Multiple polynomial correlation sequences and nilsequences

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

February 1, 2010

Abstract

A basic nilsequence is a sequence of the form $\psi(n) = f(T^n x)$, where $x$ is a point of a compact nilmanifold $X$, $T$ is a translation on $X$, and $f \in C(X)$; a nilsequence is a uniform limit of basic nilsequences. Let $X = G/\Gamma$ be a compact nilmanifold, $Y$ be a subnilmanifold of $X$, $g(n)$ be a polynomial sequence in $G$, and $f \in C(X)$; we show that the sequence $\int g(n)Y f$, $n \in \mathbb{Z}$, is the sum of a basic nilsequence and a sequence that converges to 0 in uniform density. This implies that, given an ergodic invertible measure preserving system $(W, B, \mu, T)$, with $\mu(W) < \infty$, polynomials $p_1, \ldots, p_k \in \mathbb{Z}[n]$, and sets $A_1, \ldots, A_k \in B$, the sequence $\mu(T^{p_1(n)}A_1 \cap \ldots \cap T^{p_k(n)}A_k)$ is the sum of a nilsequence and a sequence that converges to 0 in uniform density. We also get a version of this result for the case where $p_i$ are polynomials in several variables.

0. Introduction

A $(d$-step) nilmanifold is a compact homogeneous space of a $(d$-step) nilpotent Lie group; one can show that any $d$-step nilmanifold has the form $G/\Gamma$, where $G$ is a $d$-step nilpotent (not necessarily connected) Lie group and $\Gamma$ is a discrete co-compact subgroup of $G$. Elements of $G$ act on $X$ by translations; a $(d$-step) nilsystem is a $(d$-step) nilmanifold $X = G/\Gamma$ with a translation $a \in G$ on it. Nilsystems play an important role in studying “non-conventional”, or “multiple”, ergodic averages $\frac{1}{N} \sum_{n=1}^{N} T^{p_1(n)}h_1 \cdots T^{p_k(n)}h_k$, where $T$ is a transformation of a finite measure space $(W, \mu)$, $p_1, \ldots, p_k \in \mathbb{Z}[n]$, and $h_1, \ldots, h_k \in L^\infty(W)$. (See [HK1], [Z], [HK2].)

Let $X = G/\Gamma$ be a nilmanifold and $Y$ be a subnilmanifold of $X$. Let $g$ be a polynomial sequence in $G$, that is, a sequences of the form $g(n) = a_1^{p_1(n)} \cdots a_r^{p_r(n)}$, where $a_1, \ldots, a_r \in G$ and $p_1, \ldots, p_r$ are polynomials taking on integer values on the integers. It is shown in [L1] that the closure of the sequence $g(n)Y$, $X' = \bigcup_{n \in \mathbb{Z}} g(n)Y$, is a disjoint finite union of sub-nilmanifolds of $X$, and, if $X'$ is a single sub-nilmanifold, the sequence $g(n)Y$ is well distributed in $X'$. (That is, for every $f \in C(X')$,

Supported by NSF grants DMS-0600042.
\[
\frac{1}{N_2 - N_1} \sum_{n=N_1+1}^{N_2} \int_{g(n)Y} f \, d\mu_Y \xrightarrow{N_2-N_1 \to \infty} \int_X f \, d\mu_Y', \text{ where } \mu_Y \text{ and } \mu_Y' \text{ are the normalized Haar measures on } Y \text{ and on } X' \text{ respectively.}
\]

We were inspired by the following example. Let \( X \) be the 2-dimensional torus \( \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2 \) and \( G \) be the group generated by the ordinary rotations of \( X \) and by the transformation \( a(x, y) = (x, y+x) \); then \( G \) is a nilpotent Lie group acting on \( X \) transitively, which turns \( X \) to a nilmanifold. Choose an irrational \( \alpha \in \mathbb{T} \) and put \( b(x, y) = (x + \alpha, y+x) \), then \( b \in G \). Let \( Y_1 = \{(0, t), \ t \in \mathbb{T}\} \) and \( Y_2 = \{(t, 0), \ t \in \mathbb{T}\} \). Then \( b^nY_1 = \{(n\alpha, t), \ t \in \mathbb{T}\} \) and \( b^nY_2 = \{(t + na, nt + \frac{n(n-1)}{2}\alpha), \ t \in \mathbb{T}\}, \ n \in \mathbb{Z} \). Both sequences \( b^nY_1 \) and \( b^nY_2 \), \( n \in \mathbb{Z} \), are dense in \( X \), but their behaviors are different: the sequence \( b^nY_1 \) consists of congruent subtori that simply “rotate” along \( X \), whereas the members of the sequence \( b^nY_2 \), \( n \in \mathbb{Z} \), become more and more dense in \( X \). We can say that the sequence \( b^nY_1 \) converges to \( X \): \( \int_{g(n)Y_1} f \, d\mu_Y \to \int_X f \, d\mu_X \) for any \( f \in C(X) \), whereas the sequence \( b^nY_2 \) converges to \( X \) only in average: \( \frac{1}{N_2 - N_1} \sum_{n=N_1+1}^{N_2} \int_{g(n)Y_1} f \, d\mu_Y \to \int_X f \, d\mu_X \) for any \( f \in C(X) \). It is clear what difference between \( Y_1 \) and \( Y_2 \) causes this effect: \( Y_1 \) is a normal subgroup of \( G \) whereas \( Y_2 \) is not.

Our goal was to show that in the general situation the sequence \( g(n)Y \) has a “mixed” behavior: \( g(n)Y \) converges to a subnilmanifold \( Z \) (the normal closure of \( Y \)), which, in its turn, rotates along \( X \). We, however, have been unable to prove this, and only prove the weaker fact that \( g(n)Y \) converges to \( Z \) “in uniform density” (see Proposition 2.1). Our proof essentially uses a result from a recent paper by Green and Tao ([GT]) about the “uniform distribution” of subnilmanifolds (see Appendix).

In the terminology introduced in [BHK], a basic \( d \)-step nilsequence is a sequence of the form \( \psi(n) = h(R^nw) \), where \( w \) is a point of a \( d \)-step nilmanifold \( M \), \( R \) is a translation on \( M \), and \( h \in C(M) \); a \( d \)-step nilsequence is a uniform limit of basic \( d \)-step nilsequences. The algebra of nilsequences is a natural generalization of Weyl’s algebra of almost periodic sequences, which are just 1-step nilsequences. We obtain, as a corollary, that for any \( f \in C(X) \) the sequence \( \int_{g(n)Y} f \, d\mu_{g(n)}Y \) is a sum of a basic nilsequence and a sequence that tends to 0 in uniform density (Theorem 2.5 below). We apply this fact to show that for any ergodic invertible measure preserving system \((W, \mathcal{B}, \mu, T)\) with \( \mu(W) < \infty \), polynomials \( p_1, \ldots, p_k \in \mathbb{Z}[n] \), and sets \( A_1, \ldots, A_k \in \mathcal{B} \), the “multiple polynomial correlation sequence” \( \varphi(n) = \mu(T_1^{p_1(n)}A_1 \cap \ldots \cap T_k^{p_k(n)}A_k), n \in \mathbb{Z} \), is a sum of a nilsequence and a sequence that tends to 0 in uniform density (Theorem 3.1 below). (A special case of this theorem, when \( p_i(n) = in, \ i = 1, \ldots, k \), was established in [BHK].) The question whether this is true for non-ergodic systems remains open to us. We also formulate and sketch the proof of a “multiparameter” version of this result: when \( p_1, \ldots, p_k \) are polynomials of \( m \) integer variables, then the sequence \( \varphi(n) = \mu(T_1^{p_1(n)}A_1 \cap \ldots \cap T_k^{p_k(n)}A_k), n \in \mathbb{Z}^m \), is a sum of an \( (m-) \) nilsequence and a sequence that tends to 0 in (ordinary) density (Theorem 4.3).

1. Nilmanifolds and sub-nilmanifolds

We will now give necessary definitions and list some facts that we will need below; details and proofs can be found in [M], [L1], [L2], [L4], and [L5]. Throughout the paper, let
$X = G/\Gamma$ be a compact nilmanifold, where $G$ is a nilpotent Lie group and $\Gamma$ is a discrete subgroup of $G$, and let $\pi : G \rightarrow X$ be the natural projection. By $1_X$ we will denote the point $\pi(1_G)$ of $X$.

By $G^o$ we will denote the identity component of $G$. We will assume that the group $G/G^o$ is finitely generated (which is enough for our goals).

Note that if $G$ is disconnected, $X$ can be interpreted as a nilmanifold, $X = G'/\Gamma'$, in different ways; for example, if $X$ is connected, $X = G^o/(\Gamma \cap G^o)$. If $X$ is connected and we study the action on $X$ of a sequence $g(n)$ in $G$, we may always assume that $G$ is generated by $G^o$ and the elements of $G$.

Every nilpotent Lie group $G$ is a factor of a simply-connected (not necessarily connected) torsion free nilpotent Lie group. (As such, a suitable “free nilpotent Lie group” $F$ can be taken. If $G^o$ has $l_1$ generators, $G/G^o$ has $l_2$ generators, and $G$ is $d$-step nilpotent, then $F = F/F_{d+1}$, where $F$ is the free product of $l_1$ copies of $\mathbb{R}$ and $l_2$ copies of $\mathbb{Z}$, and $F_{d+1}$ is the $(d+1)$st term of the lower central series of $F$.) Thus, we may and will assume that $G$ is simply connected and torsion-free. The identity component $G^o$ of $G$ is then an exponential Lie group, which means that for every element $a \in G^o$ there exists a (unique) one-parametric subgroup $a^t$ such that $a^1 = a$.

A Malcev basis of $G$ is a finite set $\{e_1, \ldots, e_k\}$ of elements of $\Gamma$, with $e_1, \ldots, e_{k_1} \in G^o$ and $e_{k_1+1}, \ldots, e_k \notin G^o$, that generates $\Gamma$ and is such that every element $a \in G$ can be uniquely written in the form $a = e_1^{u_1} \ldots e_k^{u_k}$ with $u_1, \ldots, u_k \in \mathbb{R}$ and $u_{k_1+1}, \ldots, u_k \in \mathbb{Z}$; we call $u_1, \ldots, u_k$ the coordinates of $a$. Thus, Malcev coordinates define a homeomorphism $G \cong \mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$, $a \leftrightarrow (u_1, \ldots, u_k)$, and we may identify $G$ with $\mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$.

If $L$ is a connected closed normal subgroup of $G$ of dimension $l$ such that the lattice $L \cap \Gamma$ is co-compact in $L$, the Malcev coordinates on $G$ can be chosen so that $e_1, \ldots, e_l \in L \cap \Gamma; \}$ then $e_1^{u_1} \ldots e_k^{u_k} \in L$ iff $u_{l+1}, \ldots, u_k = 0$, and $L$ is identified with the subspace $\mathbb{R}^l \times \{0\}^{k-l} \subseteq \mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$. We will call such coordinates on $G$ compatible with $L$.

Let $X$ be connected. Then, under the identification $G^o \leftrightarrow \mathbb{R}^{k_1}$, the cube $[0,1)^{k_1}$ is the fundamental domain of $X$. We will call the closed cube $Q = [0,1]^{k_1}$ the fundamental cube of $X$ in $G^o$ and identify $X$ with $Q$. When $X$ is identified with its fundamental cube $Q$, the normalized Haar measure $\mu_X$ on $X$ coincides with the standard Lebesgue measure $\mu_Q$ on $Q$.

In Malcev coordinates, multiplication in $G$ is a polynomial operation: there are polynomials $q_1, \ldots, q_k$ in $2k$ variables with rational coefficients such that for $a = e_1^{u_1} \ldots e_k^{u_k}$ and $b = e_1^{v_1} \ldots e_k^{v_k}$ we have $ab = e_1^{q_1(u_1, v_1, \ldots, u_k, v_k)} \ldots e_k^{q_k(u_1, v_1, \ldots, u_k, v_k)}$. This implies that “life is polynomial” in nilpotent Lie groups: homomorphisms are polynomial mappings, connected closed subgroups are images of polynomial mappings and are defined by systems of polynomial equations.

A subnilmanifold $Y$ of $X$ is a closed subset of the form $Y = Hx$, where $H$ is a closed subgroup of $G$ and $x \in X$. For a closed subgroup $H$ of $G$, the set $\pi(H) = H1_X$ is closed, and so is a subnilmanifold, iff the subgroup $\Gamma \cap H$ is co-compact in $H$; we will call the subgroup $H$ with this property rational.

If $Y$ is a subnilmanifold of $X$ such that $1_X \in Y$, then $H = \pi^{-1}(Y)$ is a closed subgroup of $G$, and $Y = \pi(H) = H1_X$. $H$, however, does not have to be the minimal subgroup with this property: if $Y$ is connected, then the identity component $H^o$ of $H$ also satisfies
\[ Y = \pi(H^o). \]

Given a subnilmanifold \( Y \) of \( X \), by \( \mu_Y \) we will denote the normalized Haar measure on \( Y \); we have \( \mu_Y = \mu_a Y \) for all \( a \in G \).

Let \( Z \) be a subnilmanifold of \( X \), \( Z = Lx \), where \( L \) is a closed subgroup of \( G \). We say that \( Z \) is normal if \( L \) is normal. In this case the nilmanifold \( \tilde{X} = X/Z = G/(LG) \) is defined, and \( X \) splits into a disjoint union of fibers of the projection mapping \( X \to \tilde{X} \). (Note that if \( L \) is normal in \( G^o \) only, then the factor \( X/Z = G^o/(LG) \) is also defined, but the elements of \( G \setminus G^o \) do not act on it.)

One can show that a subgroup \( L \) is normal iff \( \gamma L \gamma^{-1} = L \) for all \( \gamma \in \Gamma \); hence, \( Z = \pi(L) \) is normal iff \( \gamma Z = Z \) for all \( \gamma \in \Gamma \).

If \( H \) is a closed rational subgroup of \( G \) then its normal closure \( L \) (the minimal normal subgroup of \( G \) containing \( H \)) is also closed and rational, thus \( Z = \pi(L) \) is a subnilmanifold of \( X \). We will call \( Z \) the normal closure of the subnilmanifold \( Y = \pi(H) \). If \( L \) is normal then the identity component of \( L \) is also normal; this implies that the normal closure of a connected subnilmanifold is connected.

Let \( X \) be connected and \( k \)-dimensional, and let \( Z \) be an \( l \)-dimensional connected normal subnilmanifold of \( X \). Let \( L \) be the connected normal closed subgroup of \( G \) such that \( Z = Lx \); choose Malcev coordinates on \( G \) compatible with \( L \), and let \( Q \) be the fundamental cube of \( X \) in \( G^o \) associated with these coordinates. Then the fundamental cube of \( Z \) is the subcube \([0,1]^k \times \{0\}^{k-l}\) of \( Q \), and the fundamental cube of \( X/Z \) is the orthogonal projection of \( Q \) to the \((k-l)\)-dimensional subspace associated with the last \( k-l \) coordinates on \( Q \).

Let \( X \) be connected. We will need the fact that “almost all” subnilmanifolds of \( X \) are “quite uniformly” distributed in \( X \). (This is in complete analogy with the situation on tori: if \( X \) is a torus, for any \( \varepsilon > 0 \) there are only finitely many subtori \( V_1,\ldots,V_r \), of codimension 1 in \( X \), such that any subtorus \( Y \) of \( X \) that contains 0 and is not contained in \( \bigcup_{i=1}^r V_i \) is \( \varepsilon \)-dense and “\( \varepsilon \)-uniformly distributed” in \( X \).) The following proposition is a corollary (of a special case) of the result obtained in [GT] (see Appendix for details):

**Proposition 1.1.** For any \( f \in C(X) \) and any \( \varepsilon > 0 \) there are finitely many subnilmanifolds \( V_1,\ldots,V_r \) of \( X \), connected, of codimension 1, and containing \( 1_X \), such that for any connected sub-nilmanifold \( Y \) of \( X \) with \( 1_X \in Y \), either \( Y \in V_i \) for some \( i \in \{1,\ldots,r\} \), or \( |f_Y f d\mu_Y - f_X f d\mu_X| < \varepsilon \), (or both).

Identifying a sub-nilmanifold \( Y \) of \( X \) with the measure \( \mu_Y \) on \( X \), we introduce the weak* topology on the set of sub-nilmanifolds of \( X \); in this topology, given sub-nilmanifolds \( Z, Y_1, Y_2,\ldots \) of \( X \), we write \( Y_n \to Z \) if \( \int_Y f d\mu_{Y_n} \to \int_Z f d\mu_Z \) for every \( f \in C(X) \). It now follows from Proposition 1.1 that if connected sub-nilmanifolds \( Y_1, Y_2,\ldots \) of \( X \), with \( 1_X \in Y_n \) for all \( n \), are such that for any proper sub-nilmanifold \( V \) of \( X \) (connected, of codimension 1, and with \( 1_X \in V \) ) the set \( \{ n \in \mathbb{Z} : Y_n \subseteq V \} \) is finite, then \( Y_n \to X \).

For a set \( S \subseteq \mathbb{Z} \), the uniform (or Banach) density of \( S \) is \( \mathcal{D}(S) = \lim_{N_2-N_1 \to \infty} \left| \frac{[S \cap [N_1,N_2]]}{N_2-N_1} \right| \) \( (\text{if it exists}) \). We will say that a sequence of points \( (\omega_n)_{n \in \mathbb{Z}} \) of a topological space \( \Omega \) converges to \( \omega \in \Omega \) in uniform density if for every neighborhood \( U \) of \( \omega \) one has \( \mathcal{D}(\{ n \in \mathbb{Z} : \omega_n \notin U \}) = 0 \). It follows from Proposition 1.1 that, given connected sub-nilmanifolds \( Y_1, Y_2,\ldots \) of \( X \) with \( 1_X \in Y_n \) for all \( n \), if for any proper sub-nilmanifold \( V \) of \( X \)
(connected, of codimension 1, and with $1_X \in V$) one has $D(\{n \in \mathbb{Z} : Y_n \subseteq V\}) = 0$, then $Y_n \rightarrow X$ in uniform density.

2. Polynomial orbits of subnilmanifolds and nilsequences

Our main technical result is the following proposition.

**Proposition 2.1.** Let $X$ be connected and let $Y = \pi(H)$ be a connected subnilmanifold of $X$, where $H$ is a connected closed subgroup of $G$. Let $g$ be a polynomial sequence in $G$ with $g(0) = 1_G$ such that $g(\mathbb{Z})Y$ is dense in $X$, and assume that $G$ is generated by $G^o$ and the elements of $g$. Let $Z$ be the normal closure of $Y$ in $X$; then $g(n)Y - g(n)Z \rightarrow 0$ in uniform density.

**Remark.** We believe that, actually, $g(n)Y - g(n)Z \rightarrow 0$ (that is, for any $f \in C(X)$, $|\int_{g(n)Y} f \, d\mu_{g(n)y} - \int_{g(n)Z} f \, d\mu_{g(n)z}| \rightarrow 0$ as $n \rightarrow \infty$).

**Proof.** Let $L$ be the identity component of $\pi^{-1}(Z)$. Choose Malcev’s coordinates in $G^o$ compatible with $L$, and let $Q$ be the corresponding fundamental cube in $G^o$. $Q$ is compact, and is as well compact with respect to the uniform norm when elements of $G$ are interpreted as transformations of $X$. Represent $g(n) = t_n \gamma_n$ so that $\gamma_n \in \Gamma$ and $t_n \in Q$, $n \in \mathbb{Z}$. Since $Z$ is normal, $\gamma_n Z = Z$ for all $n$, so that $g(n)Z = t_n \gamma_n Z = t_n Z$, $n \in \mathbb{Z}$. We have $g(n)Y = t_n \gamma_n Y$, $n \in \mathbb{Z}$, and since $Q$ is compact, we only have to show that $\gamma_n Y \rightarrow Z$ in uniform density.

Let $Q'$ be the fundamental cube of $X/Z$ and let $\tau : Q \rightarrow Q'$ be the natural projection. Since the sequence $(g(n)Z)$ is well distributed in $X$, the sequence $(\tau(t_n))$ is well distributed in $Q'$, which means that for any measurable subset $U$ of $Q'$ whose boundary is a null-set, $D(\{n \in \mathbb{Z} : \tau(t_n) \in U\}) = \mu_{Q'}(U)$.

Let $V$ be a subnilmanifold of $Z$, connected, of codimension 1 in $Z$, and with $1_X \in V$; based on Proposition 1.1, we only need to show that the set $\{n \in \mathbb{Z} : \gamma_n Y \subseteq V\}$ has zero uniform density. Let $K$ be the identity component of $\pi^{-1}(V)$; we have $\gamma_n H^o \gamma_n^{-1} \subseteq L$ for all $n \in \mathbb{Z}$, and have to prove that the set $S = \{n \in \mathbb{Z} : \gamma_n H^o \gamma_n^{-1} \subseteq K\}$ has zero uniform density.

Since $K$ is a proper subgroup of $L$, there exists $b \in G$ such that $bHb^{-1} \not\subseteq K$. By assumption, $G$ is generated by $G^o$ and $g$. The group $G^o$ is generated by $Q$, thus $tHt^{-1} \not\subseteq K$ for some $t \in Q$ or $g(n)Hg(n)^{-1} \not\subseteq K$ for some $n \in \mathbb{Z}$. So, there exists $a \in H$ such that $t^a t^{-1} \not\subseteq K$ for some $t \in Q$ or $g(n)ag(n)^{-1} \not\subseteq K$ for some $n \in \mathbb{Z}$. Let $S' = \{n \in \mathbb{Z} : \gamma_n a \gamma_n^{-1} \subseteq K\}$; since $S \subseteq S'$, it suffices to show that $D(S') = 0$. (This would not be a problem if $\gamma_n$ were a polynomial sequence, but it is not.)

Consider the mapping $\eta(n, t) = t^{-1}g(n)ag(n)t$ from $\mathbb{Z}^m \times G^o$ to $L$; this is a polynomial mapping. Let $\chi$ be a homomorphism $L \rightarrow \mathbb{R}$ such that $K = \{\chi = 0\}$. Let $\theta = \chi \eta$; then $\theta$ is a polynomial, and it is shown above that $\theta \neq 0$. Since $K$ has codimension 1 in $L$, it contains $[L, L]$, and so, is normal in $L$; hence, for any $s \in L$ we have $\theta(n, ts) = \chi(s^{-1}t^{-1}g(n)ag(n)^{-1}ts) = \chi(t^{-1}g(n)ag(n)^{-1}t) = \theta(n, t)$ for all $t \in G^o$, $n \in \mathbb{Z}$. Thus, $\theta$ is defined on $\mathbb{Z} \times (G^o/L)$: there exists a polynomial $\theta'$ on $\mathbb{Z} \times (G^o/L)$ such that $\theta(n, t) = \theta'(n, \tau(t))$, $t \in G^o$, $n \in \mathbb{Z}$. Let $P$ be the restriction of $\theta'$ to $\mathbb{Z} \times Q'$. Now, $n \in S'$ iff
\(\gamma_n a_n^{-1} = t_n^{-1} g(n) a(n)^{-1} t_n \in K, \text{ iff } \theta(n, t_n) = 0, \text{ iff } P(n, \tau(t_n)) = 0.\)

Write \(P\) in coordinates on \(Q', P(n, u) = \sum_{\alpha \in A} q_\alpha(n) u^\alpha, n \in \mathbb{Z}, u \in Q'\), where \(A\) is a set of multiindices and for each \(\alpha \in A, q_\alpha(n)\) is a polynomial in \(n\). We want to show that the set of zeroes of the polynomials \(P_\alpha(u) = P(n, u) \text{ in } Q'\) “converges”, as \(n \to \infty\), to a set of zero measure. Let \(d = \max\{\deg q_\alpha, \alpha \in A\}\). Then for any \(\alpha \in A\), a finite limit \(b_\alpha = \lim_{n \to \infty} n^{-d} q_\alpha(n)\) exists, and is nonzero for some \(\alpha\). Thus, as \(n \to \infty\), the polynomials \(n^{-d} P_\alpha(u)\) converge uniformly on \(Q'\) to the nonzero polynomial \(p(u) = \sum_{\alpha \in A} b_\alpha u^\alpha\). The set \(N = \{u \in Q' : p(u) = 0\}\) has zero measure. Given \(\varepsilon > 0\), find \(\delta > 0\) such that the set \(N_{\delta} = \{u \in Q' : |p(u)| < \delta\}\) has measure \(< \varepsilon\). Let \(n_0\) be such that \(|P(n, u) - p(u)| < \delta\) on \(Q'\) for \(|n| > n_0\); then for \(|n| > n_0\) the set \(D_n = \{u \in Q' : P(n, u) = 0\}\) is contained in \(N_{\delta}\). The sequence \(u_n = \tau(t_n), n \in \mathbb{Z},\) is well distributed in \(Q'\) and the boundary of \(N_{\delta}\) is a null-set, so \(D = \{n \in \mathbb{Z} : u_n \in N_{\delta}\} = \mu Q'(N_{\delta}) < \varepsilon\). Now,\n
\[S' = \{n \in \mathbb{Z} : P(n, u_n) = 0\} \subseteq \{n \in \mathbb{Z} : u_n \in D\} \subseteq \{-n_0, \ldots, n_0\} \cup \{n \in \mathbb{Z} : u_n \in N_{\delta}\},\]

thus \(D(S') < \varepsilon\). Hence, \(D(S') = 0\). ■

**Corollary 2.2.** Let \(X\) be connected, let \(Y\) be a connected subnilmanifold of \(X\), let \(g\) be a polynomials sequence in \(G\), let \(g(\mathbb{Z})Y\) be dense in \(X\), and let \(f \in C(X)\). There exists a factor-nilmanifold \(\hat{X}\) of \(X\), a point \(\hat{x} \in \hat{X}\), and a function \(\hat{f} \in C(\hat{X})\) such that \(\int_{g(n)Y} f d\mu_{g(n)Y} - \hat{f}(g(n)\hat{x}) \rightarrow 0\) in uniform density.

**Proof.** We may assume that \(g(0) = 1_G\), that \(G\) is generated by \(G^\circ\) and the elements of \(g\), and that \(Y \supseteq 1_X\). Let \(Z\) be the normal closure of \(Y\) in \(X\), then \(\int_{g(n)Y} f d\mu_{g(n)Y} - \int_{g(n)Z} f d\mu_{g(n)Z} \rightarrow 0\) in uniform density. Let \(\hat{X} = X/Z, \hat{x} = \{Z\} \in \hat{X}\), and \(\hat{f} = E(f|\hat{X}) \in C(\hat{X})\); then \(\int_{g(n)Y} f d\mu_{g(n)Y} - \int_{g(n)Z} f d\mu_{g(n)Z} \rightarrow 0\) in uniform density, and \(\int_{g(n)Z} f d\mu_{g(n)Z} = \hat{f}(g(n)\hat{x})\) for all \(n\). ■

We now involve nilsequences into our consideration. Recall that a basic \(d\)-step nilsequence is a sequence of the form \(\psi(n) = h(R^n w), \text{ where } w \text{ is a point of a } d\text{-step nilmanifold } M, R \text{ is a translation on } M, \text{ and } h \in C(M)\). We find it worthy to expand this notion. Given a polynomial sequence \(g(n) = a_1^{p_1(n)} \cdots a_r^{p_r(n)}\) in a nilpotent group with \(\deg p_i \leq s\) for all \(i\), we will say that \(g\) has naive degree \(\leq s\). (The term “degree” had already been reserved for another parameter of a polynomial sequence.) Let us call a sequence of the form \(\psi(n) = h(g(n)w)\), where \(w\) is a point of a \(d\)-step nilmanifold \(M = J/\Lambda, g\) is a polynomial sequence of naive degree \(\leq s\) in \(J\), and \(h \in C(M)\), a basic polynomial \(d\)-step nilsequence of degree \(\leq s\). Actually, any basic polynomial nilsequence is a basic nilsequence, as the following proposition says; the reason why we introduce this notion is that we do not want to loose the valuable information about the way a nilsequence was produced.

**Proposition 2.3.** (See [L1], Proposition 3.14) Any basic polynomial \(d\)-step nilsequence of degree \(\leq s\) is a \(ds\)-step basic nilsequence.

Clearly, basic polynomial \(d\)-step nilsequences of degree \(\leq s\) form an algebra; we will also need the following fact:
Lemma 2.4. Let \( \psi_0, \ldots, \psi_{m-1} \) be basic polynomial \( d \)-step nilsequences of degree \( \leq s \). Then the sequence \( (\ldots, \psi_0(0), \ldots, \psi_{m-1}(0), \psi_0(1), \ldots, \psi_{m-1}(1), \psi_0(2), \ldots, \psi_{m-1}(2), \ldots) \) is also a basic polynomial \( d \)-step nilsequence of degree \( \leq s \).

Proof. For each \( i = 0, \ldots, m - 1 \), let \( M_i = J_i / \Lambda_i \) be the \( d \)-step nilmanifold, \( g_i \) be the polynomial sequence in \( J_i \), \( w_i \in M_i \) is the point, and \( h_i \in C(M_i) \) be the function such that \( \psi_i(n) = h(g_i(n)w_i), n \in \mathbb{Z} \). If, for some \( i \), \( J_i \) is not connected, it is a factor-group of a free \( d \)-step nilpotent group with continuous and discrete generators, which, in its turn, is a subgroup of a free \( d \)-step nilpotent group with only continuous generators (see [L1]); thus after replacing, if needed, \( M_i \) by a larger nilmanifold and extending \( h_i \) to a continuous function on this nilmanifold we may assume that every \( J_i \) is connected. In this case for any element \( b \in J_i \) and any \( r \in \mathbb{N} \) a \( r \)-th root \( b^{1/r} \) exists in \( J_i \), and thus the polynomial sequence \( b^{p(n)} \) in \( J_i \) makes sense even if a polynomial \( p \) has non-integer rational coefficients. Thus, for each \( i \), we may construct a polynomial sequence \( g'_i \) in \( J_i \), of the same naive degree as \( g_i \), such that \( g'_i(mm + i) = g_i(n) \) for all \( n \in \mathbb{Z} \). Put \( M = \mathbb{Z}_m \times \prod_{i=0}^{m-1} M_i \), \( g = (1, g'_0, \ldots, g'_{m-1}), w = (w_0, w_1, \ldots, w_{m-1}) \in M \), and \( h(i, v_0, \ldots, v_{m-1}) = h_i(v_i), (i, v_0, \ldots, v_{m-1}) \in M \). Then \( M \) is a \( d \)-step nilmanifold, \( h \in C(M) \), and the basic polynomial nilsequence \( \psi(n) = h(g(n)w) = h_i(g'_i(n)w_i) = h_i(g_i(k)w_i) = \psi_i(k) \) whenever \( n = km + i, i = 0, 1, \ldots, m - 1 \).

We now get:

Theorem 2.5. Let \( X = G / \Gamma \) be a \( d \)-step nilmanifold, let \( Y \) be a subnilmanifold of \( X \), let \( g \) be a polynomial sequence in \( G \) of naive degree \( \leq s \), let \( f \in C(X) \), and let \( \varphi(n) = \int_{g(n)Y} f \, d\mu_{g(n)}Y \), \( n \in \mathbb{Z} \). There exists a basic polynomial \( d \)-step nilsequence \( \psi \) of degree \( \leq s \) such that \( \varphi(n) - \psi(n) \longrightarrow 0 \) in uniform density.

Proof. If both \( Y \) and \( \overline{g(\mathbb{Z})Y} \) are connected (in which case \( \overline{g(\mathbb{Z})Y} \) is a nilmanifold), the assertion follows from Corollary 2.2.

Now assume that \( Y \) is connected but \( \overline{g(\mathbb{Z})Y} \) is not. Then, by Theorem B in [L1], there exists \( m \in \mathbb{N} \) such that \( \overline{g((m\mathbb{Z} + j)Y)} \) is connected for every \( i = 0, \ldots, m - 1 \). Thus, for every \( i = 0, \ldots, m - 1 \), there exists a basic polynomial \( d \)-step nilsequence \( \psi_i \) of degree \( \leq s \) such that \( \varphi(mn + i) - \psi_i(n) \longrightarrow 0 \) in uniform density, and the assertion follows from Lemma 2.4.

Finally, if \( Y \) is disconnected and \( Y_1, \ldots, Y_l \) are the connected components of \( Y \), then

\[
\int_{g(n)Y} f \, d\mu_{g(n)}Y = \sum_{i=1}^{l} \int_{g(n)Y_i} f \, d\mu_{g(n)}Y_i, \quad n \in \mathbb{Z},
\]

and the result holds since it holds for \( Y_1, \ldots, Y_l \).

3. Multiple polynomial correlation sequences and nilsequences

Now let \((W, B, \mu)\) be a probability measure space and let \( T \) be an ergodic invertible measure preserving transformation of \( W \). Let \( p_1, \ldots, p_k \) be polynomials taking on integer values on the integers. Let \( A_1, \ldots, A_k \in B \) and let \( \varphi(n) = \mu(T^{p_1(n)}A_1 \cap \cdots \cap T^{p_k(n)}A_k), n \in \mathbb{Z} \); or, more generally, let \( h_1, \ldots, h_k \in L^\infty(W) \) and \( \varphi(n) = \int_W T^{p_1(n)}h_1 \cdots T^{p_k(n)}h_k \, d\mu, n \in \mathbb{Z} \). Using results from [HK2] it can be shown (see the argument in [BHK], Corollary 4.5) that, given \( \varepsilon > 0 \), there exist a \( d \)-step nilsystem \((X, a), X = G / \Gamma, a \in G \), and functions \( f_1, \ldots, f_k \in L^\infty(X) \) such that, for \( \phi(n) = \int_X a^{p_1(n)}f_1 \cdots a^{p_k(n)}f_k \, d\mu_X, \mathcal{D}\{n \in \mathbb{Z} : \)
Proof. We copy the proof of Theorem 1.9 in [BHK]. For each density, and polynomial \(d\) nilsequence \(\psi\) of on the integers, and let \(\phi\) of degree \(Y\) the diagonal subnilmanifold \(c\) minimal integer that tends to 0 in uniform density. 

\[ \text{We believe that Theorem 3.1 remains true without the assumption that} \]

\[ \text{Remark.} \]

\[ \text{We now switch to the multiparameter case, that is, to the situation where} \]

\[ \text{4. The multiparameter case} \]
polynomials of \( m \geq 1 \) integer variables. We say that a mapping \( g : \mathbb{Z}^m \to G \) is an \((m\text{-parameter})\) polynomial sequence in \( G \) if \( g(n) = a_{1}^{p_{1}(n)} \ldots a_{r}^{p_{r}(n)} \), where \( a_{1}, \ldots, a_{r} \in G \) and \( p_{1}, \ldots, p_{r} \) are polynomials \( \mathbb{Z}^m \to \mathbb{Z} \). It is shown in [L2] that, if \( g \) is an \( m\text{-parameter} \) polynomial sequence in \( G \) and \( Y \) is a connected submanifold of \( X \), then the closure of the sequence \( g(n)Y \), \( X' = \bigcup_{n \in \mathbb{Z}^m} g(n)Y \), is a disjoint finite union of sub-nilmanifolds of \( X \), and, if \( X' \) is a single sub-nilmanifold, the sequence \( g(n)Y \) is well distributed in \( X' \). (That is, for every \( f \in C(X') \) and any \( \text{Følner sequence} \) \( \left( \Phi_{N} \right) \) in \( \mathbb{Z}^m \),

\[
\lim_{N \to \infty} \frac{1}{|\Phi_{N}|} \sum_{n \in \Phi_{N}} f g(n)Y = \int_{X'} f d\mu_{X'}.
\]

For a subset \( S \subseteq \mathbb{Z}^m \), we define the density \( d(S) \) of \( S \) by \( d(S) = \lim_{N \to \infty} \frac{|S \cap [-N,N]^m|}{(2N)^m} \), if it exists, and say that a sequence \( \left( \omega_{n} \right)_{n \in \mathbb{Z}^m} \) of a topological space \( \Omega \) converges to \( \omega \in \Omega \) in density if for every neighborhood \( U \) of \( \omega \),

\[
d\left( \{ n \in \mathbb{Z}^m : \omega_{n} \notin U \} \right) = 0.
\]

For the case of multparameter sequences we get a result similar to Proposition 2.1, but weaker since the “ordinary” density instead of the uniform density \( D \) appears in it:

**Proposition 4.1.** Let \( X = G/\Gamma \) be a connected nilmanifold and let \( Y = \pi(H) \) be a connected subnilmanifold of \( X \), where \( H \) is a connected closed subgroup of \( G \). Let \( g : \mathbb{Z}^m \to G \) be a polynomial sequence with \( g(0) = 1_{G} \) such that \( g(\mathbb{Z}^m)Y \) is dense in \( X \), and assume that \( G \) is generated by \( G^{o} \) and the elements of \( g \). Let \( Z \) be the normal closure of \( Y \) in \( X \); then \( g(n)Y - g(n)Z \to 0 \) in density.

**Proof.** The beginning of the proof is the same as for Proposition 2.1, but we will repeat it. Let \( L \) be the identity component of \( \pi^{-1}(Z) \). Choose Malcev coordinates in \( G^{o} \) compatible with \( L \), and let \( Q \) be the corresponding fundamental cube in \( G^{o} \). \( Q \) is compact, and is as well compact with respect to the uniform norm when elements of \( G \) are interpreted as transformations of \( X \). Represent \( g(n) = t_{n}\gamma_{n} \) so that \( \gamma_{n} \in \Gamma \) and \( t_{n} \in Q \), \( n \in \mathbb{Z}^m \). Since \( Z \) is normal, \( \gamma_{n}Z = Z \) for all \( n \), so that \( g(n)Z = t_{n}\gamma_{n}Z = t_{n}Z \), \( n \in \mathbb{Z}^m \). We have \( g(n)Y = t_{n}\gamma_{n}Y \), \( n \in \mathbb{Z}^m \), and since \( Q \) is compact, we only have to show that \( \gamma_{n}Y \to Z \) in density. Let \( Q' \) be the fundamental cube of \( X/Z \) and let \( \tau : Q \to Q' \) be the natural projection. Since the sequence \( (g(n)Z) \) is well distributed in \( X \), the sequence \( (\tau(t_{n})) \) is well distributed in \( Q' \).

Let \( V \) be a subnilmanifold of \( Z \), connected, of codimension 1 in \( Z \), and with \( 1_{X} \in V \); based on Proposition 1.1, we only need to show that the set \( \{ n \in \mathbb{Z}^m : \gamma_{n}Y \subseteq V \} \) has zero density. Let \( K \) be the identity component of \( \pi^{-1}(V) \); we have \( \gamma_{n}H\gamma_{n}^{-1} \subseteq L \) for all \( n \in \mathbb{Z}^m \), and have to prove that the set \( S = \{ n \in \mathbb{Z}^m : \gamma_{n}H\gamma_{n}^{-1} \subseteq K \} \) has zero density.

Since \( K \) is a proper subgroup of \( L \) and \( L \) is the normal closure of \( H \) in \( G \) there exists \( b \in G \) such that \( bHb^{-1} \notin K \). By assumption, \( G \) is generated by \( G^{o} \) and \( g \). The group \( G^{o} \) is generated by \( Q \), thus \( tHt^{-1} \notin K \) for some \( t \in Q \) or \( g(n)Hg(n)^{-1} \notin K \) for some \( n \in \mathbb{Z}^m \). So, there exists \( a \in H \) such that \( ata^{-1} \notin K \) for some \( t \in Q \) or \( g(n)ag(n)^{-1} \notin K \) for some \( n \in \mathbb{Z}^m \). Let \( S' = \{ n \in \mathbb{Z}^m : \gamma_{n}a\gamma_{n}^{-1} \subseteq K \} \); since \( S \subseteq S' \), it suffices to show that \( d(S') = 0 \).

Consider the mapping \( \eta(n,t) = t^{-1}g(n)ag(n)^{-1}t \) from \( \mathbb{Z}^m \times G^{o} \) to \( L \); this is a polynomial mapping. Let \( \chi \) be a homomorphism \( L \to \mathbb{R} \) such that \( K = \{ \chi = 0 \} \). Let \( \theta = \chi \eta \); then \( \theta \) is a polynomial, and it is shown above that \( \theta \neq 0 \). Since \( K \) is normal in \( L \), for any \( s \in L \) we have \( \theta(n,ts) = \chi(-s^{-1}t^{-1}g(n)ag(n)^{-1}ts) = \chi(t^{-1}g(n)ag(n)^{-1}t) = \theta(n,t) \) for all \( t \in G^{o} \), \( n \in \mathbb{Z}^m \). Thus, \( \theta \) is defined on \( \mathbb{Z}^m \times (G^{o}/L) \); there exists a polynomial \( \theta' \) on
$\mathbb{Z}^m \times (G^o/L)$ such that $\theta(n, t) = \theta'(n, \tau(t))$, $t \in G^o$, $n \in \mathbb{Z}^m$. Let $P$ be the restriction of $\theta'$ to $\mathbb{Z}^m \times Q'$. Now, $n \in S'$ iff $\gamma_n \alpha_\gamma^{-1} = t_n^{-1}g(n)ag(n)^{-1}t_n \in K$, iff $\theta(n, t_n) = 0$, iff $P(n, \tau(t_n)) = 0$.

Extend $P$ to a polynomial on $\mathbb{R}^m \times Q'$. Write $P$ in coordinates: $P(w, u) = \sum_{\alpha \in A} q_\alpha(w)u^\alpha$, where $A$ is a set of multiindices and for each $\alpha \in A$, $q_\alpha$ is a polynomial on $\mathbb{R}^m$. Let $d = \max\{\deg q_\alpha, \alpha \in A\}$. For each $\alpha \in A$, let $q_\alpha^m$ be the homogeneous part of $q_\alpha$ of degree $d$. Let $S$ be the sphere $\{x \in \mathbb{R}^m : |x| = 1\}$ and let $\Xi = \{\xi \in S : q_\alpha^m(\xi) \neq 0 \text{ for some } \alpha \in A\}$. For every $\xi \in S$ and $\alpha \in A$, $\lim_{s \to \infty} s^{-d}q_\alpha(s\xi) = q_\alpha^m(\xi)$, thus the polynomials $P(s\xi, u) = s^{-d}P(s\xi, u)$ converge as $s \to \infty$ to the polynomial $P_\xi(u) = \sum_{\alpha \in A} q_\alpha^m(\xi)u^\alpha$ uniformly on $S \times Q'$. (Example: for $P((w_1, w_2), (u_1, u_2)) = (w_3^2 + w_4w_1^2u_1^2 + w_2w_1u_2w_2)$ we have $P_\xi(u_1, u_2) = w_3^2u_1^2 + 2w_1w_2u_2$, $\xi = (w_1, w_2) \in S$, and $\Xi = \{\xi \in S : P_\xi \neq 0\} = \{(u_1, u_2) \in S : w_1 \neq 0\}$.)

Fix $\varepsilon > 0$. For $\xi \in \Xi$, let $N_\xi = \{u \in Q' : P_\xi(u) = 0\}$ and let $\delta_\xi > 0$ be such that the set $N_{\xi, \delta_\xi} = \{u \in Q' : |p_\xi(u)| < \delta_\xi\}$ has measure $< \varepsilon$. Let $U_{\xi} \subset \Xi$ be an open neighborhood of $\xi$ such that $|P_\xi(u) - P_\xi(\xi)| < \delta_\xi/2$ for all $\xi \in U_{\xi}$ and $u \in Q'$. Let $s_\xi > 0$ be such that $|s^{-d}P_s(s\xi, u) - P_\xi(u)| < \delta_\xi/2$ for all $s > s_\xi$, $\xi \in U_{\xi}$, and $u \in Q'$. Then for any $s > s_\xi$ and $\xi \in U_{\xi}$, $u \in Q' : P(s\xi, u) = 0 \subset N_{\xi, \delta_\xi}$. Since the sequence $u_n = \tau(t_n), n \in \mathbb{Z}^m$, is well distributed in $Q'$, for every $\xi \in \Xi$ there exists $M_\xi \in \mathbb{N}$ such that for any $M > M_\xi$ and any $v \in \mathbb{R}^m$, $\left\lfloor \frac{1}{M^m} \right\rfloor \left\{n \in v + [1, M]^m : u_n \in N_{\xi, \delta_\xi}\right\} < 2\varepsilon$. If $v \in \mathbb{R}^m$ and $M \in \mathbb{N}$ are such that $|v| > s_\xi + \sqrt{m}M$ and $v + [1, M]^m \subset \mathbb{R}_+U_{\xi}$, then for any $w \in v + [1, M]^m$ we have $\{u \in Q' : P(w, u) = 0\} \subset N_{\xi, \delta_\xi}$. Thus, for such $v$ and $M$, $\left\lfloor \frac{1}{M^m} \right\rfloor \left\{n \in v + [1, M]^m : P(n, u_n) = 0\right\} < 2\varepsilon$, and hence, $\left\lfloor \frac{1}{M^m} \right\rfloor |S' \cap (v + [1, M]^m)| < 2\varepsilon$. Let $E = \Sigma \setminus \Xi$ is a proper algebraic subvariety of $\Sigma$, therefore there exists a compact set $D \subset \Xi$ such that $d(\mathbb{R}_+D \cap \mathbb{Z}^m) > 1 - \varepsilon$. (Indeed, $E$ can be represented as a finite union of smooth submanifolds of $\Sigma$ of dimension $\leq m - 2$, thus it can be covered by a finite union $\mathcal{E}$ of open balls with $\sigma(\mathcal{E}) < \varepsilon\sigma(\Sigma)$, where $\sigma$ is the standard $(m - 1)$-dimensional volume on $\Sigma$. For such a set $\mathcal{E}$ we have $d(\mathbb{R}_+\mathcal{E} \cap \mathbb{Z}^m) = \sigma(\mathcal{E})/\sigma(\Sigma) < \varepsilon$, and for $D = \Sigma \setminus \mathcal{E}$ we have $d(\mathbb{R}_+D \cap \mathbb{Z}^m) > 1 - \varepsilon$.) Let $\xi_1, \ldots, \xi_l$ be such that $\bigcup_{j=1}^l U_{\xi_j} \supset D$ and let $s = \max_{1 \leq j \leq l} s_{\xi_j}$, $M = \max_{1 \leq j \leq l} M_{\xi_j}$. Let $r > s + \sqrt{m}M$ be such that for any cube $C = v + [1, M]^m \subset \mathbb{R}_+D$ with $|v| > r$ we have $C \subset \mathbb{R}_+U_{\xi_j}$ for some $j$. Then for any such cube $C$ we have $\frac{1}{|C|} |S' \cap C| < 2\varepsilon$. Thus, $d(S') < 3\varepsilon$. Hence, $d(S') = 0$.}

**Remark.** The proof of Proposition 4.1 gives more information about the set $S = \{n \in \mathbb{Z}^m : |\int g(n) f - \int g(n) Z f| > \varepsilon\}$ than just the fact that $S$ has zero density. Actually, the uniform density of $S$ is zero, if we ignore a small set $\mathcal{E}$ of “bad” directions in $\mathbb{R}^m$; indeed, $S$ has uniform density 0 in $\mathbb{R}_+(\Sigma \setminus \mathcal{E}) \cap \mathbb{Z}^m$, whereas $\sigma(\mathcal{E}) < \varepsilon\sigma(\Sigma)$.

We say that a mapping $\psi: \mathbb{Z}^m \longrightarrow \mathbb{C}$ is a basic polynomial $d$-step $m$-parameter nilsequence of degree $\leq s$ if there exist a $d$-step nilmanifold $M = J/\Lambda$, a polynomial mapping $g: \mathbb{Z}^m \longrightarrow J$ of naive degree $\leq s$, a function $h \in C(M)$, and a point $w \in M$ such that $\psi(n) = h(g(n)w), n \in \mathbb{Z}^m$, and we will say that an $m$-parameter numerical sequence is a polynomial $d$-step nilsequence of degree $\leq s$ if it is a uniform limit of basic polynomial $d$-step $m$-parameter nilsequences of degree $\leq s$. The definitions and facts related to one-parameter polynomial sequences and nilsequences are translated almost literally to the
multiparameter case; one only has to use results from [L2] and [L3] instead of the corresponding results from [L1] and [HK2]. (In particular, any (basic) polynomial $m$-parameter nilsequence is a (basic) $m$-parameter nilsequence; see the proof of Theorem B* in [L2].) In the same way as we got Theorems 2.5 and 3.1, we now obtain:

**Theorem 4.2.** Let $X = G/\Gamma$ be a $d$-step nilmanifold, let $Y$ be a subnilmanifold of $X$, let $g: \mathbb{Z}^m \to G$ be a polynomials sequence of naive degree $\leq s$, let $f \in C(X)$, let $\varphi(n) = \int_{g(n)Y} f \, d\mu_{g(n)Y}$, $n \in \mathbb{Z}^m$. There exists a basic polynomial $d$-step $m$-parameter nilsequence $\psi$ of degree $\leq s$ such that $\varphi(n) - \psi(n) \to 0$ in density.

**Theorem 4.3.** Let $(W, B, \mu, T)$ be an ergodic invertible measure preserving system with $\mu(W) < \infty$, let $h_1, \ldots, h_k \in L^\infty(W)$, let $p_1, \ldots, p_k$ be polynomials $\mathbb{Z}^m \to \mathbb{Z}$, and let $\varphi(n) = \int_W T^{p_1(n)} h_1 \cdot \ldots \cdot T^{p_k(n)} h_k \, d\mu$, $n \in \mathbb{Z}^m$. Let the complexity of $\{p_1, \ldots, p_k\}$ be $c$ and let $s = \max_i (\deg p_i)$; then there exists a $(c+1)$-step $m$-parameter polynomial nilsequence $\psi$ of degree $\leq s$ such that $\varphi(n) - \psi(n) \to 0$ in density.

## 5. Appendix

We will show here how Proposition 1.1 can be derived from Green-Tao’s result in [GT].

We first need to introduce some terminology from [GT]. Let $G$ be a connected nilpotent Lie group with a discrete cocompact subgroup $\Gamma$, and let $X = G/\Gamma$.

A filtration $G_\bullet$ on $G$ is a finite decreasing sequence of subgroups $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d \supseteq G_{d+1} = \{1_G\}$ with the property that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j$.

For a sequence $g: \mathbb{Z} \to G$, “the derivative” $\partial g$ is defined by $(\partial g)(n) = g(n)^{-1} g(n+1)$, $n \in \mathbb{Z}$. Given a filtration $G_\bullet = (G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d)$ on $G$, $\text{poly}(\mathbb{Z}, G_\bullet)$ denotes the group of polynomial sequences $g$ in $G$ with the property that, for each $i = 1, \ldots, d$, $\partial^i g$ takes values in $G_i$.

Given a filtration $G_\bullet = (G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d)$ on $G$, a Malcev basis $\mathcal{M}$ adapted to this filtration can be constructed (which means that for any $i$, $\mathcal{M} \cap G_i$ is a basis in $G_i$), and this basis naturally defines a locally Euclidean metric $\rho$ on $X$.

A (horizontal) character on $X$ is a mapping $\chi: X \to \mathbb{R}/\mathbb{Z}$ induced by a character on the torus $T = [G, G]\setminus X$ (or equivalently, by a continuous homomorphism $G \to \mathbb{R}/\mathbb{Z}$ trivial on $\Gamma$). A Malcev basis in $G$ defines coordinates $(t_1, \ldots, t_l)$ on $T$, and in these coordinates any character $\chi$ on $X$ has the form $m_1 t_1 + \ldots + m_l t_l$, $(t_1, \ldots, t_l) \in T$, with $m_1, \ldots, m_l \in \mathbb{Z}$; the modulus $|\chi|$ of $\chi$ is defined by $|\chi| = |m_1| + \ldots + |m_l|$.

Given $\delta > 0$, a finite sequence $(x_1, \ldots, x_N)$ is said to be $\delta$-equidistributed in $X$ if $|\frac{1}{N} \sum_{n=1}^N f(x_n) - \int_X f \, d\mu_X| < \delta \|f\|_{\text{Lip}}$ for any Lipschitz function $f$ on $X$, where $\|f\|_{\text{Lip}} = \sup|f| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}$.

The following theorem was obtained in [GT]:

**Theorem 5.1.** ([GT] Theorem 1.16) Let $G_\bullet$ be a filtration on $G$ and let $g \in \text{poly}(\mathbb{Z}, G_\bullet)$. There exist constants $C$ and $c$, which only depend on $X$, such that for any $\delta > 0$ small enough and any $N \in \mathbb{N}$, either the sequence $(g(n))_{n=1}^N$ is $\delta$-equidistributed in $X$, or there is a nontrivial character $\chi$ on $X$ with $|\chi| < C\delta^{-c}$ such that $|\chi(g(n)) - \chi(g(n-1))| < C\delta^{-c}/N$. 

11
for all \( n \in \{1, \ldots, N\} \).

(In this theorem and below, the “either ... or ...” expression should be understood in the “inclusive” sense, that is, that both possibilities may also occur simultaneously.)

(We skipped some details; in particular, there is also a condition on the Malcev basis chosen in \( G \) and so, on the metric on \( X \); this condition is satisfied if \( \delta \) is small enough.)

We do not need much from this very strong “quantitative” theorem. Let \( X \) be connected but \( G \) not necessarily connected; represent \( X \) as \( X = G^o/(\Gamma \cap G^o) \). Define the filtrations \( G_\bullet = \{G_1 \supseteq G_2 \supseteq \ldots\} \) on \( G \) and \( G_\bullet^o = \{G_1^o \supseteq G_2^o \supseteq \ldots\} \) on \( G^o \) by \( G_1 = G_1^o \), \( G_i = [G_{i-1}, G] \) for \( i \geq 2 \), and \( G_i^o = G_i \cap G^o \). Let \( f \in C(X) \), and let \( \varepsilon > 0 \). Choose a Lipschitz function \( h \) on \( X \) with \( |h - f| < \varepsilon/3 \). Choose \( \delta > 0 \) small enough to satisfy Theorem 5.1 and such that \( \delta \|f\|_{\text{Lip}} < \varepsilon/3 \). Let \( \chi_1, \ldots, \chi_r \) be the nontrivial characters on \( X \) satisfying \( |\chi_i| < C\delta^{-c} \). Then for any \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \) and \( N \in \mathbb{N} \), either there exists \( i \) such that \( |\chi_i(g(n)1_X) - \chi_i(g(n-1)1_X)| < C\delta^{-c}/N \) for all \( n = 1, \ldots, N \), or \( \frac{1}{N} \sum_{n=1}^{N} h(g(n)1_X) - \int_X h \, d\mu_X \) \( < \delta \|f\|_{\text{Lip}} \), and then \( \frac{1}{N} \sum_{n=1}^{N} f(g(n)1_X) - \int_X f \, d\mu_X \) \( < \varepsilon \). Sending \( N \) to infinity, we get that either \( \chi_i(g(n)1_X) \equiv 1 \) for some \( i \), or \( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(g(n)1_X) - \int_X f \, d\mu_X \right| \leq \varepsilon \).

Now let \( Y \) be a connected subnilmanifold of \( X \) with \( 1_X \in Y \). Choose an element \( a \in G \) such that the sequence \( (a^n1_X)_{n \in \mathbb{N}} \) is dense in \( Y \). Choose \( \gamma \in \Gamma \) such that \( \gamma a^{-1} \in G^o \). (Such \( \gamma \) exists since \( X = G/\Gamma \) is connected.) Put \( g(n) = a^n\gamma^{-n} \), \( n \in \mathbb{N} \); then \( g(n)1_X = a^n1_X \) for all \( n \), and since \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \) and \( g(n) \in G^o \) for all \( n \), we have \( g \in \text{poly}(\mathbb{Z}, G_\bullet^o) \). Let \( \chi_1, \ldots, \chi_r \) be as above, let \( V_i' = \{x \in X : \chi_i(x) = 0\} \), \( i = 1, \ldots, r \), and for each \( i \), let \( V_i \) be the connected component of the nilmanifold \( V_i' \) that contains \( 1_X \). We have that either \( \chi_i(a^n1_X) \equiv 1 \) for some \( i \), or \( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(a^n1_X) - \int_X f \, d\mu_X \right| \leq \varepsilon \). In the first case, \( Y \subseteq V_i' \), and so, \( Y \subseteq V_i \); in the second case, since \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(a^n1_X) = \int_Y f \, d\mu_Y \) by \([\text{L1}]\) (or by one more application of Theorem 5.1), we get that \( \left| \int_Y f \, d\mu_Y - \int_X f \, d\mu_X \right| \leq \varepsilon \). We obtain

**Corollary (Proposition 1.1).** Let \( X \) be a connected nilmanifold. For any \( f \in C(X) \) and any \( \varepsilon > 0 \) there are subnilmanifolds \( V_1, \ldots, V_r \) of \( X \), connected, of codimension 1, and containing \( 1_X \), such that for any connected subnilmanifold \( Y \) of \( X \) with \( 1_X \in Y \), either \( Y \in V_i \) for some \( i \in \{1, \ldots, r\} \), or \( \left| \int_Y f \, d\mu_Y - \int_X f \, d\mu_X \right| < \varepsilon \).

**Acknowledgment.** I thank Vitaly Bergelson for corrections and good advice. I also thank an anonymous referee and Dan Rudolph, the editor, for corrections and help in preparing this paper.

**Bibliography**


[GT] B. Green and T. Tao, The quantitative behaviour of polynomial orbits on nilmanifolds,


