Ergodic components of an extension by a nilmanifold

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April 9, 2008

Abstract

We describe the structure of the ergodic decomposition of an extension of an ergodic system by a nilmanifold.

If $G$ is a compact group and $V$ a subgroup of $G$, then, under the (left) action of $V$, $G$ splits into a disjoint union of isomorphic “orbits”: if $H$ is the closure of $V$ in $G$, then the right cosets $Ha$, $a \in G$, are minimal closed $V$-invariant subsets of $G$, and the action of $V$ on each of these sets is ergodic (with respect to the Haar measure). If $X$ is a compact homogeneous space of a locally compact group $G$ and $V$ is a subgroup of $G$, then the structure of orbits of the action of $V$ on $X$ may be much more complicated. However, if $G$ is a nilpotent Lie group and $X$ is, respectively, a compact nilmanifold, then the orbit structure on $X$ is almost as simple as in the case of a compact $G$:

**Theorem 1.** Let $X$ be a compact nilmanifold and let $V$ be a group of translations of $X$. Then $X$ is a disjoint union of closed $V$-invariant (not necessarily isomorphic) subnilmanifolds, on each of which the action of $V$ is minimal and ergodic with respect to the Haar measure.

(See [Le], [L1], and [L2]; this is also a corollary of a general theory of Ratner and Shah on unipotent flows, see [Sh].)

Let us now turn to the “relative” situation. We say that a measure space $Y$ is an extension of $Y'$, and that $Y'$ is a factor of $Y$, if a measure preserving mapping $p: Y \to Y'$ is fixed. If $P$ and $P'$ are measure preserving actions of a group $V$ on $Y$ and $Y'$ respectively such that $P'_{v} \circ p = p \circ P_{v}$, $v \in V$, we say that $P$ is an extension of $P'$ on $Y$, and that $Y'$ is a factor of $Y$ under the action $P$.

Throughout the paper, $(\Omega, \nu)$ will be a probability measure space, and $S$ will be an ergodic measure preserving action of a group $V$ on $\Omega$. We will assume that $V$ is countable. (This assumption is not crucial for our argument, saves us from measure theoretical troubles: under this assumption, if something is true a.e. for every $v \in V$, then it is true a.e. for
all \( v \in V \) simultaneously.) Let \( G \) be a compact group; we say that an extension \( T \) of \( S \) on the space \( \Omega \times G \) is a group extension if \( T \) is defined by the formula \( T_v(\omega, x) = (S_v(\omega), a_{v,\omega}x) \), \( x \in G \), where \( a_{v,\omega} \in G \), \( \omega \in \Omega \), \( v \in V \), and for every \( v \in V \), the mapping \( \omega \mapsto a_{v,\omega} \) is assumed to be measurable. The family \((a_{v,\omega})_{\omega \in \Omega \atop v \in V}\) of elements of \( G \) defining \( T \) is called a cocycle; we will say that \( T \) is given by the cocycle \((a_{v,\omega})\). If \( H \) is a subgroup of \( G \) and \( a_{v,\omega} \in H \) for all \( v \in V \) and \( \omega \in \Omega \), we will say that \((a_{v,\omega})_{v \in V}\) is an \( H \)-cocycle. Clearly, if \( T \) is given by an \( H \)-cocycle, the sets \( \Omega \times (Hx) \), \( x \in G \), are \( T \)-invariant.

We will call a self-mapping of \( \Omega \times G \) defined by the formula \( (\omega, x) \mapsto (\omega, b_{\omega}x) \), \( x \in G \), where \( b_\omega \in G \), \( \omega \in \Omega \), and measurably depend on \( \omega \), a reparametrization of \( \Omega \times G \) over \( \Omega \). When reparametrizing \( \Omega \times G \) we allow ourselves to ignore a null set of \( \Omega \), so that the reparametrization function \( b_\omega \) can be only be defined on a subset \( \Omega' \) of full measure in \( \Omega \), and we substitute \( \Omega \) by \( \Omega' \). After a reparametrization given by \( b_\omega \), the cocycle \((a_{v,\omega})\), defining a group extension \( T \) of \( S \) on \( \Omega \times G \), changes to the cocycle \( (b_{S_\omega}a_{v,\omega}b_\omega^{-1}) \) (which is said to be cohomologous to \((a_{v,\omega})\)).

Let \( G \) be a compact metric group and let \( T \) be a group extension of \( S \) on \( \Omega \times G \). Then, in complete analogy with the absolute case, a simple decomposition of \( \Omega \times G \) takes place.

**Theorem 2.** (See, for example, [Z1].) There exists a closed subgroup \( H \) of \( G \) (called the Mackey group of \( T \)) such that, after a certain reparametrization of \( \Omega \times G \) over \( \Omega \), \( T \) is given by an \( H \)-cocycle and \( T \) is ergodic on the right cosets \( Ha \), \( a \in G \), with respect to \( \nu \times (\mu_Ha) \), where \( \mu_H \) is the left Haar measure on \( H \). Moreover, any \( T \)-ergodic measure on \( \Omega \times G \) whose projection to \( \Omega \) is \( \nu \) has the form \( \nu \times (\mu_Ha) \) for some \( a \in G \).

Now let \( G \) be locally compact group and let \( X \) be a compact homogeneous space of \( G \). The notion of a group extension of \( S \) on \( \Omega \times X \) given by a \( G \)-cocycle is transferred without changes to this case; we will only call it a homogeneous space extension, not a group extension. A reparametrization of \( \Omega \times X \) over \( \Omega \) with the help of a function \( b_\omega \in G^{\Omega} \) is also defined similarly. Our goal is to show that, in the framework of relative actions, compact nilmanifolds, again, behave as well as compact groups:

**Theorem 3.** Let \( X \) be a compact nilmanifold and let \( T \) be a homogeneous space extension of \( S \) on \( \Omega \times X \). There exists a closed subgroup \( H \) of \( G \) such that, after a certain reparametrization of \( \Omega \times X \) over \( \Omega \), \( T \) is given by an \( H \)-cocycle, and if \( \bigcup_{\theta \in \Theta} X_\theta \) is the partition of \( X \) into the minimal subnilmanifolds with respect to the action of \( H \), then the measures \( \nu \times \mu_{X_\theta} \), \( \theta \in \Theta \), where \( \mu_{X_\theta} \) is the Haar measure on \( X_\theta \), are \( T \)-ergodic, and are the only \( T \)-ergodic measures on \( \Omega \times X \) whose projection to \( \Omega \) is \( \nu \).

We will use the following notation and terminology. If \( a \) is a transformation of a (measure) space \( Y \) and \( f \) is a function on \( Y \), then \( a \) acts on \( f \) from the right by the rule \( (fa)(y) = f(ay) \). If a space \( Y' \) is a factor of \( Y \), then any function \( h' \) on \( Y' \) lifts to a function \( h \) on \( Y \); we identify \( h' \) with \( h \), and say that \( h \) comes from \( Y' \) in this case.

If \( Y' \) is a factor of a measure space \( Y \), \( P' \) is an action of a group \( V \) on \( Y' \), and \( P \) is an extension of \( P' \) on \( Y \), we will say that a function \( f \in L^\infty(Y) \) is an eigenfunction of \( P \) over \( Y \) if \( fP_v = \alpha_v f \), where \( \alpha_v \in L^\infty(Y') \), for every \( v \in V \). (Our definition of an eigenfunction over \( Y \) is more restricted than the standard definition of a generalized eigenfunction of \( P \).)
over $Y$, which assumes that the module spanned by the functions $fT_v, v \in V$, has finite rank over $L^\infty(\Omega)$.

$G$ will stand for a nilpotent Lie group of nilpotency class $r$, $\Gamma$ for a cocompact subgroup of $G$, and $X$ for the compact nilmanifold $G/\Gamma$. By $\mu_X$ we will denote the Haar measure on $X$, and will always mean this measure on $X$ if the opposite is not stated.

$T$ will stand for a homogeneous space extension of $S$ on $\Omega \times X$ by a cocycle $(a_{\nu,\omega})_{\nu \in V} \in \Omega \times X$.

If $Z$ is a factor of $X$ under the action of $G$, then $T$ induces an action of $V$ on $\Omega \times Z$, which is defined by the same cocycle $(a_{\nu,\omega})_{\nu \in V} \in \Omega \times X$. We will identify this action with $T$ and denote it by the same symbol.

A subnilmanifold $X'$ of $X$ is a closed subset of $X$ of the form $Kx$, where $K$ is a closed subgroup of $G$ and $x \in X$. (Note that the notion of a subnilmanifold depends on the group acting of $X$; what is a subnilmanifold of $X$ with respect to the action of $G$ may not be a subnilmanifold with respect to the action of, say, the identity component of $G$.) For a subnilmanifold $X' = Kx$ of $X$ we will denote by $\mu_{X'}$ the Haar measure on $X'$ with respect to the action of $K$, and will always mean this measure on $X'$ if the opposite is not stated.

Let $G^o$ be identity component of $G$. If $X$ is connected, then $X$ is a homogeneous space of $G^o$, $X = G^o/\Gamma \cap G^o$. If $X$ is disconnected, then $X$ is a finite union of connected subnilmanifolds; these subnilmanifolds are all isomorphic, are homogeneous spaces of $G^o$, and are permuted by elements of $G$.

We define $G_{(1)} = G^o, G_{(k)} = [G_{(k-1)}, G], k = 2, 3, \ldots, r$, and $X_{(k)} = G_{(k+1)} \setminus X, k = 0, 1, \ldots, r - 1$. When $X$ is connected, we also define $X_2 = [G^o, G^o] \setminus X$; then $X_2$ is a torus, the maximal factor-torus of $X$. We will denote by $p$ the canonical projection $\Omega \times X \rightarrow \Omega$.

A base tool in studying orbits in nilmanifolds is a lemma by W. Parry ([P1] and [P2]), that says that a shift-transformation of a compact connected nilmanifold $X$ is ergodic iff it is ergodic on the maximal factor-torus of $X$. Here is a “relative” analogue of Parry’s lemma; another proof of it can be found in [Z2].

**Proposition 4.** (Cf. [Z2], Corollary 3.4) Assume that $X$ is connected. If $T$ is ergodic on $\Omega \times X_2$, then $T$ is ergodic on $\Omega \times X$, and any eigenfunction $f$ of $T$ over $\Omega$ comes from $\Omega \times X_2$ and is such that $f(\omega, \cdot)$ is a character on $X_2$, times a constant, for a.e. $\omega \in \Omega$.

**Proof.** We will assume by induction on $r$ that $T$ is ergodic on $\Omega \times X_{(r-1)}$, and that if $g$ is an eigenfunction of $T$ on $\Omega \times X_{(r-1)}$ over $\Omega$, then $g$ comes from $\Omega \times X_2$ and $g(\omega, \cdot)$ is a character-times-a-constant on $X_2$ for a.e. $\omega \in \Omega$.

Let $f \in L^\infty(\Omega \times X)$ be an eigenfunction of $T$ over $\Omega$, $fT_v = \alpha_v(\omega)f, \alpha_v : \Omega \rightarrow \mathbb{C}, v \in V$. The action of the group $G_{(r)}$ on $\Omega \times X$ factors through an action of the compact commutative group (the torus) $G_{(r)}/(G_{(r)} \cap \Gamma)$, thus $L^2(\Omega \times X)$ is a direct sum of eigenspaces of $G_{(r)}$. Let $f'$ be a nonzero projection of $f$ to one of these eigenspaces, then $f'c = \lambda_c f'$, $\lambda_c \in \mathbb{C}$, for every $c \in G_{(r)}$. Since the eigenspaces of $G_{(r)}$ are $T$-invariant and invariant under multiplication by functions from $L^\infty(\Omega)$, we have $f'T_v = \alpha_v(\omega)f', v \in V$.

For every $b \in G$ and $c \in G_{(r)}$, $(f'b)c = f'cb = \lambda_c f'b$, so the function $f'_b = (f'b)/f'$ is $G_{(r)}$ invariant, and thus comes from $\Omega \times X_{(r-1)}$.

Assume, by induction on decreasing $k$, that for some $k \in \{2, \ldots, r\}$ we have $f'c = \lambda_c f'$, $\lambda_c \in \mathbb{C}^\Omega$, for any $c \in G_{(k)}$. Then $(f'c)(\omega, x) = \lambda_c(\omega)(\omega)f'(\omega, x), \omega \in \Omega, x \in X$, for any
\[ c = c(\omega) \in G^\Omega_{(k)}. \] Now, for any \( b \in G_{(k-1)} \) and \( v \in V, \)

\[
(f'bT_v)(\omega, x) = f'(S_v\omega, ba_v\omega x) = f'(S_v\omega, a_v\omega [a_v,\omega, b^{-1}]bx) = (f'T_v)(\omega, [a_v,\omega, b^{-1}]bx) = \alpha_v(\omega)f'(\omega, [a_v,\omega, b^{-1}]bx) = \alpha_v(\omega)\lambda_{c_v, b}(\omega)(f'b)(\omega, x),
\]

where \( c_v, b(\omega) = [a_v,\omega, b^{-1}] \in G_{(k)}, \omega \in \Omega. \) So, for any \( b \in G_{(k-1)} \) and \( v \in V, f'_bT_v = \lambda_{c_v, b}(\omega)f'_b, \) and since \( f'_b \) comes from \( X_{(r-1)}, \) by our first induction assumption, \( f'_b(\omega, \cdot) \) is a character-times-a-constant on \( X_2 \) for a.e. \( \omega \in \Omega. \) Thus, for a.e. \( \omega \in \Omega, \) we have a continuous mapping from \( G_{(k-1)} \) to the set of characters on \( X_2, \) and since this set is discrete and \( G_{(k)} \) is connected, this mapping is constant. (For a.e. \( \omega, \) the considered mapping may not be a priori defined on a null subset of \( G_{(k-1)} \), but since it is locally uniformly continuous, it extends to a continuous mapping on \( G_{(k)} \).) Hence, \( f'_b(\omega, \cdot) = \lambda_b(\omega), \lambda_b \in \mathbb{C}, \) for all \( b \in G_{(k-1)} \) and a.e. \( \omega \in \Omega, \) that is, \( f'b = \lambda_b f' \) with \( \lambda_b \in \mathbb{C}^\Omega, \) for all \( b \in G_{(k-1)} \), which gives us the induction step.

As the result of our induction on \( k \) we obtain that for every \( b \in G_{(1)} = G^o \) there exists a function \( \lambda_b \in \mathbb{C}^\Omega \) such that \( f'b = \lambda_b f'. \) Thus for any \( b_1, b_2 \in G^o \) we have \( f'[b_1, b_2] = f'. \) Hence, \( f' \) is \( [G^o, G^o] \)-invariant, and so, comes from \( \Omega \times X_2. \) The equality \( f'b = \lambda_b f', b \in G^o, \) now implies that \( f'(\omega, \cdot) \) is a character-times-a-constant on \( X_2 \) for a.e. \( \omega \in \Omega. \)

It follows that \( f \) also comes from \( \Omega \times X_2. \) In particular, there are no \( T \)-invariant functions on \( \Omega \times X \) since there are no \( T \)-invariant functions on \( \Omega \times X_2, \) so \( T \) is ergodic.

Now assume that for at least two distinct eigenspaces of \( G_{(r)} \) the projections \( f', f'' \) to these eigenspaces are nonzero. Then both \( f'T_v = \alpha_v(\omega)f' \) and \( f''T_v = \alpha_v(\omega)f'' \), \( v \in V, \) and so, \( f'/f'' \) is \( T \)-invariant, which contradicts the ergodicity of \( T. \) Hence, \( f \) belongs to one of the eigenspaces of \( G_{(r)}, \) and so, as this has been proven for \( f', f(\omega, \cdot) \) is a character-times-a-constant on \( X_2 \) for a.e. \( \omega \in \Omega. \)

**Remark.** In contrast with the absolute case (the case \( \Omega = \{ . \}), \) the stronger statement “\( T \) is ergodic if and it is ergodic on \( \Omega \times ([G, G] \backslash X) \)” (where it is assumed that \( G \) is generated by \( G^o \) and \( \{ T_v, v \in V \} \) is no longer true in the relative case. Here is an example: let \( \Omega = \mathbb{Z}_2, \) let \( X = T^2_{x_1, x_2} \) where \( T = \mathbb{R}/\mathbb{Z}, \) let \( G \) be the group of transformations of \( X \) of the form \( (x_1, x_2) \mapsto (x_1 + \alpha, x_2 + \beta), \alpha, \beta \in T, \) \( l \in \mathbb{Z}, \) and let \( V \) be the group generated by the transformation \( T(\omega, x_1, x_2) = (\omega + 1, x_1 + \omega, x_2 + (-1)^\omega x_1) \) of \( \Omega \times X, \) where \( \alpha \) is an irrational element of \( T. \) Then \([G, G] = \{(0, x_2), x_2 \in T\}, \) and \([G, G] \backslash X \simeq T_{x_1}. \) One checks that \( T \) is ergodic on \( \Omega \times ([G, G] \backslash X), \) whereas the function \( f(\omega, x_1, x_2) = \begin{cases} x_2, & \omega = 0 \\ x_2 - x_1, & \omega = 1 \end{cases} \) on \( \Omega \times X \) is \( T \)-invariant. The reason of this effect is clear, it is a “bad parametrization” of \( \Omega \times X; \) after a proper reparametrization, \( T \) acts as a rotation on \( X, \) \( G \) can be reduced to the group of rotations of \( X, \) and then \([G, G] \backslash X = X. \)

**Remark.** We do not know whether Proposition 4 can be extended to the (more general) class of generalized eigenfunctions of \( T \) over \( \Omega. \)

Let \( X \) be connected. Having Proposition 4, we may deal with the maximal factor-torus \( X_2 \) of \( X \) instead of \( X; \) indeed, if \( T \) is not ergodic on \( \Omega \times X, \) then \( T \) is not ergodic on \( T \times X_2 \) as well. The problem is that \( G, \) if disconnected, may act on \( X_2 \) not only by conventional rotations, but also by affine unipotent transformation. Thus, we will still
have to treat \( X_2 \) as a nilmanifold, not as a conventional torus. Since this does not change our argument, we will not assume that \( X \) is a torus; we will, however, call “characters” on \( X \) those on \( X_2 \).

Note that for any character \( \chi \) on \( X \) and any \( a \in G \), \( \chi a = \lambda \chi' \), where \( \chi' \) is a character on \( X \) and \( \lambda \in \mathbb{C}, |\lambda| = 1 \). On the other hand, if \( \lambda \in \mathbb{C}, |\lambda| = 1 \), and \( \chi \) is a character on \( X \), then, clearly, there exists a translation \( a \) of \( X \) such that \( \chi a = \lambda \chi \).

Rather than Proposition 4, we will actually need the following, more technical fact:

**Lemma 5.** Let \( X \) be connected. Assume that \( T \) is ergodic on \( X_{(r-1)} \) and that \( f \in L^\infty(\Omega \times X) \) is \( T \)-invariant and is an eigenfunction of \( G_{(r)} \). Then \( f(\omega, \cdot) \) is a character-times-a-constant on \( X \) for a.e. \( \omega \in \Omega \).

Of course, if \( X_2 \) is a factor of \( X_{(r-1)} \), this lemma follows from Proposition 4; otherwise it has to be proven separately, though its proof is very similar to that of Proposition 4.

**Proof.** Let \( fc = \lambda_c f \), \( \lambda_c \in \mathbb{C}, c \in G_{(r)} \). For every \( b \in G \) and \( c \in G_{(r)} \), \((fb)c = fcb = \lambda_c fb\), so the function \( fb = (fb)/f \) is \( G_{(r)} \) invariant, and thus comes from \( \Omega \times X_{(r-1)} \). Assume, by induction on decreasing \( k \), that for some \( k \in \{2, \ldots, r\} \) we have \( fc = \lambda_c f \), \( \lambda_c \in \mathbb{C}^\Omega \), for any \( c \in G_{(k)} \). Then \((fc)(\omega, x) = \lambda_c(\omega)f(\omega, x)\), \( \omega \in \Omega \), \( x \in X \), for any \( c = c(\omega) \in G_{(k)}^\Omega \). Now, for any \( b \in G_{(k-1)} \) and \( v \in V \),

\[
(fb_T)(\omega, x) = f(S_v \omega, ba_v, \omega x) = f(S_v \omega, a_v, \omega(x, b^{-1}bx)) = (fT_v)(\omega, [a_v, \omega, b^{-1}bx]) = f(\omega, [a_v, \omega, b^{-1}bx]) = \lambda_{c_v}(\omega)f(\omega, bx) = \lambda_{c_v}(\omega)(fb)(\omega, x),
\]

where \( c_v(\omega) = [a_v, \omega, b^{-1}] \in G_{(k)} \), \( \omega \in \Omega \). So, for any \( b \in G_{(k-1)} \) and \( v \in V \), \( fbT_v = \lambda_{c_v}(\omega)fb \), and since \( fb \) comes from \( X_{(r-1)} \) where \( T \) is ergodic, by Proposition 4, \( fb(\omega, \cdot) \) is a character-times-a-constant on \( X \) for a.e. \( \omega \in \Omega \). Thus, for a.e. \( \omega \in \Omega \), we have a continuous mapping from \( G_{(k-1)} \) to the set of characters on \( X \), and since this set is discrete and \( G_{(k-1)} \) is connected, this mapping is constant. Hence, \( fb(\omega, \cdot) = \lambda_b(\omega) \), \( \lambda_b \in \mathbb{C} \), for all \( b \in G_{(k-1)} \) and a.e. \( \omega \in \Omega \), that is, \( fb = \lambda_b f \) with \( \lambda_b \in \mathbb{C}^\Omega \), for all \( b \in G_{(k-1)} \), which gives us the induction step.

As the result of induction on \( k \) we obtain that for every \( b \in G_{(1)} = G^o \) there exists a function \( \lambda_b \in \mathbb{C}^\Omega \) such that \( fb = \lambda_b f \). Hence, \( f(\omega, \cdot) \) is a character-times-a-constant on \( X \) for a.e. \( \omega \in \Omega \).

We will also need the following corollary of Theorem 2.

**Lemma 6.** Let \( K \) be a compact metric group, let \( Z \) be a homogeneous space of \( K \), and let \( R \) be a homogeneous space extension of \( S \) on \( \Omega \times Z \). If \( R \) is not ergodic, then \( K \) has a proper closed subgroup \( H \) such that, after a reparametrization of \( \Omega \times Z \) over \( \Omega \), \( R \) is given by an \( H \)-cocycle.

**Proof.** The cocycle defining the action \( R \) defines a group action \( \tilde{R} \) of \( V \) on \( \Omega \times K \), for which \( \tilde{R} \) is a factor. If \( R \) is not ergodic, then \( \tilde{R} \) is not ergodic as well, and the assertion of the lemma follows from Theorem 2.

**Proposition 7.** Assume that \( T \) is not ergodic on \( \Omega \times X \). Then there exists a proper closed subgroup \( H \) of \( G \) such that, after a certain reparametrization of \( \Omega \times X \) over \( \Omega \), \( T \) is given by an \( H \)-cocycle.
Proof. We will use induction on $r$, the nilpotency class of $X$. First, for simplicity, consider the case where $X$ is connected. If $T$ is not ergodic on $\Omega \times X_{(r-1)}$, then we are done by induction on $r$. Thus, we assume that $T$ is ergodic on $\Omega \times X_{(r-1)}$. Let $f$ be a nonzero measurable $T$-invariant function on $\Omega \times X$. We replace $f$ by its nonzero projection to one of the eigenspaces of $G_{(r)}$, which is also a $T$-invariant function. By Lemma 5, $f(\omega, \cdot) = \lambda(\omega)\chi_\omega$, where $\chi_\omega$ is a character on $X$ and $\lambda(\omega) \in \mathbb{C}$, for a.e. $\omega \in \Omega$. Since $S$ is ergodic, $|\lambda(\omega)| = \text{const}$ on a subset $\Omega'$ of $\Omega$ of full measure, and we may assume that $|\lambda| \equiv 1$. There are only countably many characters on $X$, therefore a subset $\Omega''$ of full measure in $\Omega'$ is partitioned into the union of sets of positive measure where $\chi_\omega$ is constant. Since $S$ is ergodic, we can choose a character $\chi$ on $X$ and elements $b(\omega), \omega \in \Omega''$, measurably depending on $\omega$, such that for every $\omega \in \Omega'$ one has $\lambda_\omega \chi_\omega = \chi b_\omega$, so that $f(\omega, x) = \lambda(\omega)\chi_\omega(x) = \chi(b_\omega x)$, $x \in X$. After the reparametrization of $\Omega \times X$ defined by the function $\omega \mapsto (\omega, x)$ (and replacing $\Omega$ by $\Omega''$), $f$ takes the form $f(\omega, x) = \chi(x)$, $\omega \in \Omega$, $x \in X$. Let $H$ be the stabilizer of $\chi$ in $G$, $H = \{c \in G : \chi c = \chi\}$; then $H$ is a proper closed subgroup of $G$ and the cocycle defining $\Gamma$ takes values in $H$.

Now let $X$ be disconnected. $G$ acts on the finite set $\mathcal{X}$ of connected components of $X$; let $\bar{G}$ be the subgroup (of finite index) of $G$ that acts trivially on $\mathcal{X}$. Then the action of $G$ on $\mathcal{X}$ factorizes through the action of the finite group $G/\bar{G}$, and if $T$ is not ergodic on $\Omega \times \mathcal{X}$, we are done by Lemma 6. Thus, we may assume that $T$ is ergodic $\Omega \times \mathcal{X}$.

Let $X^0$ be a connected component of $X$; then $X$, under the action of $\bar{G}$, is isomorphic to $\{1, \ldots, n\} \times X^0$, where $n$ is the number of components in $X$. Consider $\Omega \times X = \Omega \times \{1, \ldots, n\} \times X^0$ as $\tilde{\Omega} \times X^0$ where $\tilde{\Omega} = \Omega \times \{1, \ldots, n\}$; by our assumption, $T$ acts ergodically on $\tilde{\Omega}$. Since $X^0$ is connected and has nilpotency class $\leq r$, we may, as in the first part of the proof, find a subset $\Omega'$ of full measure in $\tilde{\Omega}$ and a measurable $T$-invariant function $f$ on $\tilde{\Omega}' \times X^0 = \tilde{\Omega}' \times X$ such that $f(\omega, i, \cdot) = \lambda(\omega, i)\chi_{\omega, i}$, where $\chi_{\omega, i}$ is a character on $X^0$ and $\lambda(\omega, i) \in \mathbb{C}$, for all $\omega \in \Omega'$ and all $i \in \{1, \ldots, n\}$. For all $\omega \in \Omega'$ we, therefore, have the (non-ordered) set $C_\omega = \{\chi_{\omega, 1}, \ldots, \chi_{\omega, n}\}$ of characters on $X^0$ such that $T^i C_\omega = C_{S^i \omega, v}$, $v \in V$, for all $\omega \in \Omega'$, and since only countably many possibilities for $C_\omega$ exist, a certain reparametrization of $\Omega \times X$ over $\Omega$ (with replacing $\Omega$ by $\Omega''$) makes $C_\omega$ to be constant, $C_\omega = C = \{\chi_1, \ldots, \chi_n\}$ for all $\omega \in \Omega$. Moreover, since $T$ acts ergodically on $\Omega \times \mathcal{X}$, $G$ acts transitively on $C$; thus, after some change of coordinates in distinct connected components of $X$, we may make $\chi_1, \ldots, \chi_n$ to be all equal to the same character $\chi$. After this, we obtain that $\chi T_v = (\chi_{S^j o} \chi) i, j = j(v, \omega, i)$, for all $v \in V$, $\omega \in \Omega$, and $i \in \{1, \ldots, n\}$, that is, $T$ maps the fibers of $\chi$ to fibers. Let us assume, as we may, that $G$ is generated by $G^0$ and the entries of the cocycle defining $T$; then $G$ maps the fibers of $\chi$ to fibers, and we may factorize $X$ by these fibers. Let $Z$ be the factor; then $Z$ is a finite union of circles, $Z = \{1, \ldots, n\} \times \mathbb{T}$, and $G$ acts by rotations on $\mathbb{T}$, that is, for any $a \in G$, $a(i, x) = (a_i, x + \alpha_{a, i})$, $x \in \mathbb{T}$, $i \in \{1, \ldots, n\}$, with $\alpha_{a, i} \in \mathbb{T}$ (and $ai$ is defined by $X_{ai} = aX_i$). We obtain that the action of $G$ on $Z$ factorizes through the action of a compact group (the group of rotations of components of $Z$ and of permutations of these components). Since $T$ is not ergodic on $\Omega \times Z$, we are done by Lemma 6.

Lemma 8. If $T$ is ergodic on $\Omega \times X$ (with respect to $\nu \times \mu_X$), then $\nu \times \mu_X$ is the only $T$-ergodic probability measure whose projection on $\Omega$ is $\nu$. 

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Proof. Let $G_1 = G$ and $G_k = [G_{k-1}, G]$ for $k = 2, 3, \ldots, r$, let $X_{r-1} = G_r \setminus X$, and let $\pi_r: X \to X_{r-1}$ be the canonical projection. If $T$ is ergodic on $\Omega \times X$ with respect to $\nu \times \mu_X$, by induction on $r$, $\nu \times \mu_{X_{r-1}}$ is the only $T$-ergodic probability measure on $\Omega \times X_{r-1}$ whose projection on $\Omega$ is $\nu$. Thus, if $\tau$ is a $T$-ergodic probability measure on $\Omega \times X$ with $p(\tau) = \nu$, then $(\Id_\Omega \times \pi_r)(\tau) = \nu \times \mu_{X_{r-1}}$. $\Omega \times X$ is a group extension of $\Omega \times X_{r-1}$ with the fiber $F_r = G_r / (\Gamma \cap G_r)$, which is a compact commutative Lie group. Hence, by Theorem 2, $\tau = \nu \times \mu_{X_{r-1}} \times \mu_{F_r} = \nu \times \mu_X$. □

Proof of Theorem 3. Let $H$ be a minimal closed subgroup of $G$ such that there exists a reparametrization of $X \times \Omega$ over $\Omega$ after which $T$ is given by an $H$-cocycle. (Such a subgroup exists since any chain of decreasing subgroups of $G$ is finite.) Let $X = \bigcup_{\theta \in \Theta} X_\theta$ be the partition of $X$ into the union of subnilmanifolds minimal under the action of $H$, as in Theorem 1. After the reparametrization corresponding to $H$, $\Omega \times X$ splits into the disjoint union $\bigcup_{\theta \in \Theta} \Omega \times X_\theta$ of $T$-invariant subsets on each of which $T$ is given by an $H$-cocycle. If $T$ is not ergodic on one of these subsets, then by Proposition 7, $H$ contains a proper closed subgroup $H'$ such that, after a reparametrization of $\Omega \times X$ over $\Omega$, $T$ is given by an $H'$-cocycle; this contradicts the choice of $H$. Thus, $T$ is ergodic on each of $\Omega \times X_\theta$, $\theta \in \Theta$. Moreover, if $\tau$ is an ergodic measure on $\Omega \times X$ with $p(\tau) = \nu$, then $\tau$ must be supported by $\Omega \times X_\theta$ for some $\theta \in \Theta$, and thus $\tau = \nu \times \mu_{\Omega_\theta}$ by Lemma 8. □

Bibliography


