lc-degree $\leq \bar{d} \div t = s$. For $h \in S$ and $g \in G$, write

$$
\begin{aligned}
&D_h(\varphi^{-1})(g) = \varphi(g)\varphi(gh)^{-1} = \varphi(g)\big(\varphi(g)^{-1}\varphi(gh)\big)^{-1}\varphi(g)^{-1} \\
&= \varphi(g)D_h\varphi(g)^{-1}\varphi(g)^{-1} = D_h\varphi(g)^{-1}\varphi(g)[\varphi(g), D_h\varphi(g)^{-1}]\varphi(g)^{-1} \\
&= D_h\varphi(g)^{-1}[\varphi(g), D_h\varphi(g)^{-1}]\varphi(g)\big[\varphi(g), [\varphi(g), D_h\varphi(g)^{-1}]\big]\varphi(g)^{-1} \\
&\qquad\qquad\qquad\qquad \cdots \\
&= D_h\varphi(g)^{-1}[\varphi(g), D_h\varphi(g)^{-1}]\big[\varphi(g), [\varphi(g), D_h\varphi(g)^{-1}]\big]\ldots \\
&\qquad\qquad \cdot\Big[\varphi(g), \ldots, [\varphi(g), D_h\varphi(g)^{-1}]\ldots\Big]\varphi(g)\cdot C\cdot\varphi(g)^{-1},
\end{aligned}
$$

where $C = \Big[\varphi(g), \ldots, [\varphi(g), D_h\varphi(g)^{-1}]\ldots\Big] \in F_{(c+1)}$ and thus $C = \mathbf{1}_F$. Hence

$$
\begin{aligned}
D_h(\varphi^{-1})(g) = D_h\varphi(g)^{-1}[\varphi(g), D_h\varphi(g)^{-1}]\big[\varphi(g), [\varphi(g), D_h\varphi(g)^{-1}]\big]\ldots \\
\cdot\Big[\varphi(g), \ldots, [\varphi(g), D_h\varphi(g)^{-1}]\ldots\Big].
\end{aligned}
\tag{3.2}
$$

$(D_h\varphi)^{-1}$ is polynomial of lc-degree $\leq \bar{d} \div (t+1)$, and so, by (b), all factors on the right hand side of (3.2) are polynomial mappings of lc-degree $\leq \bar{d} \div (t+1)$. By (a), $D(\varphi^{-1})$ is polynomial of lc-degree $\leq \bar{d} \div (t+1)$. Hence, φ^{-1} is polynomial of lc-degree $\leq (b_1, \ldots, b_c)$ with $b_i = d_i - t$ if $d_i \geq t$. Since also $g^{-1}(G) \subseteq F_{(i+1)}$ if $d_i < t$, we are done. ∎

3.5. The following proposition shows that, when F is a nilpotent group, the polynomiality of a mapping $\varphi: G \longrightarrow F$ does not depend on the choice of a generating set in G.

Proposition. *Let F be nilpotent of class c and let $S \subseteq G$ be a generating set for G. If $\varphi: G \longrightarrow F$ is polynomial of degree $\leq d$ relative to S, then φ is polynomial of degree $\leq \frac{c^{d+1}-1}{c-1}$ relative to G.*

Proof. If φ is polynomial of degree $\leq d$ relative to S then for any $h \in S$, $D_h\varphi$ is polynomial of degree $\leq d-1$ relative to S. By induction on d, $D_h\varphi$ is polynomial of degree $\leq d_c = \frac{c^d-1}{c-1}$ relative to G and so, of lc-degree $\leq (d_c, 2d_c, \ldots, cd_c)$ relative to G. It follows from Lemma 1.2 and Proposition 3.4 that $D_h\varphi$ is polynomial of lc-degree $\leq (d_c, 2d_c, \ldots, cd_c)$ relative to G for any $h \in G$, and so φ is polynomial of degree $\leq cd_c + 1 = \frac{c^{d+1}-1}{c-1}$ relative to G. ∎

3.6. Proposition. *A mapping* $\varphi: G_1 \times G_2 \longrightarrow F$ *to a nilpotent group* F *is polynomial if and only if there are* $d_1, d_2 \in \mathbb{N}$ *such that for all* $g_2 \in G_2$ *the mapping* $\psi_{g_2}: G_1 \longrightarrow F$, $\psi_{g_2}(g_1) = \varphi(g_1, g_2)$, *is polynomial of degree* $\leq d_1$, *and for all* $g_1 \in G_1$ *the mapping* $\tau_{g_1}: G_2 \longrightarrow F$, $\tau_{g_1}(g_2) = \varphi(g_1, g_2)$, *is polynomial of degree* $\leq d_2$.

Proof. The "only if" part is clear. Assume that the mappings ψ_{g_2}, $g_2 \in G_2$, are all polynomial of lc-degrees $\leq \bar{d}_1$ relative to G_1, and the mappings τ_{g_1}, $g_1 \in G_1$, are all polynomial of lc-degrees $\leq \bar{d}_2$ relative to G_2. Then for any $h_1 \in G_1$ and any $g_2 \in G_2$, the restriction of $D_{(h_1, \mathbf{1}_{G_2})}\varphi$ on any set of the form $g_2 = \text{const}$ has lc-degree $\leq \bar{d}_1 \dot{-} 1$, and the restriction of $D_{(h_1, \mathbf{1}_{G_2})}\varphi$ on any set of the form $g_1 = \text{const}$ has lc-degree $\leq \bar{d}_2$. By induction on $(\bar{d}_1, \bar{d}_2)$ we may conclude that $D_{(h_1, \mathbf{1}_{G_2})}\varphi$ is polynomial of degree $\leq \bar{d}_1 + \bar{d}_2 \dot{-} 1$. Similarly, for any $h_2 \in G_2$, $D_{(\mathbf{1}_{G_1}, h_2)}\varphi$ is polynomial of degree $\leq \bar{d}_1 + \bar{d}_2 \dot{-} 1$. Since elements $(h_1, \mathbf{1}_{G_2})$, $(\mathbf{1}_{G_1}, h_2)$ generate $G_1 \times G_2$, φ is polynomial of lc-degree $\leq \bar{d}_1 + \bar{d}_2$. ∎

3.7. Corollary. *Let* G *be a nilpotent group. Then the operations of multiplication* $G \times G \longrightarrow G$, $(g_1, g_2) \mapsto g_1 g_2$, *and of raising to a power* $G \times \mathbb{Z} \longrightarrow G$, $(g, n) \mapsto g^n$, *are polynomial mappings.*

3.8. Let F be a finitely generated torsion-free nilpotent group. It is well known that F then possesses a subnormal (and even a central) series $\{\mathbf{1}_F\} = F_{t+1} \triangleleft F_t \triangleleft \ldots \triangleleft F_1 = F$ with infinite cyclic factors: $F_i/F_{i+1} \simeq \mathbb{Z}$, $i = 1, \ldots, t$ (see, for example, [KM]). Let f_1 be a generator of F_1 over F_2 (that is, $f_2 F_2$ be a generator of F_1/F_2), f_2 be a generator of F_2 over F_3, etc. We will call $\{f_1, \ldots, f_t\}$ **a basis of** F: every element of F is uniquely representable in the form $f_1^{a_1} \ldots f_t^{a_t}$ with $a_1, \ldots, a_t \in \mathbb{Z}$. So, we have **a coordinate mapping** $\alpha: F \longrightarrow \mathbb{Z}^t$, $\alpha(f_1^{a_1} \ldots f_t^{a_t}) = (a_1, \ldots, a_t)$.

3.9. Let F be a finitely generated torsion-free nilpotent group, let $\{f_1, \ldots, f_t\}$ be a basis of F and let $\varphi: G \longrightarrow F$ be a mapping. We can write $\varphi(g) = f_1^{p_1(g)} \ldots f_t^{p_t(g)}$, where $p_1, \ldots, p_t$ are mappings $G \longrightarrow \mathbb{Z}$.

Proposition. φ *is polynomial if and only if all* $p_1, \ldots, p_t$ *are polynomial mappings.*

Proof. If $p_1, \ldots, p_t$ are polynomial, then $\varphi(g) = f_1^{p_1(g)} \ldots f_k^{p_k(g)}$ is polynomial as a product of polynomial mappings. Conversely, let φ be polynomial. Then the mapping $\tilde{\varphi}_1 : G \longrightarrow F_1/F_2$ induced by φ is also polynomial. Let $\tilde{f}_1 = f_1F_2$; then $\tilde{\varphi}_1(g) = \tilde{f}_1^{p_1(g)}$. It follows that p_1 is polynomial. Hence, the mapping $\varphi_1(g) = f_1^{p_1(g)}$ is also polynomial, and so $\varphi_1^{-1}\varphi$ is polynomial and maps G into F_2. Now, we may apply induction on i for which $\varphi(G) \subseteq F_i$. ∎

3.10. Corollary. *Let F be a finitely generated torsion-free nilpotent group and let $\varphi\colon \mathbb{Z} \longrightarrow F$ be a nonconstant polynomial mapping of degree $\leq d$. Then for any $f \in F$, $\#\varphi^{-1}(f) \leq d$.*

Proof. We may assume that $f = \mathbf{1}_G$. Let $\{f_1, \ldots, f_k\}$ be a basis of F and let $\varphi(n) = f_1^{p_1(n)} \ldots f_t^{p_t(n)}$, $p_1, \ldots, p_k\colon \mathbb{Z} \longrightarrow \mathbb{Z}$. Then $\varphi(n) = \mathbf{1}_F$ if and only if $p_1(n) = \ldots = p_k(n) = 0$. Let i be the minimal index for which p_i is nonconstant and let F_{i+1} be the subgroup of F generated by $f_{i+1}, \ldots, f_k$. Since $D_1\varphi(n) \equiv f_i^{p_i(n+1)-p_i(n)} \bmod F_{i+1}$, we have $\deg p_i \leq d$. ∎

3.11. The following is a reformulation of Proposition 3.9.

Proposition. *Let F be a finitely generated torsion-free nilpotent group and $\alpha\colon F \longrightarrow \mathbb{Z}^t$ be a coordinate mapping. A mapping $\varphi\colon G \longrightarrow F$ is polynomial if and only if $\alpha \circ \varphi$ is polynomial. In particular, α is polynomial.*

Let us also note that the inverse mapping $\alpha^{-1}\colon \mathbb{Z}^t \longrightarrow F$, $(a_1, \ldots, a_t) \mapsto f_1^{a_1} \ldots f_t^{a_t}$, is polynomial as a product of polynomial mappings.

3.12. Proposition. *Let G and F be finitely generated torsion-free nilpotent groups, let $\beta\colon G \longrightarrow \mathbb{Z}^s$ and $\alpha\colon F \longrightarrow \mathbb{Z}^t$ be their coordinate mappings. Then a mapping $\varphi\colon G \longrightarrow F$ is polynomial if and only if the mapping $\psi\colon \mathbb{Z}^s \longrightarrow \mathbb{Z}^t$, $\psi = \alpha \circ \varphi \circ \beta^{-1}$, is polynomial.*

Proof. By Proposition 3.9, we may ignore α and assume that $F = \mathbb{Z}^t$. Then, if ψ is polynomial, $\varphi = \psi \circ \beta$ is polynomial by Proposition 2.6.

Let us assume that φ is polynomial of degree $\leq d$ relative to G, and let $\{g_1, \ldots, g_s\}$ be the basis of G corresponding to β. Fix $1 \leq j \leq s$ and $b_1, \ldots, b_{j-1}$, $b_{j+1}, \ldots, b_s \in \mathbb{Z}$. The restriction of ψ on the line $\bigl\{(b_1, \ldots, b_{j-1}, b, b_{j+1}, \ldots, b_s) \bigm| b \in \mathbb{Z}\bigr\}$,

$$b \mapsto \varphi(g_1^{b_1} \ldots g_{j-1}^{b_{j-1}} g_j^b g_{j+1}^{b_{j+1}} \ldots g_s^{b_s}),$$

is a polynomial mapping of degree $\leq d$. By Proposition 3.6, ψ is polynomial. ∎

3.13. Propositions 3.7 and 3.12 imply as a corollary the well known fact that, in a nilpotent group, the operations of multiplication and arising to power, being written in coordinates, are represented by polynomials:

Corollary. *Let G be a finitely generated torsion-free nilpotent group and let $\{g_1, \dots, g_s\}$ be a basis of G. Then there are polynomials $P_1, \dots, P_s: \mathbb{Z}^{2s} \longrightarrow \mathbb{Z}$ and $Q_1, \dots, Q_s: \mathbb{Z}^{s+1} \longrightarrow \mathbb{Z}$ such that for any $a_1, \dots, a_s, b_1, \dots, b_s, n \in \mathbb{Z}$ one has*

$$(g_1^{a_1} \dots g_s^{a_s})(g_1^{b_1} \dots g_s^{b_s}) = g_1^{P_1(a_1,\dots,a_s,b_1,\dots,b_s)} \dots g_s^{P_s(a_1,\dots,a_s,b_1,\dots,b_s)} \quad \textit{and}$$
$$(g_1^{a_1} \dots g_s^{a_s})^n = g_1^{Q_1(a_1,\dots,a_s,n)} \dots g_s^{Q_s(a_1,\dots,a_s,n)}.$$

3.14. We will need a bound on the degrees of the polynomial mappings p_i in 3.9. Such a bound is easily obtainable when the basis of F is compatible with the lower central series of F:

Proposition. *Let F be a finitely generated torsion-free nilpotent group of class c and let $F = F_{(1)} \supset F_{(2)} \supset \dots \supset F_{(c)} \supset F_{(c+1)}$ be the lower central series of F. Assume that for all $i = 1, \dots, c$ the (finitely generated abelian) groups F_i/F_{i+1} have no torsion; let $\{f_1, \dots, f_{t_1}\} \subseteq F_{(1)}$ be a basis of $F_{(1)}$ over $F_{(2)}$ (that is, $\{f_1F_{(2)}, \dots, f_{t_1}F_{(2)}\}$ is a basis of $F_{(1)}/F_{(2)}$), let $\{f_{t_1+1}, \dots, f_{t_2}\} \subseteq F_{(2)}$ be a basis of $F_{(2)}$ over $F_{(3)}$, and so on. Let $\varphi: G \longrightarrow F$, $\varphi(g) = f_1^{p_1(g)} \dots f_t^{p_{t_c}(g)}$, be a polynomial mapping of degree $\leq d$. Then, for any $j \leq t_c$, if $t_{i-1} + 1 \leq j \leq t_i$, then $\deg p_j \leq id$.*

Proof. Since $\deg \varphi \leq d$, lc-$\deg \varphi \leq (d, 2d, \dots, cd)$. Assume by induction on i that $\varphi(G) \subseteq F_{(i)}$, that is, $\varphi(g) = f_{t_{i-1}+1}^{p_{t_{i-1}+1}(g)} \dots f_{t_c}^{p_{t_c}(g)}$. Then the mapping $\tilde{\varphi}_i: G \longrightarrow F_i/F_{i+1}$ induced by φ, $\tilde{\varphi}_i = f_{t_{i-1}+1}^{p_{t_{i-1}+1}} \dots f_{t_c}^{p_{t_i}} \bmod F_{(i+1)}$, has degree $\leq id$ and so, all the mappings $p_{t_{i-1}+1}, \dots, p_{t_i}$ have degrees $\leq id$. Thus the mappings $f_{t_{i-1}+1}^{p_{t_{i-1}+1}}, \dots, f_{t_i}^{p_{t_i}}: G \longrightarrow F$ have degrees $\leq id$, so they have lc-degrees $\leq (d, 2d, \dots, cd)$, and so their product $\varphi_i = f_{t_{i-1}-1}^{p_{t_{i-1}-1}} \dots f_{t_i}^{p_{t_i}}$ has lc-degree $\leq (d, 2d, \dots, cd)$. Hence, the mapping $\varphi_i^{-1}\varphi = f_{t_i+1}^{p_{t_i+1}} \dots f_{t_c}^{p_{t_c}}$, which sends G into $F_{(i+1)}$, also has lc-degree $\leq (d, 2d, \dots, cd)$, which gives the step of induction. ■

3.15. Not all torsion-free nilpotent groups possess lower central series with torsion -free factors. However, every finitely generated torsion-free nilpotent group is contained in a finitely generated nilpotent group of the same nilpotency class which already satisfies this property (see, for example, [KM] §17). This allows us to generalize Proposition 3.14:

Proposition. *Let F be a finitely generated torsion-free nilpotent group of class*

c and let $\varphi: G \longrightarrow F$ be a polynomial mapping of degree $\leq d$. Then there is a basis $\{f_1, \ldots, f_t\}$ of F such that in the representation $\varphi(g) = f_1^{p_1(g)} \ldots f_t^{p_t(g)}$ the polynomial mappings $p_1, \ldots, p_t$ are all of degrees $\leq dc$.

Proof. Let $\tilde{F}$ be a finitely generated nilpotent group of class c containing F and such that the factors $\tilde{F}_{(i)}/\tilde{F}_{(i+1)}$, $i = 1, \ldots, c$, of the lower central series $\tilde{F} = \tilde{F}_{(1)} \supset \tilde{F}_{(2)} \supset \ldots \supset \tilde{F}_{(c)} \supset \tilde{F}_{(c+1)} = \{\mathbf{1}_F\}$ have no torsion. Put $F_i = \tilde{F}_{(i)} \cap F$, $i = 1, \ldots, c+1$. Then the factors F_i/F_{i+1} have no torsion as well.

Let $\{f_1, \ldots, f_{t_1}\} \subseteq F_1$ be a basis of F_1 over F_2, $\{f_{t_1+1}, \ldots, f_{t_2}\} \subseteq F_2$ be a basis of F_2 over F_3, and so on. Then the basis $\{f_1, \ldots, f_{t_c}\}$ of F satisfies the requirements of the proposition: for any j with $t_{i-1}+1 \leq j \leq t_i$, one has $\deg p_j \leq id$. The proof is the same as in Proposition 3.14, with the only distinction that one has to consider the lc-degree of the arising polynomial mappings with respect to the lower central series of $\tilde{F}$ instead of F. ∎

3.16. Given $h \in G$, we define **the left h-derivative** of $\varphi: R \longrightarrow F$ by $D_h^L \varphi(g) = \varphi(hg)\varphi(g)^{-1}$. Let S be a generating set for G; we say that φ **is left-polynomial of degree $\leq d$ (relative to S)** if for any $h_1, \ldots, h_{d+1} \in S$, $D_{h_1}^L \ldots D_{h_{d+1}}^L \varphi \equiv \mathbf{1}_F$.

Proposition. *If F is nilpotent then $\varphi: G \longrightarrow F$ is (right-)polynomial if and only if φ is left-polynomial.*

Proof. It suffices to check only one implication. Let φ be polynomial of degree $\leq d$ and let F have nilpotency class c; we will show that φ is left-polynomial of degree $\leq dc^2$.

We may replace G by the free group generated by S. F is a factor of a torsion-free nilpotent group $\tilde{F}$; by Corollary 1.20, we may replace F by $\tilde{F}$ and assume that F is torsion-free. We have to check that φ satisfies $D_{h_1}^L \ldots D_{h_{dc^2+1}}^L \varphi(g) = \mathbf{1}_G$ for all $h_1, \ldots, h_{dc^2+1} \in S$, $g \in G$. Any such identity involves finitely many elements and so, we may assume that S is finite. By Corollary 1.18, we may assume that F is finitely generated.

Using Proposition 3.15, find a basis $\{f_1, \ldots, f_t\}$ of F such that for $\varphi(g) = f_1^{p_1(g)} \ldots f_t^{p_t(g)}$ one has $\deg p_i \leq dc$, $i = 1, \ldots, t$. By Corollary 2.13, the polynomial mappings $p_i: G \longrightarrow \mathbb{Z}$, $i = 1 \ldots, t$, are also left-polynomial of degree $\leq dc$. By the "left" version of Proposition 3.4, φ is left-polynomial of degree $\leq dc^2$. ∎

3.17. Let F be a finitely generated nilpotent group and let H be a subgroup of F. Then H is closed in F (see 1.25) if and only if F possesses a subnormal (and even a central) series over H, $H = F_{r+1} \lhd F_r \lhd \ldots \lhd F_1 = F$, with infinite cyclic factors: $F_i/F_{i+1} \simeq \mathbb{Z}$, $i = 1, \ldots, r$ (see [BL], Proposition 1.17). Let f_1 be a generator of F_1 over F_2, f_2 be a generator of F_2 over F_3, etc. Then $\{f_1, \ldots, f_r\}$ is **a basis of F over H**: every element of F is uniquely representable in the form $f_1^{a_1} \ldots f_r^{a_r} h$ with $a_1, \ldots, a_t \in \mathbb{Z}$ and $h \in H$.

3.18. Proposition 3.9 is extendible to the case of a basis over a closed subgroup:

Proposition. *Let G be a group, let F be a finitely generated nilpotent group, let H be a closed subgroup of F, let $\{f_1, \ldots, f_r\}$ be a basis of F over H, and let $\varphi: G \longrightarrow F$ be a mapping, $\varphi(g) = f_1^{p_1(g)} \ldots f_r^{p_r(g)} \psi(g)$, $p_1, \ldots, p_r: G \longrightarrow \mathbb{Z}$ and $\psi: G \longrightarrow H$. Then φ is polynomial if and only if all $p_1, \ldots, p_r$ and ψ are polynomial mappings.*

The proof is completely analogous to the proof of Proposition 3.9.

3.19. Corollary. *Let F be a finitely generated nilpotent group, let H be a closed subgroup of F and let $\varphi: \mathbb{Z} \longrightarrow F$ be a polynomial mapping of degree $\leq d$. Then for any $f \in F$, $\varphi(\mathbb{Z}) \not\subseteq fH$ implies $\#\varphi^{-1}(fH) \leq d$.*

3.20. Propositions in this section were formulated for polynomial mappings of general groups to special (nilpotent) groups. However, the structure of the range of a polynomial mapping puts some restrictions on the structure of the domain of the mapping:

Proposition. *Let F be a solvable group and let $\varphi: G \longrightarrow F$ be a polynomial mapping (relative to a set S generating G). Then φ is representable as the composition $\varphi' \circ \pi$ of a homomorphism π of G onto a solvable group G' and a polynomial mapping $\varphi': G' \longrightarrow F$.*

Proof. Let $G^{(1)} = G$ and $G^{(j)} = [G^{(j-1)}, G^{(j-1)}]$, $j = 2, 3, \ldots$. Assume that φ has degree $\leq d$ and that F has solvability class c. We will show that φ is defined on $G/G^{(d(c+1))}$. Since the shifts $\varphi'(g) = \varphi(g_0 g)$ of φ are also polynomial of degree $\leq d$, this is enough to prove that φ is constant on $G^{(d(c+1))}$.

We may assume that $\varphi(\mathbf{1}_G) = \mathbf{1}_F$. By induction on d, for any $h \in S$ the mapping $D_h\varphi$ is constant on cosets of the subgroup $G_{d-1} = G^{((d-1)(c+1))}$ in G. So, for any $g \in G_{d-1}$ and $g_0 \in G$, $D_h(gg_0) = D_h(g_0)$. By Lemma 1.23, $\varphi(gg_0) = \varphi(g)\varphi(g_0)$ for all $g \in G_{d-1}$ and $g_0 \in G$; in particular, $\varphi|_{G_{d-1}}$ is a homomorphism. Since F has solvability class c, $\varphi|_{G_{d-1}}$ is trivial on the $(c+1)$-st commutator subgroup of G_{d-1}, that is, on $G^{(d(c+1))}$. ∎

3.21. Proposition. *Let F be a nilpotent group and let $\varphi: G \longrightarrow F$ be a polynomial mapping (relative to a set S generating G). Then φ is representable as the composition $\varphi = \pi \circ \varphi'$ of a homomorphism π of G onto a nilpotent group G' and a polynomial mapping $\varphi': G' \longrightarrow F$.*

Proof. Let $G = G_{(1)} \supset G_{(2)} \supset \ldots$ be the lower central series of G, $G_{(i+1)} = [G, G_{(i)}]$. We have to show that for k large enough, φ is constant on cosets of $G_{(k)}$ in G. G is a factor, $\varphi: \tilde{G} \longrightarrow G$, of the free group $\tilde{G}$ generated by S; since $\pi(\tilde{G}_{(k)}) = \pi(G_{(k)})$ for all $k \in \mathbb{N}$, we may replace G by $\tilde{G}$. In its turn, F is a factor of a torsion-free nilpotent group $\tilde{F}$; by Corollary 1.20, we may replace F by $\tilde{F}$ and assume that F is torsion-free.

Let F have nilpotency class c and let the degree of φ relative to S be $\leq d$; we will show that φ is constant on cosets of the subgroup $G_{(cd+1)}$ in G. We have to check that for any $g \in G_{(cd+1)}$ and $g_0 \in G$, $\varphi(g_0 g) = \varphi(g_0)$; we may therefore assume that G is finitely generated (that is, that S is finite). By Corollary 1.18, we may assume that F is finitely generated as well. It then follows from Proposition 3.15 that the proposition is reducible to the case where F is abelian and $\varphi: G \longrightarrow F$ has degree $\leq cd$; this case is covered by Proposition 2.15. ∎

3.22. In conclusion, we can generalize Proposition 2.6:

Proposition. *Let $G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{k-1}} G_k$ be a sequence of polynomial mappings such that G_k is a nilpotent group. Then the composition $\varphi_{k-1} \circ \ldots \circ \varphi_2 \circ \varphi_1$ is also a polynomial mapping.*

Proof. When all $G_1, \ldots, G_k$ are finitely generated torsion-free nilpotent groups, the statement follows from Proposition 3.12 and Proposition 2.6. The general case can be reduced to this special case in the following way. First, Proposition 1.12 allows to reduce the problem to the case where G_1 is finitely generated. It then follows from Corollary 1.18 that $G_2, \ldots, G_k$ may also be assumed to be finitely generated. Next, Corollary 1.20 (combined with Propositions 1.10 and Proposition 1.11) allows to replace $G_1, \ldots, G_{k-1}$ by finitely generated free groups. Then, G_k is a factor of a finitely generated torsion-free nilpotent group and, again, Corollary 1.20 allows replace G_k by this group. Then, by Proposition 3.21, G_{k-1} may be replaced by some its nilpotent factor. Induction on k finishes the proof. ∎

3.23. Question: How does the structure of G affect the structure of F? For example, if $\varphi\colon \mathbb{Z} \longrightarrow F$ is a polynomial mapping of degree $\leq d$ relative to $\mathbb{Z}$ and with $\varphi(0) = \mathbf{1}_F$, what can one say about the group generated by $\varphi(\mathbb{Z})$?

4. Polynomial mappings of amenable groups

For any conventional nonzero polynomial $P\colon \mathbb{Z}^k \longrightarrow \mathbb{Z}^l$, the set of zeroes of P has zero density in $\mathbb{Z}^k$. We will show in this section that an analogous fact holds for polynomial mappings of any countable amenable group.

4.1. From now on, let G be a countable amenable group. G possesses a (right) Følner sequence, namely, a sequence $\Phi_1, \Phi_2, \ldots \subseteq G$ of finite subsets satisfying $\frac{|\Phi_k g \triangle \Phi_k|}{|\Phi_k|} \underset{k\to\infty}{\longrightarrow} 0$ for any $g \in G$. We fix a Følner sequence $\Phi_1, \Phi_2, \ldots$ in G; a set $Q \subseteq G$ is said to be **of zero density in** G if $\frac{|Q\cap\Phi_k|}{|\Phi_k|} \underset{k\to\infty}{\longrightarrow} 0$, and **of density one** if $G \setminus Q$ is of zero density.

4.2. The following proposition is formulated under some restrictions on the generating set S; we do not know if it remains true without these restrictions.

Proposition. *Let G be a countable amenable group and let S be a generating set for G satisfying the following property: $h \in S$ implies $h^n \in S$ for all $n \in \mathbb{N}$. Let F be a torsion-free group and let $\varphi\colon G \longrightarrow F$ be a nontrivial mapping polynomial relative to S. Then the preimage $Q = \varphi^{-1}(\mathbf{1}_F)$ has zero density in G.*

Proof. We will call a subset of G of the form $L(g,h) = \{gh^l\}_{l\in\mathbb{Z}}$ with $g, h \in G$, **a line**, and a set $I(g,h,l) = \{gh^l\}_{l=0}^{l}$ with $g, h \in G$, $l \geq 0$, **an interval**.

We may assume that φ is nonconstant. By Proposition 1.24, φ is constant on any finite line, that is, on any line $L(g,h)$ with h having finite order. Thus, there must exist $h \in S$ of infinite order such that φ is nonconstant on a line $L(g,h)$; $D_h\varphi$ is then nontrivial for this h. We fix such $h \in S$; by induction on the degree of φ, the set $Q' = \big\{g \in G \bigm| \varphi|_{L(g,h)} = \text{const}\big\} \subseteq \big\{g \in G \bigm| D_h\varphi(g) = \mathbf{1}_F\big\}$ has zero density in G.

We will use the following fact:

Szemerédi's Theorem on arithmetic progressions. ([Sz]) *For any $r \in \mathbb{N}$ and any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $M \geq N$, any set $R \subseteq \{1, \ldots, M\}$ with $|R| \geq \varepsilon M$ contains an r-term arithmetic progression.*

Fix $\varepsilon > 0$. Let $\deg\varphi \le d$; let, by the Szemerédi theorem, $N \in \mathbb{N}$ be such that for any $M \ge N$ any set $R \subseteq \{1, \ldots, M\}$ with $|R| > \varepsilon M$ contains a $d+1$-term arithmetic progression. Let $k \in \mathbb{N}$ be such that, for the k-th term of our Følner sequence, $|\Phi_k h \mathbin{\triangle} \Phi_k| < \frac{2\varepsilon}{N}|\Phi_k|$ and $|Q' \cap \Phi_k| < \varepsilon|\Phi_k|$.

Represent Φ_k as a disjoint union $\Phi_k = \bigcup_{\beta\in B} I_\beta$ of *maximal* intervals $I_\beta = I(g_\beta, h, l_\beta)$, that is, such that $g_\beta h^{-1}, g_\beta h^{l_\beta+1} \notin \Phi_k$. Let $B_1 = \{\beta \in B \mid |I_\beta| < N\}$, $B_2 = \{\beta \in B \setminus B_1 \mid |Q \cap I_\beta| < \varepsilon|I_\beta|\}$, $B_3 = B \setminus (B_1 \cup B_2)$, and let $A_i = \bigcup_{\beta\in B_i} I_\beta$, $i = 1, 2, 3$. Φ_k is then partitioned $\Phi_k = A_1 \cup A_2 \cup A_3$.

We have, first,

$$\frac{2\varepsilon}{N}|\Phi_k| > |\Phi_k h \mathbin{\triangle} \Phi_k| \ge |A_1 h \mathbin{\triangle} A_1| \ge \frac{2}{N}|A_1|,$$

so $|A_1| < \varepsilon|\Phi_k|$. Secondly, $|Q \cap A_2| < \varepsilon|A_2| \le \varepsilon|\Phi_k|$. And finally, let $\beta \in B_3$ and $I_\beta = I(g, h, l)$. Then, by the Szemerédi theorem, there are $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $gh^m, gh^{m+n}, \ldots, gh^{m+dn} \in Q$. By Proposition 1.22, applied to the line $L(g, h)$ (here we use our restrictions on S), φ is constant on $L(g, h)$. So, $I_\beta \subset L(g, h) \subseteq Q'$, which implies $A_3 \subseteq Q'$ and so, $|A_3 \cap \Phi_k| < \varepsilon|\Phi_k|$. Summarizing, $|Q \cap \Phi_k| \le |A_1| + |Q \cap A_2| + |A_3| < 3\varepsilon|\Phi_k|$. ∎

4.3. In the case where F is nilpotent we can generalize Proposition 4.2:

Proposition. *Let G be a countable amenable group, let F be a nilpotent group, let $\varphi: G \longrightarrow F$ be a polynomial mapping relative to some S generating G, and let H be a closed subgroup of F. For $f \in F$, if $\varphi(G) \not\subseteq fH$, then $Q = \varphi^{-1}(fH)$ has zero density in G.*

Proof. We may assume that $f = \mathbf{1}_F$. Like in the proof of Proposition 4.2, there is $h \in S$ of infinite order such that $D_h\varphi(G) \not\subseteq H$. By induction on the degree of φ, $Q' = (D_g\varphi)^{-1}(H)$ has zero density in G.

Fix $\varepsilon > 0$, put $a = \frac{d+1}{\varepsilon}$, and let $k \in \mathbb{N}$ be such that $|\Phi_k g \mathbin{\triangle} \Phi_k| < \frac{2\varepsilon}{a}|\Phi_k|$ and $|Q' \cap \Phi_k| < \varepsilon|\Phi_k|$. Represent Φ_k as a disjoint union of maximal intervals, $\Phi_k = \bigcup_{\beta\in B} I_\beta$. Put $B_1 = \{\beta \in B \mid |I_\beta| < a\}$, $B_2 = \{\beta \in B \setminus B_1 \mid |Q \cap I_\beta| < d+1\}$, $B_3 = B \setminus (B_1 \cup B_2)$, and $A_i = \bigcup_{\beta\in B_i} I_\beta$, $i = 1, 2, 3$. Then, first,

$$\frac{2\varepsilon}{a}|\Phi_k| > |\Phi_k h \mathbin{\triangle} \Phi_k| \ge |A_1 h \mathbin{\triangle} A_1| \ge \frac{2}{a}|A_1|,$$

so $|A_1| < \varepsilon|\Phi_k|$. Secondly,

$$|Q \cap A_2| < (d+1)\frac{|A_2|}{a} = \varepsilon|A_2| \le \varepsilon|\Phi_k|.$$

And finally, for $\beta \in B_3$, by Corollary 3.19 we have $\varphi(I_\beta) \subseteq H$, which implies $I_\beta \subseteq Q'$. Thus, $A_3 \subseteq Q'$ and so, $|A_3 \cap \Phi_k| < \varepsilon|\Phi_k|$. ■

4.4. We will need the following technical corollary of Proposition 4.3:

Corollary. *Let K be a group, let G be a countable amenable group, let F be a nilpotent group, let $\xi: G \times K \longrightarrow F$ be a polynomial mapping, let H be a closed subgroup of F and let $f \in F$. If $\xi(G \times K) \not\subseteq fH$, then the set $Q = \bigl\{g \in G \bigm| \xi(\{g\} \times K) \subseteq H\bigr\}$ has zero density in G.*

Proof. Let $h \in K$ be such that $\xi(G \times \{h\}) \not\subseteq fH$; we may replace K by the group generated by h and so, assume that $G \times K$ is amenable. If Q had nonzero density in G, then $Q \times K \subseteq \xi^{-1}(H)$ would have nonzero density in $G \times K$. ■

4.5. Question: How many zeroes may a polynomial mapping $\varphi: \mathbb{Z} \longrightarrow F$ of degree $\leq d$ have? May it have more than d zeroes? (The positive answer to this question would allow to avoid the usage of the Szemerédi theorem in the proof of Proposition 4.2 and remove the restriction on S from its formulation.)

5. An application: Unitary polynomial actions of amenable groups

5.1. Let G be an amenable group, let φ be a mapping of G to the group of unitary operators on a Hilbert space $\mathcal{H}$. We will say that φ **is compact on** $u \in \mathcal{H}$, or that u **is a compact vector for** φ, if the orbit $\varphi(G)u = \bigl\{\varphi(g)u \bigm| g \in G\bigr\}$ is precompact. Let $\mathcal{L}$ be a subspace of $\mathcal{H}$. We will say that φ **is compact on** $\mathcal{L}$ if φ is compact on all $u \in \mathcal{L}$. We will say that φ **is weakly mixing on** $\mathcal{L}$ if for every $u \in \mathcal{L}$, $u' \in \mathcal{H}$ and every $\varepsilon > 0$, the set $\bigl\{g \in G \bigm| |\langle\varphi(g)u, u'\rangle| > \varepsilon\bigr\}$ has zero density in G (with respect to a fixed Følner sequence).

If φ is a mapping of G to the group of measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$, we say that φ is compact on X if the induced mapping of G to the group of unitary operators on $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$ is compact on $\mathcal{H}$, and that φ is weakly mixing on X if the induced mapping is weakly mixing on the orthogonal complement $\mathcal{H} \ominus \mathbb{C}$ of the subspace of constants in $\mathcal{H}$.

5.2. We call a polynomial mapping of a group G to a group of transformations of a (topological, linear, measure, etc.) space X **a polynomial action of** G **on** X. We will consider **polynomial unitary actions** of a group on a Hilbert space and **polynomial measure preserving actions** on a probability space, under the assumption that the images of these actions are contained in finitely generated *nilpotent* groups of transformations.

5.3. We will need the following "structure theorem" for unitary actions of finitely generated nilpotent groups:

Theorem. ([L3]) *Let F be a finitely generated nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$, $\mathcal{H} = \bigoplus_{\alpha\in A} \mathcal{L}_\alpha$, into a direct sum of pairwise orthogonal subspaces such that elements of F permute these subspaces: for any $T \in F$ and $\alpha \in A$, $T(\mathcal{L}_\alpha) = \mathcal{L}_\beta$, $\beta \in A$, and for $H_\alpha = \{T \in F \mid T(\mathcal{L}_\alpha) = \mathcal{L}_\alpha\}$ the following holds:*
(a) *H_α is closed in G;*
(b) *H_α contains a closed normal subgroup E_α such that*
  (i) *the action of E_α on $\mathcal{L}_\alpha$ is compact, and*
  (ii) *every $T \in H_\alpha \setminus E_\alpha$ is weakly mixing on $\mathcal{L}_\alpha$.*

We will call the decomposition above **a primitive decomposition of** $\mathcal{H}$ (relative to the action of F).

5.4. Let G be a finitely generated amenable group, let F be a nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$ and let $\varphi\colon G \longrightarrow F$ be a polynomial mapping with $\varphi(\mathbf{1}_G) = \mathbf{1}_F = \mathrm{Id}_{\mathcal{H}}$. By Corollary 1.18, we may assume that F is finitely generated as well. Let $\mathcal{H} = \bigoplus_{\alpha\in A} \mathcal{L}_\alpha$ be a primitive decomposition of $\mathcal{H}$ relative to the action of F. Fix $\alpha \in A$, let $H_\alpha = \{T \in F \mid T(\mathcal{L}_\alpha) = \mathcal{L}_\alpha\}$ and let E_α be the maximal subgroup of H_α whose action on $\mathcal{L}_\alpha$ is compact.

Theorem. (a) *For any $\beta \in A \setminus \{\alpha\}$, $\varphi(g)(\mathcal{L}_\alpha) \perp \mathcal{L}_\beta$ for all $g \in G$ but a set of zero density, and if $\varphi(G) \not\subseteq H_\alpha$, then also $\varphi(g)(\mathcal{L}_\alpha) \perp \mathcal{L}_\alpha$ for all $g \in G$ but a set of zero density.*
(b) *If $\varphi(G) \subseteq H_\alpha \setminus E_\alpha$, then φ is weakly mixing on $\mathcal{L}_\alpha$.*
(c) *If $\varphi(G) \subseteq E_\alpha$, then φ is compact on $\mathcal{L}_\alpha$.*

Proof. (c) is trivial. Since H_α is closed in F, (a) follows from Proposition 4.3. We only have to prove (b). Let $\varphi(G) \subseteq H_\alpha \setminus E_\alpha$. Consider the mapping $\xi\colon G \times G \longrightarrow F$ and the mappings $\varphi_h\colon G \longrightarrow F$, $h \in G$, defined by $\xi(h, g) = \varphi_h(g) = \varphi(h)^{-1} D_h\varphi(g) = \varphi(h)^{-1}\varphi(g)^{-1}\varphi(gh)$. There may be two cases:

(i) $\xi(G \times G) \not\subseteq E_\alpha$. By Corollary 4.4, $\varphi_h(G) = \xi(h, G) \not\subseteq E_\alpha$ for all $h \in G \setminus Q$, where Q is a set of zero density in G. By induction on the degree of φ, φ_h and so, $D_h\varphi = \varphi(h)\varphi_h$ are weakly mixing on $\mathcal{L}_\alpha$ for all $h \in G \setminus Q$. Hence, for any $u \in \mathcal{L}_\alpha$, any $\varepsilon > 0$ and any $h \in G \setminus Q$, the set $\bigl\{g \in G \bigm| |\langle D_h\varphi(g)u, u\rangle| > \varepsilon\bigr\} = \bigl\{g \in G \bigm| |\langle \varphi(gh)u, \varphi(g)u\rangle| > \varepsilon\bigr\}$ has zero density in G. The result now follows from the following lemma (it is proven in [F] for the case $G = \mathbb{Z}$, but the proof is almost verbatim transferable to the case of the general amenable G):

Lemma. ([F], Lemma 4.9) *Let $g \mapsto u_g$ be a mapping of an amenable group G to a Hilbert space $\mathcal{L}$ such that the set*

$$\Bigl\{h \in G \Bigm| \ \bigl\{g \in G \bigm| |\langle u_{gh}, u_g\rangle| < \varepsilon\bigr\} \text{ has density one}\Bigr\}$$

has density one in G. Then for any $u' \in \mathcal{L}$ and $\varepsilon > 0$, the set $\bigl\{g \in G \bigm| |\langle u_g, u'\rangle| < \varepsilon\bigr\}$ has density one in G.

(ii) $\xi(G \times G) \subseteq E_\alpha$. Then for any $g, h \in G$, $\varphi(gh) = \varphi(g)\varphi(h)$ modulo E_α, that is, φ induces a homomorphism $\tilde{\varphi}\colon G \longrightarrow H_\alpha/E_\alpha$. Let G be of solvability class c_1, let H_α be of solvability class c_2 and let $c = \max\{c_1, c_2\}$; then G is a factor of "the universal solvable group of class c" $\tilde{G}/\tilde{G}^{(c+1)}$, where $\tilde{G}$ is the free group with the same generating set as G, $\tilde{G}^{(1)} = \tilde{G}$, and $\tilde{G}^{(j+1)} = [\tilde{G}^{(j)}, \tilde{G}^{(j)}]$, $j = 1, \ldots, c$. We may replace G by this group; then $\tilde{\varphi}$ is extendible to a homomorphism $\psi\colon G \longrightarrow H_\alpha$. $\eta = \psi^{-1}\varphi$ maps G into E_α and hence, is compact on $\mathcal{L}_\alpha$. On the other hand, ψ can not have compact vectors in $\mathcal{L}_\alpha$, since for such a vector u, $\varphi(G)u$ would be precompact. Hence, ψ is weakly mixing on $\mathcal{L}_\alpha$. It remains to apply the following lemma:

Lemma. *Let ψ, η be mappings of an amenable group G to the group of unitary operators on a Hilbert space $\mathcal{L}$, let ψ be weakly mixing and η be compact on $\mathcal{L}$. Then $\varphi = \psi\eta$ is weakly mixing.*

Proof. Let $u, u' \in \mathcal{L}$ and $\varepsilon > 0$, and let $v_1, \ldots, v_k$ be an $\frac{\varepsilon}{2\|u'\|}$-net for $\eta(G)u$. Then

$$\begin{aligned}\bigl\{g \in G \bigm| |\langle \varphi(g)u, u'\rangle| > \varepsilon\bigr\} &= \bigl\{g \in G \bigm| |\langle \psi(g)\eta(g)u, u'\rangle| > \varepsilon\bigr\} \\ &\subseteq \bigcup_{i=1}^{k} \bigl\{g \in G \bigm| |\langle \psi(g)v_i, u'\rangle| > \varepsilon/2\bigr\},\end{aligned}$$

which is a union of sets of zero density in G and so, is of zero density itself. ∎

5.5. Corollary. *Let φ be a polynomial mapping of a finitely generated amenable group G to a nilpotent group F of unitary operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}=\mathcal{H}^{\mathrm{c}}(\varphi)\oplus\mathcal{H}^{\mathrm{wm}}(\varphi)$ so that $\mathcal{H}^{\mathrm{c}}(\varphi)$ and $\mathcal{H}^{\mathrm{wm}}(\varphi)$ are $\varphi(G)$-invariant, φ is compact on $\mathcal{H}^{\mathrm{c}}(\varphi)$ and is weakly mixing on $\mathcal{H}^{\mathrm{wm}}(\varphi)$.*

Proof. In the notation of 5.4, define $A^{\mathrm{c}}(\varphi)=\bigl\{\alpha\in A \bigm| \varphi(G)\subseteq E_\alpha\bigr\}$, $A^{\mathrm{wm}}=A\setminus A^{\mathrm{c}}$, and put $\mathcal{H}^{\mathrm{c}}(\varphi)=\bigoplus_{\alpha\in A^{\mathrm{c}}(\varphi)}\mathcal{L}_\alpha$ and $\mathcal{H}^{\mathrm{wm}}(\varphi)=\bigoplus_{\alpha\in A^{\mathrm{wm}}(\varphi)}\mathcal{L}_\alpha$. ∎

5.6. Corollary. *Let G be a finitely generated group and let F be a nilpotent group of unitary operators on a Hilbert space $\mathcal{H}$. For $u\in\mathcal{H}$, the polynomial mappings $G\longrightarrow F$ which are compact on u form a group.*

Proof. By Corollary 1.18, we may assume that F is finitely generated. By Proposition 3.21, we may assume that G is nilpotent and so, amenable. Let $\mathcal{H}=\bigoplus_{\alpha\in A}\mathcal{L}_\alpha$ be the primitive decomposition of $\mathcal{H}$ relative to the action of F, and let $u=\sum_{\alpha\in A}u_\alpha$, $u_\alpha\in\mathcal{L}_\alpha$, $\alpha\in A$. Put $A(u)=\bigl\{\alpha\in A \bigm| u_\alpha\neq 0\bigr\}$; then, for the notation of 5.4, a polynomial mapping $\varphi\colon G\longrightarrow F$ is compact on u if and only if $\varphi(G)\subseteq E_\alpha$ for all $\alpha\in A(u)$. ∎

5.7. Now, let us turn to polynomial measure preserving actions. Let G be a finitely generated amenable group, let F be a nilpotent group of measure preserving transformations of a probability space $(X,\mathcal{B},\mu)$, and let $\varphi\colon G\longrightarrow F$ be a polynomial mapping. Then φ induces a polynomial unitary action of G on the Hilbert space $\mathcal{H}=L^2(X,\mathcal{B},\mu)$; let $\mathcal{H}=\mathcal{H}^{\mathrm{c}}(\varphi)\oplus\mathcal{H}^{\mathrm{wm}}(\varphi)$ be the corresponding decomposition of $\mathcal{H}$. It is clear that $\mathcal{H}^{\mathrm{c}}(\varphi)\cap L^\infty(X)$ is a $\varphi(G)$-invariant algebra closed under the operation of taking pointwise maximum of its elements. It follows that $\mathcal{H}^{\mathrm{wm}}(\varphi)$ corresponds to a factor of $(X,\mathcal{B},\mu,\varphi)$: there is a probability space $(Y,\mathcal{D},\nu)$ along with a measurable mapping $\pi\colon X\longrightarrow Y$ satisfying $\mu\bigl(\pi^{-1}(Q)\bigr)=\nu(Q)$ for all $Q\in\mathcal{D}$, and a polynomial measure preserving action ψ of G on $(Y,\mathcal{D},\nu)$ satisfying $\psi(g)\circ\pi=\pi\circ\varphi(g)$ for all $g\in G$, such that $\mathcal{H}^{\mathrm{wm}}(\varphi)=\pi^*\bigl(L^2(Y,\mathcal{D},\nu)\bigr)$. We have, consequently, the following theorem:

Theorem. *Let φ be a polynomial mapping of a finitely generated amenable group G to a nilpotent group of measure preserving transformations of a probability space $(X,\mathcal{B},\mu)$. Then the system $(X,\mathcal{B},\mu,\varphi)$ possesses a factor $(Y,\mathcal{D},\nu,\psi)$ such that ψ is compact on Y, and φ is compact on $u\in L^2(X,\mathcal{B},\mu)$ if and only if $u\in\pi^*\bigl(L^2(Y,\mathcal{D},\nu)\bigr)$.*

5.8. In conclusion, we want to bring an analogue of Theorem 5.4 in the case of polynomial measure preserving actions. It is based on the following "structure theorem":

Theorem. ([L2], Theorem 11.11) *Let F be a finitely generated nilpotent group of measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$. Then there is a nontrivial factor $(Z, \mathcal{D}, \nu, F)$ of $(X, \mathcal{B}, \mu, F)$ with the following properties. $(Z, \mathcal{D}, \nu)$ is representable as a direct product of probability spaces $(Z, \mathcal{D}, \nu) = \prod_{\alpha \in A}(Z_\alpha, \mathcal{D}_\alpha, \nu_\alpha)$ so that elements of F permute the spaces Z_α: for any $T \in F$ and $\alpha \in A$, $T(Z_\alpha) = Z_\beta$, $\beta \in A$. For every $\alpha \in A$, F contains a subgroup E_α such that*
(a) *the action of E_α on $\mathcal{L}_\alpha$ is compact;*
(b) *the stabilizer of Z_α, $H_\alpha = \{T \in F \mid T(Z_\alpha) = Z_\alpha\}$, coincides with the normalizer of E_α in F;*
(c) *every $T \in H_\alpha \setminus E_\alpha$ is weakly mixing on Z_α.*

It is clear that for any $\alpha \in A$ the group E_α is closed in G. By [BL] Proposition 1.16, the groups H_α, $\alpha \in A$, are also closed in G.

5.9. Theorem. *Let G be an amenable group, let φ be a polynomial mapping of G to a finitely generated nilpotent group F of measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$ satisfying $\varphi(\mathbf{1}_G) = \mathbf{1}_F = \mathrm{Id}_X$, and let $(Z, \mathcal{D}, \nu, F)$, $(Z, \mathcal{D}, \nu) = \prod_{\alpha \in A}(Z_\alpha, \mathcal{D}_\alpha, \nu_\alpha)$, be the factor of $(X, \mathcal{B}, \mu, F)$ described in Theorem 5.8. Then for any $\alpha \in A$ one has:*
(a) *for any $\beta \in A \setminus \{\alpha\}$, $\varphi(g)(Z_\alpha) \neq Z_\beta$ for all $g \in G$ but a set of zero density, and if $\varphi(G) \not\subseteq H_\alpha$, then also $\varphi(g)(Z_\alpha) \neq Z_\alpha$ for all $g \in G$ but a set of zero density;*
(b) *if $\varphi(G) \subseteq H_\alpha \setminus E_\alpha$, then φ is weakly mixing on Z_α;*
(c) *if $\varphi(G) \subseteq E_\alpha$, then φ is compact on Z_α.*

Proof. Let us consider the spaces $\mathcal{L}_\alpha = L^2(Z_\alpha, \mathcal{D}_\alpha, \nu_\alpha) \ominus \mathbb{C}$, $\alpha \in A$, as subspaces of the Hilbert space $L^2(Z, \mathcal{D}, \nu)$. Then the action of F on the space $\bigoplus_{\alpha \in A} \mathcal{L}_\alpha$ is as described in Theorem 5.3. Thus, Theorem 5.4 may be applied to this space. ■

5.10. Question: Do the results in this section remain true for non-finitely generated groups?

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