Ergodic components of an extension by a nilmanifold

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Abstract

We describe the structure of the ergodic decomposition of an extension of an ergodic system by a nilmanifold.

If G is a compact group and V a subgroup of G, then, under the (left) action of V, G splits into a disjoint union of isomorphic "orbits": if H is the closure of V in G, then the right cosets Ha, $a \in G$, are minimal closed V-invariant subsets of G, and the action of V on each of these sets is ergodic (with respect to the Haar measure). If X is a compact homogeneous space of a locally compact group G and V is a subgroup of G, then the structure of orbits of the action of V on X may be much more complicated. However, if G is a nilpotent Lie group and X is, respectively, a compact *nilmanifold*, then the orbit structure on X is almost as simple as in the case of a compact G:

Theorem 1. *Let X be a compact nilmanifold and let V be a group of translations of X. Then X is a disjoint union of closed V-invariant (not necessarily isomorphic) subnilmanifolds, on each of which the action of V is minimal and ergodic with respect to the Haar measure.*

(See [Le], [L1], and [L2]; this is also a corollary of a general theory of Ratner and Shah on unipotent flows, see [Sh].)

Let us now turn to the "relative" situation. We say that a measure space Y is *an extension* of Y', and that Y' is *a factor* of Y, if a measure preserving mapping $p: Y \longrightarrow Y'$ is fixed. If P and P' are measure preserving actions of a group V on Y and Y' respectively such that $P'_v \circ p = p \circ P_v$, $v \in V$, we say that P *is an extension of* P' *on* Y, and that Y' is *a factor of* Y *under the action* P.

Throughout the paper, (Ω, ν) will be a probability measure space, and S will be an ergodic measure preserving action of a group V on Ω. We will assume that V is countable. (This assumption is not crucial for our argument, saves us from measure theoretical troubles: under this assumption, if something is true a.e. for every $v \in V$, then it is true a.e. for

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2000 Mathematical Subject Classification: 22F10, 22D40 all $v \in V$ simultaneously.) Let G be a compact group; we say that an extension T of S on the space $\Omega \times G$ is *a group extension* if T is defined by the formula $T_v(\omega, x) = (S_v\omega, a_{v,\omega}x)$, $x \in G$, where $a_{v,\omega} \in G$, $\omega \in \Omega$, $v \in V$, and for every $v \in V$, the mapping $\omega \mapsto a_{v,\omega}$ is assumed to be measurable. The family $(a_{v,\omega})_{\substack{v\in V\\ \omega\in\Omega}}$ of elements of G defining T is called *a cocycle*; we will say that T *is given by* the cocycle $(a_{v,\omega})$. If H is a subgroup of G and $a_{v,\omega} \in H$ for all $v \in V$ and $\omega \in \Omega$, we will say that $(a_{v,\omega})_{\substack{v\in V\\ \omega\in\Omega}}$ is *an H-cocycle.* Clearly, if T is given by an H-cocycle, the sets $\Omega \times (Hx)$, $x \in G$, are T-invariant.

We will call a self-mapping of $\Omega \times G$ defined by the formula $(\omega, x) \mapsto (\omega, b_\omega x)$, $x \in G$, where $b_\omega \in G$, $\omega \in \Omega$, and measurably depend on ω, *a reparametrization of* $\Omega \times G$ *over* Ω. When reparametrizing $\Omega \times G$ we allow ourself to ignore a null set of Ω, so that the reparametrization function b_ω can be only be defined on a subset Ω' of full measure in Ω, and we substitute Ω by Ω'. After a reparametrization given by b_ω, the cocycle $(a_{v,\omega})$, defining a group extension T of S on $\Omega \times G$, changes to the cocycle $\big(b_{S_v\omega}a_{v,\omega}b_\omega^{-1}\big)$ (which is said to be *cohomologous* to $(a_{v,\omega})$).

Let G be a compact metric group and let T be a group extension of S on $\Omega \times G$. Then, in complete analogy with the absolute case, a simple decomposition of $\Omega \times G$ takes place.

Theorem 2. (See, for example, [Z1].) *There exists a closed subgroup H of G (called* the Mackey group *of T) such that, after a certain reparametrization of $\Omega \times G$ over Ω, T is given by an H-cocycle and T is ergodic on the right cosets Ha, $a \in G$, with respect to $\nu \times (\mu_H a)$, where μ_H is the left Haar measure on H. Moreover, any T-ergodic measure on $\Omega \times G$ whose projection to Ω is ν has the form $\nu \times (\mu_H a)$ for some $a \in G$.*

Now let G be locally compact group and let X be a compact homogeneous space of G. The notion of a group extension of S on $\Omega \times X$ given by a G-cocycle is transferred without changes to this case; we will only call it *a homogeneous space extension*, not a group extension. A reparametrization of $\Omega \times X$ over Ω with the help of a function $b_\omega \in G^\Omega$ is also defined similarly. Our goal is to show that, in the framework of relative actions, compact nilmanifolds, again, behave as well as compact groups:

Theorem 3. *Let X be a compact nilmanifold and let T be a homogeneous space extension of S on $\Omega \times X$. There exists a closed subgroup H of G such that, after a certain reparametrization of $\Omega \times X$ over Ω, T is given by an H-cocycle, and if $\bigcup_{\theta\in\Theta} X_\theta$ is the partition of X into the minimal subnilmanifolds with respect to the action of H, then the measures $\nu \times \mu_{X_\theta}$, $\theta \in \Theta$, where μ_{X_θ} is the Haar measure on X_θ, are T-ergodic, and are the only T-ergodic measures on $\Omega \times X$ whose projection to Ω is ν.*

We will use the following notation and terminology. If a is a transformation of a (measure) space Y and f is a function on Y, then a acts on f from the right by the rule $(fa)(y) = f(ay)$. If a space Y' is a factor of Y, then any function h' on Y' lifts to a function h on Y; we identify h' with h, and say that h *comes from* Y' in this case.

If Y' is a factor of a measure space Y, P' is an action of a group V on Y', and P is an extension of P' on Y, we will say that a function $f \in L^\infty(Y)$ is *an eigenfunction of P over Y* if $fP_v = \alpha_v f$, where $\alpha_v \in L^\infty(Y')$, for every $v \in V$. (Our definition of an eigenfunction over Y is more restricted than the standard definition of a *generalized eigenfunction of P*

over Y, which assumes that the module spanned by the functions fT_v, $v \in V$, has finite rank over $L^\infty(\Omega)$.)

G will stand for a nilpotent Lie group of nilpotency class r, Γ for a cocompact subgroup of G, and X for the compact nilmanifold G/Γ. By μ_X we will denote the Haar measure on X, and will always mean this measure on X if the opposite is not stated.

T will stand for a homogeneous space extension of S on $\Omega \times X$ by a cocycle $(a_{v,\omega})_{\substack{v\in V\\ \omega\in\Omega}}$.

If Z is a factor of X under the action of G, then T induces an action of V on $\Omega \times Z$, which is defined by the same cocycle $(a_{v,\omega})_{\substack{v\in V\\ \omega\in\Omega}}$. We will identify this action with T and denote it by the same symbol.

A subnilmanifold X' of X is a closed subset of X of the form Kx, where K is a closed subgroup of G and $x \in X$. (Note that the notion of a subnilmanifold depends on the group acting of X; what is a subnilmanifold of X with respect to the action of G may not be a subnilmanifold with respect to the action of, say, the identity component of G.) For a subnilmanifold $X' = Kx$ of X we will denote by $\mu_{X'}$ the Haar measure on X' with respect to the action of K, and will always mean this measure on X' if the opposite is not stated.

Let $G^{\rm o}$ be identity component of G. If X is connected, then X is a homogeneous space of $G^{\rm o}$, $X = G^{\rm o}/(\Gamma \cap G^{\rm o})$. If X is disconnected, then X is a finite union of connected subnilmanifolds; this subnilmanifolds are all isomorphic, are homogeneous spaces of $G^{\rm o}$, and are permuted by elements of G.

We define $G_{(1)} = G^{\rm o}$, $G_{(k)} = [G_{(k-1)}, G]$, $k = 2, 3, \ldots, r$, and $X_{(k)} = G_{(k+1)}\backslash X$, $k = 0, 1, \ldots, r-1$. When X is connected, we also define $X_2 = [G^{\rm o}, G^{\rm o}]\backslash X$; then X_2 is a torus, *the maximal factor-torus* of X. We will denote by p the canonical projection $\Omega \times X \longrightarrow \Omega$.

A base tool in studying orbits in nilmanifolds is a lemma by W. Parry ([P1] and [P2]), that says that a shift-transformation of a compact connected nilmanifold X is ergodic iff it is ergodic on the maximal factor-torus of X. Here is a "relative" analogue of Parry's lemma; another proof of it can be found in [Z2].

Proposition 4. (Cf. [Z2], Corollary 3.4) *Assume that X is connected. If T is ergodic on $\Omega \times X_2$, then T is ergodic on $\Omega \times X$, and any eigenfunction f of T over Ω comes from $\Omega \times X_2$ and is such that $f(\omega, \cdot)$ is a character on X_2, times a constant, for a.e. $\omega \in \Omega$.*

Proof. We will assume by induction on r that T is ergodic on $\Omega \times X_{(r-1)}$, and that if g is an eigenfunction of T on $\Omega \times X_{(r-1)}$ over Ω, then g comes from $\Omega \times X_2$ and $g(\omega, \cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega \in \Omega$.

Let $f \in L^\infty(\Omega \times X)$ be an eigenfunction of T over Ω, $fT_v = \alpha_v(\omega)f$, $\alpha_v{:}\,\Omega \longrightarrow \mathbb{C}$, $v \in V$. The action of the group $G_{(r)}$ on $\Omega \times X$ factors through an action of the compact commutative group (the torus) $G_{(r)}/(G_{(r)} \cap \Gamma)$, thus $L^2(\Omega \times X)$ is a direct sum of eigenspaces of $G_{(r)}$. Let f' be a nonzero projection of f to one of these eigenspaces, then $f'c = \lambda_c f'$, $\lambda_c \in \mathbb{C}$, for every $c \in G_{(r)}$. Since the eigenspaces of $G_{(r)}$ are T-invariant and invariant under multiplication by functions from $L^\infty(\Omega)$, we have $f'T_v = \alpha_v(\omega)f'$, $v \in V$.

For every $b \in G$ and $c \in G_{(r)}$, $(f'b)c = f'cb = \lambda_c f'b$, so the function $f'_b = (f'b)/f'$ is $G_{(r)}$ invariant, and thus comes from $\Omega \times X_{(r-1)}$.

Assume, by induction on decreasing k, that for some $k \in \{2, \ldots, r\}$ we have $f'c = \lambda_c f'$, $\lambda_c \in \mathbb{C}^\Omega$, for any $c \in G_{(k)}$. Then $(f'\mathbf{c})(\omega, x) = \lambda_{c(\omega)}(\omega)f'(\omega, x)$, $\omega \in \Omega$, $x \in X$, for any

$\mathbf{c} = c(\omega) \in G_{(k)}^{\Omega}$. Now, for any $b \in G_{(k-1)}$ and $v \in V$,

$$(f'bT_v)(\omega,x) = f'(S_v\omega, ba_{v,\omega}x) = f'\big(S_v\omega, a_{v,\omega}[a_{v,\omega},b^{-1}]bx\big) = (f'T_v)\big(\omega,[a_{v,\omega},b^{-1}]bx\big)$$
$$= \alpha_v(\omega)f'\big(\omega,[a_{v,\omega},b^{-1}]bx\big) = \alpha_v(\omega)\lambda_{c_{v,b}(\omega)}(\omega)f'(\omega,bx) = \alpha_v(\omega)\lambda_{c_{v,b}(\omega)}(\omega)(f'b)(\omega,x),$$

where $c_{v,b}(\omega) = [a_{v,\omega},b^{-1}] \in G_{(k)}$, $\omega \in \Omega$. So, for any $b \in G_{(k-1)}$ and $v \in V$, $f'_bT_v = \lambda_{c_{v,b}(\omega)}(\omega)f'_b$, and since f'_b comes from $X_{(r-1)}$, by our first induction assumption, $f'_b(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega \in \Omega$. Thus, for a.e. $\omega \in \Omega$, we have a continuous mapping from $G_{(k-1)}$ to the set of characters on X_2, and since this set is discrete and $G_{(k-1)}$ is connected, this mapping is constant. (For a.e. ω, the considered mapping may not be a priori defined on a null subset of $G_{(k-1)}$, but since it is locally uniformly continuous, it extends to a continuous mapping on $G_{(k-1)}$.) Hence, $f'_b(\omega,\cdot) = \lambda_b(\omega)$, $\lambda_b \in \mathbb{C}$, for all $b \in G_{(k-1)}$ and a.e. $\omega \in \Omega$, that is, $f'b = \lambda_b f'$ with $\lambda_b \in \mathbb{C}^{\Omega}$, for all $b \in G_{(k-1)}$, which gives us the induction step.

As the result of our induction on k we obtain that for every $b \in G_{(1)} = G^{\mathrm{o}}$ there exists a function $\lambda_b \in \mathbb{C}^{\Omega}$ such that $f'b = \lambda_b f'$. Thus for any $b_1,b_2 \in G^{\mathrm{o}}$ we have $f'[b_1,b_2] = f'$. Hence, f' is $[G^{\mathrm{o}},G^{\mathrm{o}}]$-invariant, and so, comes from $\Omega \times X_2$. The equality $f'b = \lambda_b f'$, $b \in G^{\mathrm{o}}$, now implies that $f'(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega \in \Omega$.

It follows that f also comes from $\Omega \times X_2$. In particular, there are no T-invariant functions on $\Omega \times X$ since there are no T-invariant functions on $\Omega \times X_2$, so T is ergodic.

Now assume that for at least two distinct eigenspaces of $G_{(r)}$ the projections f', f'' of f to these eigenspaces are nonzero. Then both $f'T_v = \alpha_v(\omega)f'$ and $f''T_v = \alpha_v(\omega)f''$, $v \in V$, and so, f'/f'' is T-invariant, which contradicts the ergodicity of T. Hence, f belongs to one of the eigenspaces of $G_{(r)}$, and so, as this has been proven for f', $f(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega \in \Omega$. ■

Remark. In contrast with the absolute case (the case $\Omega = \{.\}$), the stronger statement "T is ergodic if it is ergodic on $\Omega \times \big([G,G]\backslash X\big)$" (where it is assumed that G is generated by G^{o} and $\{T_v,\ v \in V\}$) is no longer true in the relative case. Here is an example: let $\Omega = \mathbb{Z}_2$, let $X = \mathbb{T}^2_{x_1,x_2}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, let G be the group of transformations of X of the form $(x_1,x_2) \mapsto (x_1+\alpha, x_2+lx_1+\beta)$, $\alpha,\beta \in \mathbb{T}$, $l \in \mathbb{Z}$, and let V be the group generated by the transformation $T(\omega,x_1,x_2) = \big(\omega+1, x_1+\omega\alpha, x_2+(-1)^{\omega}x_1\big)$ of $\Omega \times X$, where α is an irrational element of $\mathbb{T}$. Then $[G,G] = \big\{(0,x_2),\ x_2 \in \mathbb{T}\big\}$, and $[G,G]\backslash X \simeq \mathbb{T}_{x_1}$. One checks that T is ergodic on $\Omega \times \big([G,G]\backslash X\big)$, whereas the function $f(\omega,x_1,x_2) = \begin{cases} x_2, & \omega = 0 \\ x_2 - x_1, & \omega = 1 \end{cases}$ on $\Omega \times X$ is T-invariant. The reason of this effect is clear, it is a "bad parametrization" of $\Omega \times X$; after a proper reparametrization, T acts as a rotation on X, G can be reduced to the group of rotations of X, and then $[G,G]\backslash X = X$.

Remark. We do not know whether Proposition 4 can be extended to the (more general) class of generalized eigenfunctions of T over Ω.

Let X be connected. Having Proposition 4, we may deal with the maximal factor-torus X_2 of X instead of X; indeed, if T is not ergodic on $\Omega \times X$, then T is not ergodic on $T \times X_2$ as well. The problem is that G, if disconnected, may act on X_2 not only by conventional rotations, but also by affine unipotent transformation. Thus, we will still have to treat X_2 as a nilmanifold, not as a conventional torus. Since this does not change our argument, we will not assume that X is a torus; we will, however, call "characters" on X those on X_2.

Note that for any character χ on X and any $a \in G$, $\chi a = \lambda \chi'$, where χ' is a character on X and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. On the other hand, if $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and χ is a character on X, then, clearly, there exists a translation a of X such that $\chi a = \lambda \chi$.

Rather than Proposition 4, we will actually need the following, more technical fact:

Lemma 5. *Let X be connected. Assume that T is ergodic on $X_{(r-1)}$ and that $f \in L^\infty(\Omega \times X)$ is T-invariant and is an eigenfunction of $G_{(r)}$. Then $f(\omega, \cdot)$ is a character-times-a-constant on X for a.e. $\omega \in \Omega$.*

Of course, if X_2 is a factor of $X_{(r-1)}$, this lemma follows from Proposition 4; otherwise it has to be proven separately, though its proof is very similar to that of Proposition 4.

Proof. Let $fc = \lambda_c f$, $\lambda_c \in \mathbb{C}$, $c \in G_{(r)}$. For every $b \in G$ and $c \in G_{(r)}$, $(fb)c = fcb = \lambda_c fb$, so the function $f_b = (fb)/f$ is $G_{(r)}$ invariant, and thus comes from $\Omega \times X_{(r-1)}$. Assume, by induction on decreasing k, that for some $k \in \{2, \dots, r\}$ we have $fc = \lambda_c f$, $\lambda_c \in \mathbb{C}^\Omega$, for any $c \in G_{(k)}$. Then $(f\mathbf{c})(\omega, x) = \lambda_{c(\omega)}(\omega) f(\omega, x)$, $\omega \in \Omega$, $x \in X$, for any $\mathbf{c} = c(\omega) \in G_{(k)}^\Omega$. Now, for any $b \in G_{(k-1)}$ and $v \in V$,

$$(fbT_v)(\omega, x) = f(S_v\omega, ba_{v,\omega}x) = f\big(S_v\omega, a_{v,\omega}[a_{v,\omega}, b^{-1}]bx\big) = (fT_v)\big(\omega, [a_{v,\omega}, b^{-1}]bx\big)$$
$$= f\big(\omega, [a_{v,\omega}, b^{-1}]bx\big) = \lambda_{c_{v,b}(\omega)}(\omega) f(\omega, bx) = \lambda_{c_{v,b}(\omega)}(\omega)(fb)(\omega, x),$$

where $c_{v,b}(\omega) = [a_{v,\omega}, b^{-1}] \in G_{(k)}$, $\omega \in \Omega$. So, for any $b \in G_{(k-1)}$ and $v \in V$, $f_b T_v = \lambda_{c_{v,b}(\omega)}(\omega) f_b$, and since f_b comes from $X_{(r-1)}$ where T is ergodic, by Proposition 4, $f_b(\omega, \cdot)$ is a character-times-a-constant on X for a.e. $\omega \in \Omega$. Thus, for a.e. $\omega \in \Omega$, we have a continuous mapping from $G_{(k-1)}$ to the set of characters on X, and since this set is discrete and $G_{(k-1)}$ is connected, this mapping is constant. Hence, $f_b(\omega, \cdot) = \lambda_b(\omega)$, $\lambda_b \in \mathbb{C}$, for all $b \in G_{(k-1)}$ and a.e. $\omega \in \Omega$, that is, $fb = \lambda_b f$ with $\lambda_b \in \mathbb{C}^\Omega$, for all $b \in G_{(k-1)}$, which gives us the induction step.

As the result of induction on k we obtain that for every $b \in G_{(1)} = G^{\mathrm{o}}$ there exists a function $\lambda_b \in \mathbb{C}^\Omega$ such that $fb = \lambda_b f$. Hence, $f(\omega, \cdot)$ is a character-times-a-constant on X for a.e. $\omega \in \Omega$. ∎

We will also need the following corollary of Theorem 2.

Lemma 6. *Let K be a compact metric group, let Z be a homogeneous space of K, and let R be a homogeneous space extension of S on $\Omega \times Z$. If R is not ergodic, then K has a proper closed subgroup H such that, after a reparametrization of $\Omega \times Z$ over Ω, R is given by an H-cocycle.*

Proof. The cocycle defining the action R defines a group action $\widetilde{R}$ of V on $\Omega \times K$, for which R is a factor. If R is not ergodic, then $\widetilde{R}$ is not ergodic as well, and the assertion of the lemma follows from Theorem 2. ∎

Proposition 7. *Assume that T is not ergodic on $\Omega \times X$. Then there exists a proper closed subgroup H of G such that, after a certain reparametrization of $\Omega \times X$ over Ω, T is given by an H-cocycle.*

Proof. We will use induction on r, the nilpotency class of X. First, for simplicity, consider the case where X is connected. If T is not ergodic on $\Omega \times X_{(r-1)}$, then we are done by induction on r. Thus, we assume that T is ergodic on $\Omega \times X_{(r-1)}$. Let f be a nonzero measurable T-invariant function on $\Omega \times X$. We replace f by its nonzero projection to one of the eigenspaces of $G_{(r)}$, which is also a T-invariant function. By Lemma 5, $f(\omega, \cdot) = \lambda(\omega)\chi_\omega$, where χ_ω is a character on X and $\lambda(\omega) \in \mathbb{C}$, for a.e. $\omega \in \Omega$. Since S is ergodic, $|\lambda(\omega)| = \text{const}$ on a subset Ω' of Ω of full measure, and we may assume that $|\lambda| \equiv 1$. There are only countably many characters on X, therefore a subset Ω'' of full measure in Ω' is partitioned into the union of sets of positive measure where χ_ω is constant. Since S is ergodic, we can choose a character χ on X and elements $b(\omega)$, $\omega \in \Omega''$, measurably depending on ω, such that for every $\omega \in \Omega'$ one has $\lambda_\omega \chi_\omega = \chi b_\omega$, so that $f(\omega, x) = \lambda(\omega)\chi_\omega(x) = \chi(b_\omega x)$, $x \in X$. After the reparametrization of $\Omega \times X$ defined by the function b_ω (and replacing Ω by Ω''), f takes the form $f(\omega, x) = \chi(x)$, $\omega \in \Omega$, $x \in X$. Let H be the stabilizer of χ in G, $H = \{c \in G : \chi c = \chi\}$; then H is a proper closed subgroup of G and the cocycle defining T takes values in H.

Now let X be disconnected. G acts on the finite set $\mathcal{X}$ of connected components of X; let $\widetilde{G}$ be the subgroup (of finite index) of G that acts trivially on $\mathcal{X}$. Then the action of G on $\mathcal{X}$ factorizes through the action of the finite group $G/\widetilde{G}$, and if T is not ergodic on $\Omega \times \mathcal{X}$, we are done by Lemma 6. Thus, we may assume that T is ergodic $\Omega \times \mathcal{X}$.

Let X^{o} be a connected component of X; then X, under the action of $\widetilde{G}$, is isomorphic to $\{1, \ldots, n\} \times X^{\mathrm{o}}$, where n is the number of components in X. Consider $\Omega \times X = \Omega \times \{1, \ldots, n\} \times X^{\mathrm{o}}$ as $\widetilde{\Omega} \times X^{\mathrm{o}}$ where $\widetilde{\Omega} = \Omega \times \{1, \ldots, n\}$; by our assumption, T acts ergodically on $\widetilde{\Omega}$. Since X^{o} is connected and has nilpotency class $\leq r$, we may, as in the first part of the proof, find a subset Ω' of full measure in Ω and a measurable T-invariant function f on $\widetilde{\Omega}' \times X^{\mathrm{o}} = \Omega' \times X$ such that $f(\omega, i, \cdot) = \lambda(\omega, i)\chi_{\omega,i}$, where $\chi_{\omega,i}$ is a character on X^{o} and $\lambda(\omega, i) \in \mathbb{C}$, for all $\omega \in \Omega'$ and all $i \in \{1, \ldots, n\}$. For all $\omega \in \Omega'$ we, therefore, have the (non-ordered) set $C_\omega = \{\chi_{\omega,1}, \ldots, \chi_{\omega,n}\}$ of characters on X^{o} such that $T_v C_\omega = C_{S_v \omega}$, $v \in V$, for all $\omega \in \Omega'$, and since only countably many possibilities for C_ω exist, a certain reparametrization of $\Omega \times X$ over Ω (with replacing Ω by Ω') makes C_ω to be constant, $C_\omega = C = \{\chi_1, \ldots, \chi_n\}$ for all $\omega \in \Omega$. Moreover, since T acts ergodically on $\Omega \times \mathcal{X}$, G acts transitively on C; thus, after some change of coordinates in distinct connected components of X, we may make $\chi_1, \ldots, \chi_n$ to be all equal to the same character χ. After this, we obtain that $\chi T_v = \frac{\lambda(\omega,i)}{\lambda(S_v\omega,j)}\chi$, $j = j(v, \omega, i)$, for all $v \in V$, $\omega \in \Omega$, and $i \in \{1, \ldots, n\}$, that is, T maps the fibers of χ to fibers. Let us assume, as we may, that G is generated by G^{o} and the entries of the cocycle defining T; then G maps the fibers of χ to fibers, and we may factorize X by these fibers. Let Z be the factor; then Z is a finite union of circles, $Z = \{1, \ldots, n\} \times \mathbb{T}$, and G acts by rotations on $\mathbb{T}$, that is, for any $a \in G$, $a(i, x) = (ai, x + \alpha_{a,i})$, $x \in \mathbb{T}$, $i \in \{1, \ldots, n\}$, with $\alpha_{a,i} \in \mathbb{T}$ (and ai is defined by $X_{ai} = aX_i$). We obtain that the action of G on Z factorizes through the action of a compact group (the group of rotations of components of Z and of permutations of these components). Since T is not ergodic on $\Omega \times Z$, we are done by Lemma 6. ∎

Lemma 8. *If T is ergodic on $\Omega \times X$ (with respect to $\nu \times \mu_X$), then $\nu \times \mu_X$ is the only T-ergodic probability measure whose projection on Ω is ν.*

Proof. Let $G_1 = G$ and $G_k = [G_{k-1}, G]$ for $k = 2, 3, \ldots, r$, let $X_{r-1} = G_r \backslash X$, and let $\pi_r \colon X \longrightarrow X_{r-1}$ be the canonical projection. If T is ergodic on $\Omega \times X$ with respect to $\nu \times \mu_X$, by induction on r, $\nu \times \mu_{X_{r-1}}$ is the only T-ergodic probability measure on $\Omega \times X_{r-1}$ whose projection on Ω is ν. Thus, if τ is a T-ergodic probability measure on $\Omega \times X$ with $p(\tau) = \nu$, then $(\mathrm{Id}_\Omega \times \pi_r)(\tau) = \nu \times \mu_{X_{r-1}}$. $\Omega \times X$ is a group extension of $\Omega \times X_{r-1}$ with the fiber $F_r = G_r/(\Gamma \cap G_r)$, which is a compact commutative Lie group. Hence, by Theorem 2, $\tau = \nu \times \mu_{X_{r-1}} \times \mu_{F_r} = \nu \times \mu_X$. ■

Proof of Theorem 3. Let H be a minimal closed subgroup of G such that there exists a reparametrization of $X \times \Omega$ over Ω after which T is given by an H-cocycle. (Such a subgroup exists since any chain of decreasing subgroups of G is finite.) Let $X = \bigcup_{\theta \in \Theta} X_\theta$ be the partition of X into the union of subnilmanifolds minimal under the action of H, as in Theorem 1. After the reparametrization corresponding to H, $\Omega \times X$ splits into the disjoint union $\bigcup_{\theta \in \Theta} \Omega \times X_\theta$ of T-invariant subsets on each of which T is given by an H-cocycle. If T is not ergodic on one of these subsets, then by Proposition 7, H contains a proper closed subgroup H' such that, after a reparametrization of $\Omega \times X$ over Ω, T is given by an H'-cocycle; this contradicts the choice of H. Thus, T is ergodic on each of $\Omega \times X_\theta$, $\theta \in \Theta$. Moreover, if τ is an ergodic measure on $\Omega \times X$ with $p(\tau) = \nu$, then τ must be supported by $\Omega \times X_\theta$ for some $\theta \in \Theta$, and thus $\tau = \nu \times \mu_{\Omega_\theta}$ by Lemma 8. ■

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