MATH 148 TEXTBOOK - 2010

ERIC CONRAD, DAVID GEORGE, KEVIN SCHOENECKER, DAN SHAPIRO

3.1 Functions

Functions and functional notation are used everywhere in modern mathematics. In this Chapter you will learn definitions, the use of functional notation, and several ways to describe mathematical functions. You will also learn to construct and interpret graphs of functions.

To introduce the idea of mathematical functions let's consider a couple of examples where one numerical quantity depends on another.

Example 1

Cousin Billy's height depends on his age. Aunt Rhody measured his height every year since he was a baby, and kept track of the results in a table. Here are a few entries.

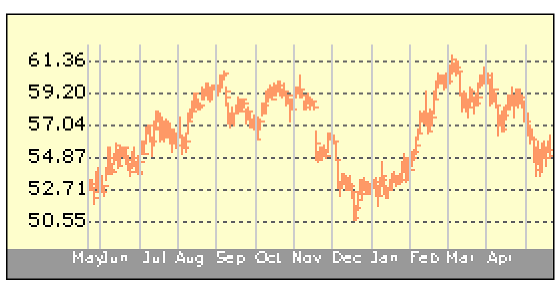
Age	Height
2	24
3	36
5	44
10	52
16	72

Example 2

A stone is dropped from the top of a tower. The distance it falls in t seconds equals $16t^2$ feet.

Example 3

The price of one share of stock in LMN Corporation went through some dramatic changes over a one year period. The trend was presented to stockholders as the following graph:



These examples are quite different but have basic features in common. Each has input values and corresponding output values. For instance, Billy's height (the output) corresponds to his age (the input). The rule is given by the information in Aunt Rhody's table. Moreover, for each input value there is no guesswork involved in determining the corresponding output. Every input determines exactly one output.

What are the input and output values in the other two cases?

These examples illustrate the idea of a mathematical function. Here is a formal definition (as required in math courses).

A *function* consists of:

A set of input values (that set is the *domain*).
A rule: for every input value this rule assigns exactly one output value.
A set of output values (that set is the *range*).

The function concept is often described using the analogy of a function "machine". You enter an input value, the machine operates on that input, and then it outputs a value. For each input value, it produces one and only one output value. A calculator is an example of a function machine. If an input number is entered and some key is pressed, the display provides the function value.

The rule used for a function can be given in various forms: a table, or an algebraic formula, or a graph, or in some other way. Examples of these three methods are seen in Examples 1, 2, 3.

Representations of Functions

Functional Notation (Formulas)

To discuss different functions we assign them names. Typical names for functions are letters like f, g, and h. Some special function names use more than one letter, like *sin* (the sine function) and *log* (the logarithm function). However, in this section, we will stick with simpler names. If c is in the domain of f we write the corresponding output value as $f(c)$. [This term is read "f of c".]

Here is an example. Suppose f is the function given by the algebraic formula $f(x) = 3x + 5$. Writing the function in this way tells us the rule to use to match the items in the domain with the items in the range. Here the "rule" is that we pair eachinput number x with the output number that is 5 more than 3 times itself. It also tells us that x is the independent variable, which is the input, and that $f(x)$ is the dependent variable, which is the output. The notation $f(x)$ is read as "f of x" and it is the value obtained by applying the rule to given value of x. Note: f(x) does NOT represent a multiplication. Table 1 illustrates this functional notation.

x: value from the domain	**3x + 5: rule**	**f (x): value from the range**
–1	$\xrightarrow{3(-1)+5}$	$f(-1) = 2$
0	$\xrightarrow{3(0)+5}$	$f(0) = 5$
1	$\xrightarrow{3(1)+5}$	$f(1) = 8$
2	$\xrightarrow{3(2)+5}$	$f(2) = 11$
3	$\xrightarrow{3(3)+5}$	$f(3) = 14$

Table 1

In Example 2, the formula there tells us that for input value t the corresponding output value is $16t^2$. If g is the name of that function, this can be written as:

$$g(t) = 16t^2.$$

Not every formula describes a function. For example, the formula $h(x) = x \pm 2$ does not define a function h. This is seen since giving one input value x can result in more than one output. For instance, if $x = 3$ then $h(3) = 3 \pm 2$ which equals either 5 or 1. Two different values of $h(x)$ correspond to that one value x.

Example 4 (Expressing Formulas as Functions)

I. Determine if the formula $u + v^2 = 9$ expresses u as a function of v.
II. Likewise, determine if this formula expresses v as a function of u.

Solution:

I. We can solve this equation for u to obtain: $u = 6 - v^2$. This expresses u as a function of v because for any input v the formula provides an explicit rule for computing one output u. We could call this function q, defining $q(v) = 6 - v^2$ and say that the relation is described by: $u = q(v)$.

II. However this same equation does NOT express v as a function of u. To see this let's try to solve for v. We find $v^2 = 9 - u$. Then if $9 - u \geq 0$ we obtain: $v = \pm\sqrt{9-u}$. Since one input u can lead to more than one output v this expression is not a function of u. For instance if the input is $u = 5$ then the output is $v = \pm\sqrt{4}$ which equals 2 or –2. ■

Mathematical functions are important and are used in every area of mathematics (although sometimes not by name). The use of functions helps unify and simplify many advanced concepts. The price for that unification is learning the terminology and notation for functions.

Some functions cannot be described by a simple algebraic formula. For instance, the function in Example 1 expressing Billy's height in terms of his age isn't given by a formula. Similarly, if you knew a simple formula for stock prices on given dates, you could make some serious profits.

Graphs and Tables (Ordered-pair)

Some functions can be described in English sentences, others are easily represented using algebraic formulas. A function f can also be represented as a set of ordered-pairs (x, y), where x is the independent variable and $y = f(x)$ is the dependent variable. Displaying that information in a table can provide useful information even though only a few values are listed. Alternatively that list of ordered pairs can be displayed geometrically by plotting each pair (x, y) as a point in the coordinate plane. We plot all the pairs (x, y) where x lies in the domain and $y = f(x)$ is the corresponding value in the range. That is the graph of f.

Figure 1 contains a table and sketch of the graph of a quadratic function.

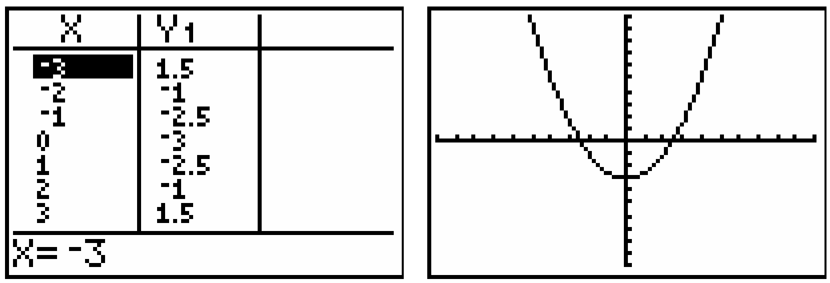


Figure 1

[For instance, the ordered-pair (2, –1) is a point on the graph.]

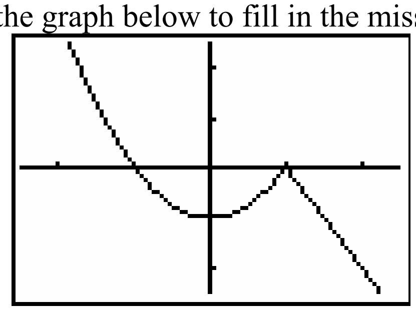
The table gives you very specific information about a few ordered pairs, while the graph provides less accurate information about a large number of ordered-pairs. Note that both of these representations display the function as a set of ordered-pairs.

The relationship between the table and the graph is useful, to say the least. When first learning the concept of graphing an equation, we are often instructed to construct a table of values and plot the points. The graph then takes shape by "connecting the dots." When calculators display graphs they behave in much the same way, connecting the dots (when you use "line" mode rather than "point" mode for graphs).

The opposite is also true. Given a graph you can construct a table of values (or at least, estimates of the values). Finding a table of values is often an intermediate step used when switching between the graph and the formula.

Example 5 (Finding Values Using the Graph)

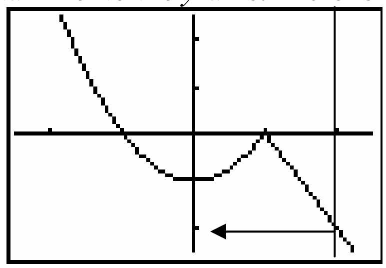
Use the graph below to fill in the missing table values.



x	$f(x)$
0	
2	
–1	
0.5	
	2
	0

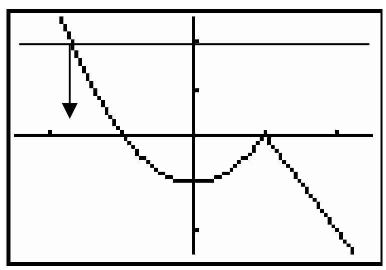
Solution:

To find the missing values for $f(x)$, we consider a vertical line through the given x-value and note where it intersects the graph. We then need to find the y-coordinate of that point. That is found using a horizontal line to the y-axis. Here is a diagram for when $x = 2$.



The x value 2 corresponds to the y value –2, so we know that $f(2) = -2$.

To find the missing x-values we draw a horizontal line through the given functional value on the y-axis and note where it intersects the graph. We then need to find the x-coordinate of that point. That's done by using a vertical line to the x-axis. Here is a diagram for when $f(x) = 2$.



We can not read off the exact value of x here, but it is approximately –1.7. The missing values are listed in the table below. That is, $f(-1.7) \approx 2$. (That symbol $\approx$ means approximately equal.)

x	***f*(*x*)**
0	**–1**
2	**–2**
–1	**0**
0.5	**–0.75**
≈ –1.7	2
1 & –1	0

Note that there are two x-values that produce a functional value 0. Is this consistent with the definition of a function? ■

We mentioned four different techniques for describing a function: (1) an English sentence, (2) an algebraic formula, (3) a table of values, and (4) a graph.

Each of these representations can be useful, and the different methods can provide different insights into the behavior of a given function.

Domain and Range

Suppose numbers x and y are related by some rule enabling us to express y as a function of x. If f is the name of that function then that rule becomes: $y = f(x)$. Recall.

The **domain** of f is the set of valid input values.
The **range** of f is the set of possible output values.

Finding Domain and Range when given a Graph

If the graph of a function is given, how can we read off the domain and range? We first note that not every picture is the graph of a function, and then we'll consider domains and ranges.

Suppose that you are given a picture in the plane and want to know whether it represents the graph of a function. Since the infinitely long number line cannot be easily drawn on paper we can never get the whole graph of a function defined on the set of real numbers. Instead we try to draw a large enough part of the graph so that the interesting features are displayed and the omitted part of the graph has no surprises, in other words, a "complete graph". Of course it takes experience and practice to know whether all the interesting bits have been included in a given graph picture.

In order for a picture in the coordinate plane to represent a function, it must satisfy the definition of a function:

Each number in the domain is paired with one and only one number from the range.

The values in the domain are represented by the independent variable x, so they are found along the x-axis. Similarly, the values for the range are represented by the dependent variable and are located along the y-axis.

In order for a graph (a set of points in the coordinate plane) to represent a function, each valid input x must correspond to exactly one output y. That is: a given x-coordinate cannot have two corresponding y-coordinates. Geometrically this says: A vertical line through any input value on the x axis cannot meet the graph in two or more points. This is the "vertical line test":

A graph represents a function if every vertical line intersects that picture in at most one point.

Figure 2 displays a graph representing a function and one that does not. The graph on the left passes the vertical line test but the one on the right has many places where it fails the vertical line test.

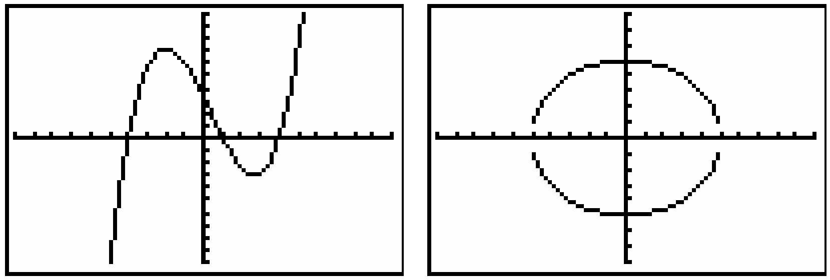


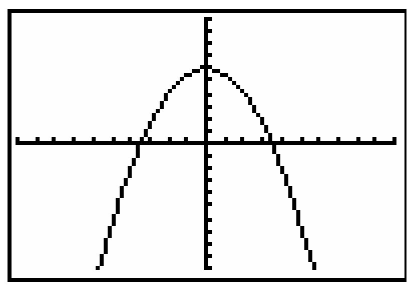
Figure 2

Once we know that the graph represents a function then we can also read off the domain and the range directly from the graph. The graph of a function f is the set of all ordered pairs (x, y) where $y = f(x)$. The set of all of the x-coordinates of points on the graph is the domain, and the set of all of the y-coordinates is the range. Then the domain is the set of points along the x-axis that lie above or below points on the graph. Similarly, the range is the set of all the points along the y-axis that lie to the left or right of points on the graph.

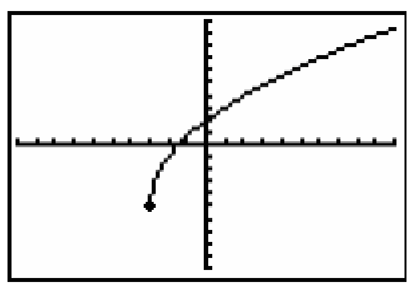
Example 6 (Finding Domain and Range from the Graph)

Find the domain and range of the functions represented by the following pictures of a "complete" graph in the standard viewing window.

I.



II.

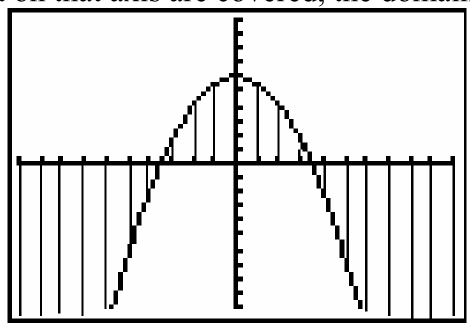


Solution:

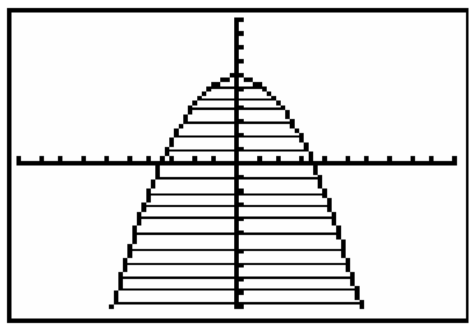
First we check that these are graphs of functions: they do satisfy the vertical line test. To find the domain, we look along the x-axis. To find the range, we must look along the y-axis. Here are some pictures illustrating the process. (Compare the process used in Example 5.)

I. This is a parabola that opens downward with vertex at (0, 6). How will this help us find the range?

Domain: Draw vertical lines from points on the graph to the x-axis. Since every point on that axis are covered, the domain is $(-\infty, \infty)$.

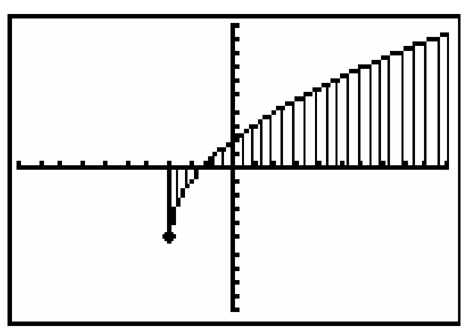


Range: Drawing horizontal lines from points on the graph to the y-axis results in all the points y with $y \le 6$. Then the range is the interval $(-\infty, 6]$. ■

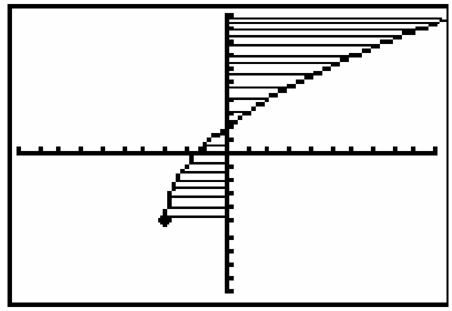


II. Here the picture looks like a shifter version of the graph of the square root function.

Domain: Vertical lines drawn from the graph to the x-axis show that the domain is the interval $[-3, \infty)$. (How do we know to include -3?)



Range: Horizontal lines drawn from the graph to the y-axis show that the range is the interval $[-5, \infty)$. ■



Remember, that a function consists of three parts, the domain, range, and a rule that pairs them together. Given the graph you can identify whether it is a function, you can determine the domain and range, and in simple cases you might be able to find the rule. For instance suppose the given graph is a straight line (which isn't vertical). If you find two points on the line, you can compute the slope and use the point-slope form to find the equation of the line. That expression for y provides the formula for the function.

Finding Domain when given a Formula

The domain and range of a function can be easily found if a complete list of values (x, y) is available. The first entries (x values) make up the domain, and the second entries (y values) make up the range. However a complete list is not usually available, especially when we write functions as formulas. In those cases we can often do some algebra to determine the domain and range.

Example 7 (Finding the Domain using the Formula)

I. Find the domain of the function $p(s) = \sqrt{s-8}$.

<u>**Solution:**</u>

Well $p(8) = 0$ and $p(11) = \sqrt{3}$, but $p(5)$ is not defined (as a real number). In order for the formula for $p(s)$ to make sense we require $s - 8 \geq 0$, so that $s \geq 8$. [Recall that only non-negative numbers have square roots.] This says that the domain of this function p is the set of all valid input values s, which is the interval $[8, \infty)$. ■

II. Find the domain of the function $h(b) = \dfrac{12}{b^2 - 4}$.

<u>**Solution:**</u>

To determine the domain find all input values which fail to work. The only values for b causing a problem here are the ones making the denominator zero. Then the domain is the set of all b except for those where $b^2 - 4 = 0$. Solving that equation we find $b^2 = 4$ or: $b = 2$ or -2. Then the domain of h is the set of all numbers b except for 2 and -2. In interval notation, we have that the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.■

III. Find the domain of the function $f(x) = \dfrac{x+2}{\sqrt{3x-7}}$.

Solution:
To find this domain, remember two important properties of the real numbers:

- Division by 0 is undefined
- The square root of a number makes sense only when the number is ≥ 0.

Because of these two properties, we need to be sure that we are using values of x with $3x - 7 > 0$. Solving this gives use that $x > \dfrac{7}{3}$ so the domain is the interval $\left(\dfrac{7}{3}, \infty\right)$. ∎

When we write functions using formulas, the domain and range are often not specified, even though they are two of the three pieces needed to define a function. This is one reason why it is important to be able to find them, either graphically or algebraically.

- **Rule:** When a function is described by a formula, its domain is assumed to be the set of all input numbers where that formula makes sense.

Note:
1. Using algebra to find the domain is often more accurate than inspecting the graph. For instance, graph any of the equations from Example 7 on the graphing calculator and see if the picture accurately displays the domain.
2. It is often harder to find the range of a function which is given by an algebraic formula. We won't discuss such examples here.

Piecewise-Defined Functions

Some functions are defined by giving algebraic rules for different pieces of the domain. The rule of the function is defined based on the value for x. Because of this, these graphs can have sharp corners, and may have gaps between different pieces. Using a more mathematical description, these functions whose graphs are disconnected are "discontinuous" for some of the values in the domain.

Example 8 (Piecewise-Defined Functions)

I. $f(x) = \begin{cases} 1 & \text{if } x \in (-\infty, 0) \\ -1 & \text{if } x \in [0, \infty) \end{cases}$

a. Find the domain of the function.
b. Find $f(-2), f(0), f(2)$
c. Graph the function.

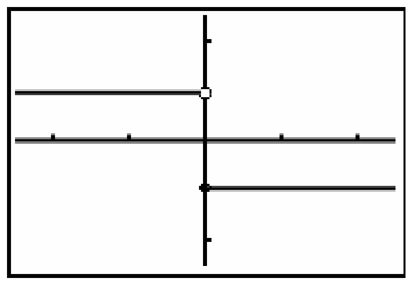
Solution:

1. The formulas make sense for the input values x given by those cases: for any x in one of the two intervals $(-\infty, 0)$ and $[0, \infty)$. Then the domain for this function is the union of these two sets, which is the set $(-\infty, \infty)$ of all real numbers.

2. To find $f(-2)$, we want $f(x)$ when $x = -2$. That x value lies in $(-\infty, 0)$ and the definition of f tells us which rule to use: Since $-2 \in (-\infty, 0)$ we find $f(-2) = 1$.

   To find $f(0)$, note that 0 lies in $[0, \infty)$, so the rule says: $f(0) = -1$. Similarly, $2 \in [0, \infty)$, so that $f(2) = -1$.

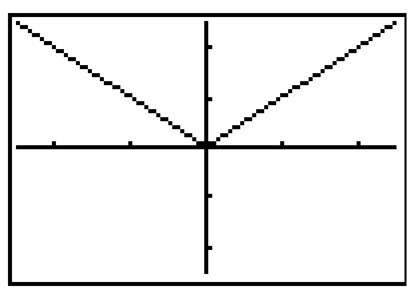
3. This graph will have two separate pieces to it. The first is the piece along the x-axis from $(-\infty, 0)$, which will be the graph of $f(x) = 1$. The second is the piece along the x-axis from $[0, \infty)$, which will be the graph of $f(x) = -1$. A picture of the graph is sketched below. ■



II. Graph the function $f(x) = \begin{cases} -x \text{ if } x \in (-\infty, 0) \\ x \quad \text{if } x \in [0, \infty) \end{cases}$

Solution:

Thispiecewise-defined function is defined for every value of x. Notice that this function behaves like $f(x) = -x$ on the interval $(-\infty, 0)$ and that it behaves like $f(x) = x$ on the interval $[0, \infty)$. . Here is a picture of the graph. ■



That graph should look familiar: it seems similar to the absolute value function. How can you check that this function g is identical with the absolute value function: $g(x) = |x|$ for every number x ?

Let's think for a moment about those functions that have a finite domain (and hence a finite range). Having a finite domain means that, in theory, you could count all of the items in domain (of course if the domain is large you may not live long enough to do so). Since the domain is finite, every ordered pair of the function can be listed. This is impossible for a function with an infinite domain since the number of ordered pairs would also be infinite.

Since you can represent such a "finite" function completely using a list of ordered pairs, you don't need to know an algebraic "rule" for this function: just look through the ordered pairs until you find the correct one. These functions can be viewed as piecewise-defined function in which each number in the domain has a unique "rule" for matching it with the number in the range.

The graph of such a "finite" function is just a collection of disconnected points. There are no line segments connecting the points together. Functions with finite domains are examples of "discrete functions".

Example 9 (Discrete Functions)

Let the relation h be defined by the set of ordered-pairs:

$$\{(1, 2), (2, 7), (3, 4), (7, 2)\}$$

1. Is h a function?
2. Find the domain and range.
3. Find $h(1)$ and $h(3)$.
4. Represent h using piecewise-defined notation.
5. Graph the function.

Solution:

1. Yes, h is a function. Why?
2. The domain of the function h is the set $\{1, 2, 3, 7\}$. The range is the set $\{2, 4, 7\}$.
3. $h(1) = 2$ and $h(3) = 4$.
4. $h(x) = \begin{cases} 2 & if \;\; x = 1, 7 \\ 4 & if \;\; x = 3 \\ 7 & if \;\; x = 2 \end{cases}$. Notice that the two of the values (1 and 7) in the domain have the same "rule" that you would use to match it with a value in the range.
5. The graph of this function is the points (1, 2), (2, 7), (3, 4) and (7, 2). ■

In mathematics, we can't function without functions. The notion of a function is crucial to the understanding of advanced mathematics. There are many different types of functions and many different ways to represent them, but all functions must satisfy the defining condition: Each number in the domain is paired with one and only one number in the range.

Exercises 3.1

1. State the definition of a function and explain the purpose of each of the three parts.
2. Write a brief statement about why and how we can use the "Vertical Line Test" in determining whether a graph is not a function. What are its limitations in showing us that a graph is a function?

For problems 3 – 6, state whether the domain and range are the correct ones for the given "rule". If they aren't correct, describe why they fail.

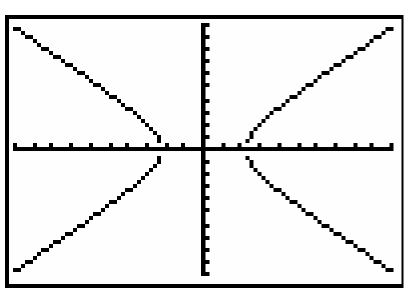
3. Rule: $f(x) = 2x + 3$ Domain: $(-\infty, \infty)$ Range: $[0, \infty)$
4. Rule: $f(x) = \sqrt{5x}$ Domain: $[0, \infty)$ Range: $(-\infty, \infty)$
5. Rule: $f(x) = \sqrt{x^2}$ Domain: $(-\infty, \infty)$ Range: $[0, \infty)$
6. Rule: $f(x) = \dfrac{x}{x^2 - 4}$ Domain: $(-\infty, 2) \cup (2, \infty)$ Range: $(-\infty, \infty)$

For problems 7 – 12, solve the formula for each of the variables and in each case state whether it defines a function. (See Example 4.) If it is a function, state the domain.

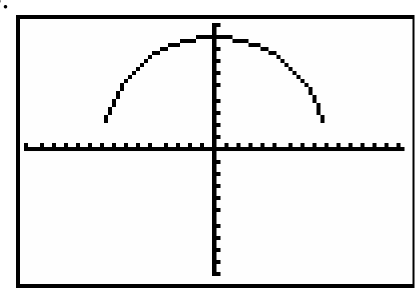
7. $\frac{1}{4}r - s = 5$
8. $3x + 2xy = y$
9. $y - 9 = \sqrt{4x}$
10. $75 = \dfrac{.5x - 10}{12y}$, $y \neq 0$
11. $A = \dfrac{4}{3}\pi r^3$
12. $40 = \pi rh + \pi r^2$

For problems 13 & 14, a picture of a complete graph is given. Determine whether the picture is a graph of a function.

13.

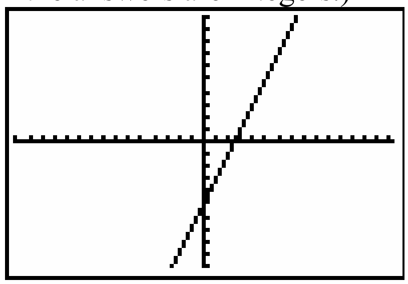


14.



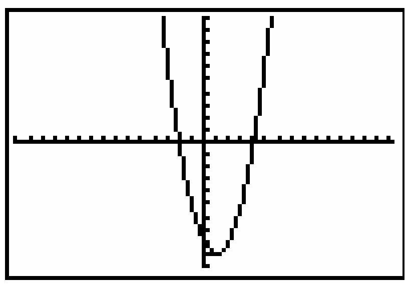
For problems 15 – 17, use the graph of the function f (in the square window) to answer the questions.

15. (All the answers are integers.)



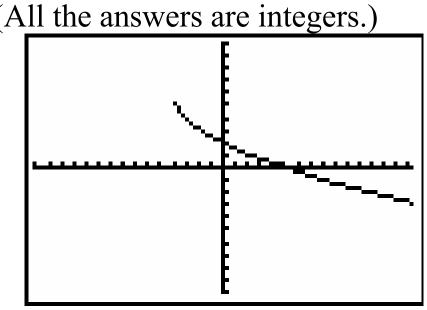
i. Find the Domain of f.
ii. Find the Range of f.
iii. Make a table and enter the information for the rows $f(2)$, $f(4)$, and $f(0)$.
iv. Find the "Rule" for f.
v. Which domain values produce negative range values?

16.



i. Find the Domain of f.
ii. Find the Range of f.
iii. Find all x such that $f(x) = 0$. Find when $f(x) = 4$.
iv. Find all x such that $f(x) < 0$.

17. (All the answers are integers.)



i. Find the Domain of f.
ii. Find the Range of f.
iii. Find $f(0)$ and $f(5)$.
iv. Find all x such that $f(x) > 0$.

For problems 18 – 27, determine whether the following statements about the functions *h* and *g* are true. Please explain your reasoning.

$$h(x) = \frac{x^2+7}{x^2-13} \qquad g(x) = \frac{x^2-7}{\sqrt{x^2+3}}$$

18. $h(2) = \frac{-11}{9}$
19. $g(1) = -3$
20. $\left(4, \frac{43}{3}\right)$ is on the graph of *h*.
21. For some value of *x*, $h(x) = 0$.
22. For some value of *x*, $g(x) = 0$.
23. The domain of *h* is $(-\infty, \infty)$.
24. The domain of *g* is $(-\infty, \infty)$.
25. The range of *h* is $(-\infty, \infty)$.
26. The range of *g* is $(-\infty, \infty)$.
27. For some value of *x*, $h(x) = g(x)$.

For problems 28 – 31, use the given piecewise-defined function to answer the questions.

28. $f(x) = \begin{cases} 2x+3 & \text{if } x \le 0 \\ -0.5x+3 & \text{if } x > 0 \end{cases}$

i. Find the domain of *f*.
ii. Graph the function (by hand) and find the range of *f*.
iii. If possible, fill in the missing table values.

x	***f*(*x*)**
–5	
0	
4	
	4
	0

29. $f(x) = \begin{cases} 2x-1 & \text{if } x \le -3 \\ x^2 & \text{if } x \ge 4 \end{cases}$

i. Find the domain of *f*.
ii. Graph the function and find the range of *f*.
iii. If possible, fill in the missing table values.

x	***f*(*x*)**
–5	
0	
4	
	–21
	0

30. $f(x) = \begin{cases} -4x^2 + 8 & if\ x \le 0 \\ 4x^2 - 8 & if\ x > 0 \end{cases}$

i. Find the domain of f.
ii. Graph the function and find the range of f.
iii. If possible, fill in the missing table values.

x	$f(x)$
-2	
0	
2	
	0
	4

31. Let h be the function defined by the set of ordered-pairs as follows:
$\{(1, 3), (2, 5), (3, 7), (4, -2), (10, 5), (-6, 7)\}$

i. Find the domain of h.
ii. Graph the function and find the range of h.
iii. Write the function as a piecewise-defined function.

For questions 32 – 34, use the function T given by the Federal tax table below.

Taxable Income (x)		**Amount of Tax Owed (y)**
Is Over	**But Not Over**	**Tax formula**
0	7,000	$.10(x)$
7,000	28,400	$700 + .15(x - 7{,}000)$
28,400	68,800	$3{,}910 + .25(x - 28{,}400)$
68,800	143,500	$14{,}010 + .28\ (x - 68{,}800)$
143,500	311,950	$34{,}926 + .33\ (x - 143{,}500)$
311,950	–	$90{,}514.50 + .35(x - 311{,}950)$

32. Write T as a piecewise-defined function.
33. Find $T(1000)$, $T(24596.95)$, $T(456791)$
34. For each of the amounts above, find the percentage of taxable income that is paid in tax.

35. In the following statement, is y a function of x? Is x a function of y?
"y is the product of the square of x and 25"

36. To rent a car for the weekend from the Rent-A-Wreck company, you pay a flat fee of \$25. You are also charged for mileage using the following formula:
\$0.25 per mile if you travel less than 100 miles
\$5 plus \$0.20 per mile if you travel 100 miles or more

i. Write the price that you pay as a function of the miles that you drive.
ii. How much do you pay if you travel 30 miles? 130 miles?
iii. How many miles can you travel if you only have \$75 to spend on a car rental?

3.2 Significance and Uncertain Numbers

3.2.1 The Art of Estimating

In physical sciences and in other applications, it is necessary to do calculations with uncertain data, such as measurements and statistics obtained from physical observations. The results of these calculations will carry uncertainty. A full treatment requires tools such as statistics and calculus, but we can give a few basic guidelines for making these estimates. Some variations on our treatment are mentioned in the footnotes and in the exercises.

Contents

3.2.2 Uncertainty in Applications

In applications of mathematics to real-world problems, small errors are often introduced by the act of measurement. Consider the following problem:

> **(Room Area Problem - First Attempt)** The floor of a rectangular room is 5.261 meters long and 3.741 meters wide. Determine its area.

If the room were perfectly rectangular and the measurements of the length and width were exact, then we could say that the area of the floor is just the length times the width:

$$(5.261 \text{ m})(3.741 \text{ m}) = 19.681401 \text{ m}^2.$$

> **Units of Measure:** The symbol "m^2" stands for the square meter, the standard unit of area in the Système Internationale (SI, commonly known as the metric system). Since a meter is about 3.28 feet, a square meter is about 10.8 square feet.

But the measurements probably have errors. Some of these may be purely physical, for example, rulers and tape measures expand slightly in higher temperatures. Some may involve problems with the way the the measuring instrument is used, such as allowing too much slack in a tape measure – or stretching the tape too much. Other errors arise because of the precision of the measuring instrument. For example, my metric tape measure has marks 1 millimeter apart, so it cannot easily be used for measurements much more precise than that. And of course the room is probably not a perfect rectangle.

We might need to have a good estimate of the area of our room in order to apply a polyurethane finish to the floor. If the polyurethane is expensive, overestimating the room size could be costly. On the other hand, if we underestimate the room size, we will not have enough finishing fluid – we might incur extra costs if the job needs to be redone or if contractors need to be called back for another day's work.

So about how reliable is our estimate of 19.681401 square meters for the area of our floor? Do all those decimal places really provide useful information?

> **(Room Area Problem - Refinement)** After taking many independent measurements of the room, I find the length is between 5.15 m and 5.38 m, and the width is between 3.61 m and 3.82 m. What are the smallest and largest possible areas for the room?

Using the measurements provided:

$$A_{\text{min}} = \text{Minimum Area} = (5.15 \text{ m})(3.61 \text{ m}) = 18.5915 \text{ m}^2$$
$$A_{\text{max}} = \text{Maximum Area} = (5.38 \text{ m})(3.82 \text{ m}) = 20.5516 \text{ m}^2$$

It seems silly to provide an answer like 19.681401 square meters when the correct value could be anywhere from roughly 18.6 to 20.6 square meters. Don't give six extra decimal places when answer is only reliable to the nearest square meter.

In addition, the measurements of the length and the width in the refinement of the problem suggest that our measurments are still too precise. How can we approach the problem to reduce the workload and come up with an answer which is as precise as the data allows, but no more? The given measurements for length and width suggest that these measurements are only reliable to about one tenth of a meter. If the width could be anywhere between 5.152 and 5.377 meters, then it is misleading to say that the width is 5.261 meters. It would be better to say that the width is about 5.25 plus or minus about 0.1 meters. The dimensions and the area of our rectangle all seem to be reliable to only about two digits.

These considerations give us a new statement of our room area problem and an outline of its solution:

(Room Area Problem) The floor of a rectangular room is approximately 5.3 meters long by 3.7 meters wide. Find a good estimate for its area.

Calculations: (5.3 m)(3.7 m) = 19.61 m^2 = about $2\underline{0}$ m^2.

Answer: $\boxed{2\underline{0} \text{ square meters}}$. (The underscored zero will be explained in the next section.)

There are two technical questions that need to be answered:

- How do we decide to round from 19.61 square meters to 20 square meters instead of something else? For example, why not round to 19.6 square meters?
- How do we interpret the answer of 20 square meters?

Fortunately both questions can be answered with very little additional calculation. We will refer back to this problem several times in the next section.

3.2.3 Reporting Uncertain Numbers

3.2.3.1 Intervals of Uncertainty

When reporting data, it is important to know how precise the data are. In our room area problem, we were originally given that the length of the room was 5.261 meters. After making additional measurements, we found that the length of the floor was between 5.152 meters and 5.377 meters. Perhaps the room wasn't quite rectangular, or perhaps the floorboard made it hard to get precise measurements – it would appear our measurements are only precise to about a tenth of a meter. The interval (5.152 m, 5.377 m) could be taken as a rough indication of how precise the room length measurements are. An interval of this sort is called an **interval of uncertainty**.

For example, in measuring the depth of the deep end of a swimming pool, we might find that measurements fall between 12.4 and 12.7 feet. Here the interval of uncertainty is (12.4 ft, 12.7 ft).

We can write down an uncertainty interval in another way using the "±" symbol. Instead of saying that the measured depth is between 12.4 and 12.7 feet, we might say that the measured depth at the deep end is 12.55 ± 0.15 feet.

Definition: When reporting uncertain data:

$$x = c \pm r \qquad \text{means} \qquad c - r < x < c + r.$$

One way of putting this into words is to say that measurements x that fall outside the interval $(c - r, c + r)$ are **significantly** in error. Here the value c is the **center** of the interval and the value r is the **radius** of the interval.[1] Plus or minus the radius ($\pm r$) of the uncertainty interval is called the **uncertainty**. Differences between values inside the uncertainty interval are treated as **insignificant**.

> **Problem:** The power consumption of a handheld electronic game is between 11.2 Watts and 11.7 Watts. Express this as uncertainty interval in the form $P \pm r$.
> **Solution:** The uncertainty interval is (11.2 W, 11.7 W). The center of the interval or the average power consumption is:
>
> $$P = \frac{11.2 + 11.7}{2} = 11.45 \text{ W}.$$
>
> The radius or uncertainty is half the width of the interval:
>
> $$r = \frac{11.7 - 11.2}{2} = 2.5 \text{ W}.$$
>
> So power consumption is:
>
> $$P \pm r = \boxed{11.45 \pm 2.5 \text{ W}}.$$

(The interval is called an uncertainty interval because, although we are reasonably certain that our data point lies somewhere in the interval, we are not certain where in the interval the data point lies. We usually assume that it is more likely to be near the center of the interval than near the ends.[2])

3.2.3.2 Significant Digits.

One system that works reasonably well and saves a considerable amount of writing is based on the way we write numbers in the base 10 (or decimal) system. For instance 36,500 has three significant positions – the most significant is the ten thousands position (digit 3) and the least significant position is the hundreds position (digit 5). More generally, if an interval of uncertainty is not given, we adopt the following guidelines:

[1] The radius is half of the width or the diameter of the interval.

[2] A different approach to uncertainties in data is the *confidence interval* in statistics. A confidence interval is associated with a stated *level of confidence* that a given data point lies in the interval. With an interval of uncertainty, we are not given a level of confidence.

- The nonzero digits of any number are always significant. (But some of the the zeros may be significant.)
- The **most significant position** (MSP) in a nonzero base 10 number is the position with first or leftmost nonzero digit. For example, in the numbers 4860 and 0.005210, the MSP's are thousands and thousandths, respectively. When we want to indicate the most significant position, we will use an overscore, for example $\overline{4}860$ or $0.00\overline{5}210$.
- For an inexact number, the **least significant position** (LSP) is found as follows:
  - If the number is written with a decimal point, the LSP is the position of the rightmost digit, whether zero or not.
  - If the number is written without a decimal point, the LSP is the position of the rightmost *nonzero* digit.

In the number 4860, the zero is insignificant, so the LSP is the tens position (the digit 6). For 0.005210, the zero in the last position is significant, so the LSP is the millionths position. When we want to indicate the least significant position, we will use an underscore, for example, $48\underline{6}0$ or $0.00521\underline{0}$.

The numbers 4860 ($\overline{4}8\underline{6}0$) has three significant digits (the 4, 8 and 6 in the thousands, hundreds and tens positions). The number 0.005210 ($0.00\overline{5}21\underline{0}$) has four significant digits (from left to right: 5, 2, 1 and 0). The **number of significant digits** is the the number of positions counting from left to right from the most significant position (overscored) to the least significant position (underscored).

Then the inexact number 980 indicates a value between about 970 and 990, that is about 980±10 while the inexact number 976 indicates a value of about 976±1. But what about a value like 900±1? That shouldn't be written as 900 since that suggests a value ranging between 800 and 1000. Instead we could write 900±1 in scientific notation as 9.00×10^2.

The inexact values 100, 1.0×10^2, 1.00×10^2 and 1.000×10^2 have 1, 2, 3, and 4 significant digits, respectively. All of these are "about 100", with varying degrees of uncertainty. (The last of these, 1.000×10^2, could also be written as 100.0.) We can also use the underscore to change the least significant position.

In our room area problem, we wrote our answer as $2\underline{0}$ square meters – the underscored zero tells us that the digit 0 is significant – the answer has two significant digits. In the expression $1.2\underline{3}4$, the underscore indicates that the 4 in the thousandths position is insignificant.

inexact number	most significant position	least significant position	significant digits	approximate uncertainty	interval of uncertainty
1200	10^3	10^2	2	±100	1200 ± 100
$1\underline{2}30$	10^3	10^2	2	±100	1230 ± 100
$12\underline{0}0$	10^3	10^1	3	±10	1200 ± 10
1230	10^3	10^1	3	±10	1230 ± 10
1030	10^3	10^1	3	±10	1030 ± 10
1	10^0	10^0	1	±1	1 ± 1
1.0	10^0	10^{-1}	2	±0.1	1.0 ± 0.1
1.2340	10^0	10^{-4}	5	±0.0001	1.2340 ± 0.0001
0.0340	10^{-2}	10^{-4}	3	±0.0001	0.0340 ± 0.0001
−0.0340	10^{-2}	10^{-4}	3	±0.0001	−0.0340 ± 0.0001

Table 1: Significant digits and uncertainty intervals

Significant digits for negative numbers are handled just as for positive numbers. So -4860 amperes represents an electric current between about -4870 and about -4850 amperes. Paradoxically, because the uncertain number 0 has no nonzero digits, it has no significant positions. (We can use an underscore to mark a least sigificant position if we need to do that.)

To get an uncertainty interval based on significant digits, we added a value of ± 1 in the least significant position. So a measured mass of 2.040 grams indicates a mass of about 2.040 ± 0.001 grams.[3] Table 1 illustrates these ideas with a few examples.

In the room area problem in the previous section, the dimensions of the room were obtained by measurement and given as 5.3 by 3.7 meters. Since we have no additional information about uncertainty, we apply the guidelines above. Using $\pm$ notation, the dimensions are 5.3 ± 0.1 by 3.7 ± 0.1 meters.

The floor area of the room (about 20 square meters) was obtained by calculation rather than by direct measurement so we have additional information about the interval of uncertainty. (The additional information comes from the measured dimensions of the room.) Using guidelines for multiplication which follow, the area has two significant digits. That suggests an uncertainty interval of about 19 to 21 square meters, in rough agreement with the interval of 18.6 square meters minimum to 20.6 square meters maximum obtained from the refinement of the original problem.

[3]Some writers prefer an uncertainty of half this value. A measurement of 2.040 grams would be treated as 2.040 ± 0.0005 grams instead of 2.040 ± 0.001 grams.

3.2.3.3 Significant Digits and Mathematical Operations

When inexact numbers are used in mathematical processes (like addition), we cannot use the same kinds of guidelines for estimating the uncertainty of the results. The following *guidelines* are commonly used in science courses:

1. When adding or subtracting a small collection of inexact numbers, the **least significant position** of the result is the **leftmost** of the least significant positions of the terms.

   Example: In the inexact sum $12.123459 + 2.3$, the tenths position is significant, but the hundredths position is not. The result has three significant figures. (It would therefore be more appropriate to write the sum as 14.4 rather than 14.423459.)

   Cautionary Example: In the inexact difference $1.22 - 1.2 = 0.\underline{0}\overline{2}$, where we are using the overscore to mark the most significant position of the result. Since the most significant position is to the right of the least significant position, the number of significant digits is undefined. The uncertainty interval contains both positive and negative real numbers.[4]

2. When multiplying or dividing a small collection of inexact numbers, the **number of significant digits** of the result is the **smallest** of the numbers of significant digits of the factors and divisors.

   Example: The inexact product 2.117×4.1 has two significant digits. Using a calculator, the product is 8.6797. But since the result has just two significant digits, it would be more appropriate to write the result as 8.7 (or perhaps $8.\underline{6}8$).

   Caution: Don't divide by an inexact number whose interval of uncertainty includes zero.

We have a few final words on the room area problem:

- In the room area problem, the dimensions of the floor were about 5.3 by 3.7 meters. Both length and width have 2 significant positions. The number of significant positions in the product (19.61 square meters) is the minimum of 2 (the number of significant digits in the length) and 2 (the number of significant digits) in the width. So the area has two significant positions. It can be rounded to 20 square meters with an uncertainty of ± 1 square meter. (But since we may be using this value in another calculation, *i.e.* the amount of floor finish, we might prefer to keep some extra precision and use the uncertainty interval 19.6 ± 1 square meters, that is between 18.6 and 20.6 square meters.)

- We can also use the data to estimate the uncertainty interval for the area of the floor with a little more work. Since the length is between 5.2 and 5.4 meters and the width is between 3.6 and 3.8 meters:

  $$A_{\text{min}} = (5.2\text{ m})(3.6\text{ m}) = 18.72\text{ m}^2\text{, and}$$
  $$A_{\text{max}} = (5.4\text{ m})(3.8\text{ m}) = 20.52\text{ m}^2.$$

  The area is between 18.7 and 20.5 square meters. This "squares well" with our estimated area of between 19 and 21 meters, and even better with our alternate estimate of between 18.6 and 20.6 meters.

Now let us consider a similar problem with a few complications:...

[4] An uncertain number of this sort presents serious problems when used in products and quotients. Since the number could be either positive or negative, the sign of the product or quotient is unknown. Since zero is in the interval of uncertainty, division by the number may be undefined.

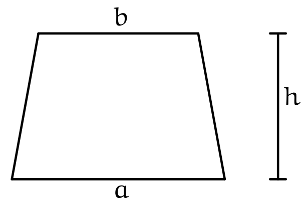


Figure 1: Gotham City Community Theatre stage

Problem: The stage of the Gotham City Community Theatre has a trapezoidal floor (see Figure 1) with dimensions $a = 18.2$ meters, $b = 15.1$ meters and $h = 14.3$ meters. The floor needs to be refinished with a special finish (Formula X) for an upcoming production. If one can of finish will cover $c = 0.43$ square meters of floor, then how much finish should be ordered?
(Do the calculations in two ways: (a) estimate the number of cans needed by using significant digits and (b) find the maximum number of cans of varnish that might be needed.)
Solution: The area of the floor is $A = h(a + b)/2$. Let x be the number of cans needed. Then x cans will cover $A = cx$ square meters of floor. Then $x = A/c$ or:

$$x = \frac{h(a+b)}{2c}$$

(Method a) The sum $a + b$ has three significant digits. The number 2 is exact, so the number of significant digits in x is the minimum of the number of significant digits in h, in $a + b$ and in c. Since h and $a + b$ both have three significant digits and c has two significant digits, the result has two significant digits.

$$x = \frac{14.3\ \text{m}(18.2\ \text{m} + 15.1\ \text{m})}{2(0.43\ \text{m}^2/\text{can})} = 553.7 = \text{ about 550 cans.}$$

Since the uncertainty interval seems to be about 550 ± 10 cans, to be safe, we should order 560 cans. (From the calculations, we expect to use about 554 cans, so we have an expected waste of six cans.[5])

Algebra Fact: To increase the size of a fraction $\frac{A}{B}$, we increase the size of the numerator A and decrease the size of the denominator B. To find the maximum value of $\frac{A}{B}$, we find $\frac{A_{max}}{B_{min}}$.
Caution: You have to do some tweaking if some of the numbers are negative.

(Method b) To obtain a better estimate of the maximum value, we need to use the maximum values of the lengths in the numerator and the minimum value of the coverage rate in the denominator. Since the number of cans must be an integer, fractional results should be rounded **upward**:

$$x_{max} = \frac{14.4\ \text{m}(18.3\ \text{m} + 15.2\ \text{m})}{2(0.42\ \text{m}^2/\text{can})} = 574.3 = \text{ about 575 cans.}$$

We would be less likely to run out of varnish with a larger order. However, since we only expect to use 554 cans, we expect to waste twenty-one cans.

There is a trade-off here. Since the estimate in part (a) is lower than the estimated maximum in part (b), there is some risk of running out of varnish. On the other hand, there is additional cost for ordering too much varnish. Evaluating this sort of trade-off is outside of the scope of this course.

[5] If we feel uneasy about rounding down, we might use an uncertainty interval of 554 ± 10 cans, making our order 564 cans with an expected waste of 10 cans.

• **Notation:** We often use a squiggly equals sign to express the idea "equals about". So instead of writing "$x =$ about 575 cans", we can save space by writing "$x \approx 575$ cans".

Powers of numbers are generally treated as repeated multiplications, but care must be taken. Large exponents, both positive and negative, do cause dramatic loss of significance: even very small errors in the value of the base can cause a loss of significance.[6]

Example: Suppose that $y = a^n$ where $a = 1.05 \pm 0.01$. We will look at errors when $n = 2$ and when $n = 100$.
(a) Suppose that n is exactly 2. Using our guidelines for uncertainty, 1.05^2 is about 1.10. What happens at the endpoints of the interval of uncertainty for a?

$$y_{min} = 1.04^2 = 1.0816, \qquad y_{max} = 1.06^2 = 1.1236.$$

Both values are slightly outside the estimated interval of uncertainty, with a small error in the least significant position. The uncertainty guidelines give a reasonable estimate of the error.
(b) Suppose that n is exactly 100. Since 100 is a large power, the guidelines don't apply. Rounded to three significant positions, 1.05^{100} is 132. What happens at the endpoints?

$$y_{min} = 1.04^{100} \approx 50.5, \qquad y_{max} = 1.06^{100} \approx 339.$$

Saying that the answer is between 50.5 and 339 is not being very informative. Both values are well outside the interval of uncertainty, with disagreement even in the **most** significant position. Even with more precision, the situation does not improve much. For example, for 1.04±0.005,

$$y_{min} = 1.045^{100} \approx 81.6, \quad y_{max} = 1.055^{100} \approx 211.$$

You can experiment with 1.04±0.001 by comparing 1.049^{100} and 1.051^{100} on your calculator.

3.2.3.4 Significance and Rounding

When carrying out calculations, it is best to use all available precision. In short, don't round intermediate results. However, when reporting results to others, it is important to convey the uncertainty in the calculated data. This can be done by rounding or by giving an uncertainty interval.

Sometimes the result of a calculation can serve as both a final result and an intermediate result. In this case, the unrounded value should be used in later calculations, but the rounded value or the uncertainty interval should be reported.

When we report an inexact number as a final result, we normally round to the least significant digit. For example:

$$2.117 \times 4.1 = 8.6797 \pm 0.1 \approx 8.7.$$

In applications such as engineering, handbooks are often used to to supply standard reference values for various quantities. Since these values can be used in many calculations, it is often desirable to retain additional digits, for example 8.68 ± 0.10 or $8.\underline{6}8$. As we saw in the room area problem, an extra digit of precision can provide a better center for the uncertainty interval.

[6] Better estimates of uncertainty for powers can be obtained using differentials, normally covered in a calculus course.

3.2.4 Approximate Equality

The symbol ≈ is often used to provide an estimate or an approximation of an exact value. An exact number is approximately equal to (≈) an inexact number if it lies in the uncertainty interval for the inexact number. For example:

$$\pi \approx 3.14, \quad \pi \approx 3.14159, \quad \sqrt{2} \approx 1.414, \quad \sqrt{2} \approx 1.415.$$

To see that $\sqrt{2} \approx 1.415$, we verify that:

$$1.414 < \sqrt{2} < 1.416.$$

Two inexact numbers are approximately equal if their uncertainty intervals overlap. For example:

$$3.14 \approx 3.14159, \qquad 1.414 \pm 0.002 \approx 1.416, \qquad 1.414 \approx 1.415.$$

On the other hand:

$$\pi \not\approx 3.17 \qquad \text{and} \qquad 1.414 \not\approx 1.416$$

Caution

Unlike equality, approximate equality is not transitive. For example:

$$3.14158 \approx 3.14159 \approx 3.14160.$$

But:

$$3.14158 \not\approx 3.14160.$$

3.2.5 Summary

Guidelines for Locating Significant Digits. In the absence of other information about an inexact decimal number:

- The nonzero digits are significant.
- The most significant position is the location of the leftmost nonzero digit.
- If the number has a decimal point, the least significant position is the location of the rightmost digit.
- If the number does not have a decimal point, the least significant position is the location of the rightmost nonzero digit.

Guidelines for Inexact Arithmetic.

- When adding or subtracting a small collection of inexact numbers, the least significant position of the result is *leftmost* least significant position of the terms being added or subtracted.
- When multiplying or dividing a small collection of inexact numbers, the number of significant digits in the result is smallest of the numbers of significant digits of the factors and divisors. Division is illegal if an inexact divisor contains zero in its uncertainty interval.
- Two inexact numbers are approximately equal if their intervals of uncertainty overlap. When two inexact numbers are approximately equal, their difference is "insignificant" (close to zero).

3.2.6 Exercises

1. Express the given uncertainty intervals using ± notation.
   (a) $(2.1, 2.4)$, (b) 1.781, (c) between 2 and 3 teaspoons,
   (d) about 1.4 inches.

2. Express the given uncertainty intervals using ± notation.
   (a) $(3.31, 3.42)$, (b) 3.1416, (c) between 15.1 and 15.2 inches of rain,
   (d) about 2.8 square feet.

3. Express the following uncertainty intervals using interval notation.
   (a) 3.14 ± 0.005, (b) 3.14, (c) 1.5 feet, give or take an inch.

4. Express the following uncertainty intervals using interval notation.
   (a) 1.421 ± 0.003, (b) 2.718, (c) 1 quart, give or take a cup.

In problems 5-6, use the given measurements to determine the area A of a trapezoid with height h and bases a and b. ($A = \frac{1}{2}h(a+b)$. In this formula, the constant $\frac{1}{2}$ is treated as an exact number.) Round your answer to the least significant position.

5. (a) $h = 1.2$ in. $a = 1.01$ in. $b = 2.001$ in.
   (b) $h = 1.20$ in. $a = 1.01$ in. $b = 2.001$ in.
   (c) $h = 1.200$ in. $a = 1.01$ in. $b = 2.001$ in.

6. (a) $h = 1.234$ cm. $a = 1.221$ cm. $b = 0.123$ cm.
   (b) $h = 1.23$ cm. $a = 1.221$ cm. $b = 0.123$ cm.
   (c) $h = 1.234$ cm. $a = 1.22$ cm. $b = 0.123$ cm.

7. The work W done by a constant force F acting over a fixed displacement d is given by $W = Fd$.[7]
   (a) $F = 2.1$ lb, $d = 39.7$ ft. Find the work W done by the force, rounded to the least significant digit.
   (b) $W = 16$ J, $d = 18.1$ m. Find the force F, rounded to the least significant digit.
   (c) $d = 39.7$ ft. If the work W needs to have three significant positions, then how many significant positions do we need in our measurement of the force F?
   (d) $d = 120$ m. If the work W needs to have three significant positions, then how many significant positions do we need in our measurement of the force F?

8. A farmer wishes to paint a 53.1 foot tall silo with a diameter of 37 feet. According to the manufacturer, one gallon of paint will cover about 230 square feet of surface. Since the paint is on sale, the farmer wants purchase all the paint now, but he doesn't want to buy too much paint.
   (a) How much paint does the farmer expect to use? (Round your answer to the least significant digit.)
   (b) Use only your answer to part (a) to get a rough estimate of the maximum amount of paint that he may need.
   (c) Use the original data to estimate the maximum amount of paint that he may need.
   (d) *Briefly* answer the following questions: When would the answer to (b) be a better answer to the farmer's problem? When would (c) be more appropriate?

9. For an electrical circuit with two parallel resistors, the total current I in amperes [A] is given by:
   $$I = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

[7] In English gravitational units, displacements are given in feet [ft], forces in pounds [lb], and work in foot-pounds [ft·lb]. In SI units, displacements are given in meters [m], forces in newtons [N], and work in joules [J].

where V is the potential difference in volts [V] and R_1 and R_2 are the resistances in ohms [Ω].
(a) Find I and I_{max} if $V = 12$ V, $R_1 = 15\ \Omega$, and $R_2 = 11\ \Omega$.
(b) Find I if $V = 120$ V, $R_1 = 1000\ \Omega$, and $R_2 = 11\ \Omega$.
(c) Find I if $V = 120$ V, $R_1 = 1600\ \Omega$, and $R_2 = 11\ \Omega$. Is the answer significantly different from your answer in part (b)? Why or why not?

10. A certain rectangle is about $w = 15$ meters by $\ell = 20$ meters. Its area A can be measured with 5 significant digits of precision.
(a) How many significant digits are needed in measurements of its width in order to calculate its length with three significant digits of precision?
(b) If we calculate length using $\ell = A/w$, what is the maximum number of significant digits of precision?

For some purposes, it may be better to use **relative** or **percentage uncertainties**. In the specification for a family of resistors, the manufacturer may state that the resistances have a percentage uncertainty of about 5%. According to the specification, a 40-ohm [Ω] resistor has an uncertainty of:

$$(40\ \Omega) \times 0.05 = 2\ \Omega$$

The uncertainty interval is (38 Ω, 42 Ω) or $40 \pm 2\ \Omega$.

A weight of 22 pounds (22 ± 1 lb) has a percentage uncertainty of:

$$\frac{1 \text{ lb}}{22 \text{ lb}} = 0.04545 \approx 0.05 = 5\%$$

Problems 11-12 deal with percentage uncertainties.

11. (a) A room has a length of 5 meters, a width of 4 meters and a height of 3 meters, all measured with a percentage uncertainty of 1%. The maximum volume is:

$$V_{max} = (5.05)(4.04)(3.03) = 61.81806 \text{ m}^3.$$

Determine the minimum volume V_{min}. Based on the values for V_{min} and V_{max}, which of the following is the best (closest) estimate for the percentage uncertainty for the volume?
(i) 0.1% (ii) 1% (iii) 10% (iv) 100%

(b) A room has length 5 meters, width 4 meters and height 3 meters. The percentage error in the length and width is 1%, but room height is harder to measure and has a percentage error of 10%. Find values for V_{min} and V_{max}. Which of the following is now the best estimate for the percentage uncertainty for the volume?
(i) 0.1% (ii) 1% (iii) 10% (iv) 100%

(c) Using a laser device, we find that a room has a length of 5 meters with a percentage error of 0.1%. Because of the layout of the room, we have to use more conventional means to get a width of 4 meters and a height of 3 meters with percentage errors of 1% and 10%, respectively. Find values for V_{min} and V_{max}. Which of the following is now the best estimate for the percentage uncertainty for the volume?
(i) 0.1% (ii) 1% (iii) 10% (iv) 100%

Comment: When multiplying or dividing a small collection of inexact numbers, we can estimate the percentage uncertainty by taking the largest percentage uncertainty.

12. (a) The inexact numbers 10.0, 3.14 and 9.00 all have three significant digits. What are their percentage uncertainties?

(b) Suppose instead that the measurements 10.0, 3.14 and 9.00 all have a percentage uncertainty of 1%. What are their uncertainty intervals?

(c) Using the uncertainty rules in the text, we obtain the estimate:

$$\pi^2 \approx (3.14)^2 \approx 9.8596 \pm 0.01 \approx 9.86. \text{ Uncertainty} = 0.01.$$

When doing multiplication or division using relative uncertainties, the relative uncertainty of the result is approximately the larger of the relative uncertainties of the inputs. What is the uncertainty (to one significant digit) if we use this new rule?

---

13. **Ballpark Estimates:** Without using a calculator, estimate the following quantities. Round things off to obtain an uncertainty of about one significant digit.

    (Using scientific notation will make the estimates easier.)

    (a) A heart rate of about 60 to 100 beats per minute (BPM) is considered normal for most adults. At a rate of 70 BPM, how many times will your heart beat in 10 years?

    (b) Rip Van Winkle slept for twenty years. In deep sleep, the heart rate drops to about 40 BPM. How times did Rip's heart beat during his twenty-year nap?

    (c) The second-nearest star, *Proxima Centauri*, is about 4.3 light-years from the earth. Light travels at about 190,000 miles per second. About how many miles is it from Earth to *Proxima Centauri*?

    (d) A car is travelling 60 miles per hour. What is is speed in feet per second? A foot is 0.3048 meters. What is the speed of the car in meters per second?

    (e) A fathom is 6 feet. A fortnight is 14 days. If one snail's pace is one fathom per fortnight, then about how many miles per hour are there in one snail's pace? About how many snail's paces are there in one mile per hour?

3.2.7 Solutions to Selected Exercises

1. (a) 2.25 ± 0.15, (b) 1.781 ± 0.001, (c) 2.5 ± 0.5 tsp, (d) 1.4 ± 0.1 in.

3. (a) $(3.135, 3.145)$, (b) $(3.13, 3.15)$, (c) $(17 \text{ in}, 19 \text{ in})$.

5. (a) 1.8 in^2, (b) 1.81 in^2, (c) 1.81 in^2.

7. (a) $W = 83$ ft·lb. (b) $F = 0.88$ N. (c) three. (d) No solution. Since thre are only two significant positions in the displacement, the calculated force can have at most two significant positions.

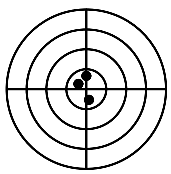
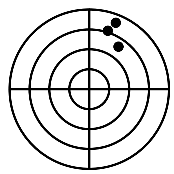
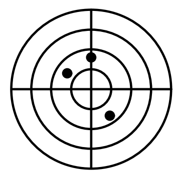
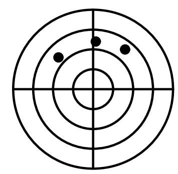
9. (a) 1.9 A. (b) 11 A. (c) 11 A. No, the difference in the two answers is insignificant because the reciprocals of the resistances R_1 are both much less than 0.01.

12. (a) For 10.0: 1%. For 3.14: About 0.3%. (b) For 10: 10.0±0.1. For 3.14: 3.14±0.03. (c) The percentage uncertainty is about 0.318%. The uncertainty is about $0.00318 \cdot \pi^2 \approx 0.03$.

3.2.8 Appendix: Precision and Accuracy

In informal speech, we often use the terms "accuracy" and "precision" interchangeably. In technical usage, their meanings are quite different. The following example illustrates the difference.

Example: In a recent dart tournament, four players (Al, Bea, Chaz and Don) managed to throw their darts as follows. (They were trying to hit the centers of their targets.)

Al	Bea	Chaz	Don
			
precise	precise	imprecise	imprecise
accurate	inaccurate	accurate	inaccurate

Al's darts were close to center (accurate) and close to each other (precise). Bea's darts were also close to each other, but off-center, so Bea was precise but inaccurate. Chaz's darts were distributed around the center of the target, but not very close to each other. Chaz's throws were thus accurate but imprecise. Don's throws were neither distributed around the center nor close to each other, so Don's throws were both inaccurate and imprecise.

More generally, when we say a collection of estimates is accurate, we mean that the average of the collection is close to a true value. But in practice, true values are often unknown. When we say a collection of estimates is precise, we mean that they don't deviate much from each other or they tend to stay close to their average value. It is not necessary to know a true value in order to talk about precision.

In practical situations, we often do not know the true values of the quantities that we are measuring or calculating. In these situations, we usually speak of precision rather than accuracy.