Math 152 Calculus and Analytic Geometry II

Sec 5.2 The Definite Integral

Definition of a Definite Integral:
If f(x) is a continuous function defined on [a,b], we divide the interval into n subintervals of equal width $\Delta x = \frac{b-a}{n}$

We let $x_0(=a), x_1, x_2, \cdots, x_{n-1}, x_n(=b)$ be the endpoints of the subintervals and we let $x^*_1, x^*_2, \cdots, x^*_{n-1}, x^*_n$ be any sample points in each of the subintervals,
(whew... take a breath...)

Then the <u>definite integral of f from a to b</u> is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)\Delta x}$$

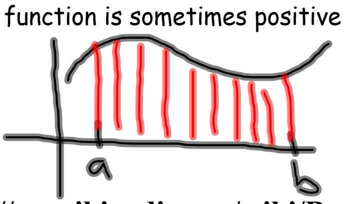
height times width of each rectangle

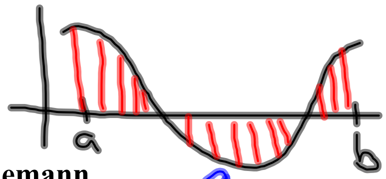
"The integral of f(x) from a to b"

Notes:

- Since f(x) is continuous, the limit always exists.
- The limit does not depend on the sample points chosen (Since f(x) is continuous, any sample point in a small interval must have almost the same y-value)
- The limit also exists even if the function f(x) has a finite number of jumps, holes or breaks.
- But it <u>does not always exist</u> if there is a <u>vertical asymptote in the interval</u> !!!!
- The definite integral $\int_a^b f(x)dx$ is a number. It does not depend on 'x'.

- The sum is called a "Reimann sum" after Bernhard Riemann (1822-1866)
- If the function is always positive, then the sum can be thought of as the area under the curve.
- If the function is sometimes positive and sometimes negative, we'll have to rethink that...





http://en.wikipedia.org/wiki/Bernhard_Riemann

count as 'negative'

Express the following Limit as a definite integral on the interval $[0,\pi]$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+3x_i^{\,2}\right)Sin(x_i)\Delta x = \int_0^{\pi}(1+3x^2)Sin(x)\,dx$$

How can we evaluate these limits?

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} \qquad \boxed{n=5} \qquad 1+2+3+4+5 = \frac{5\cdot 6}{2}$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\begin{array}{r} 1+2+3+4+5 \\ 5+4+3+2+1 \\ \hline 6+6+6+6+6 = 5\cdot 6 \end{array}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

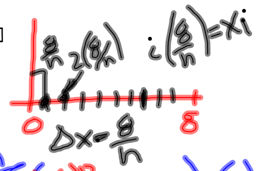
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+1+\cdots+1}_{n \text{ terms}} = n$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

http://www.math.utah.edu/~cherk/ccli/bob/sums.html

Evaluate the following using rules for summations. Use the interval [2,8]



$$f(x)=x^2-5x+2$$

$$\int x^2-5x+2dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(x_i^2-5x_i+2\right)\frac{b-a}{n}$$

$$\int_0^8 x^2-5x+2\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(i\left(\tfrac{8}{n}\right)\right)^2\Big(\quad\Big)\Big(\quad\Big)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(i\left(\tfrac{8}{n}\right)\right)^2-5\left(\tfrac{8}{n}\cdot i\right)+2\right]\left(\tfrac{8}{n}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(i^2\left(\tfrac{8}{n}\right)^2\cdot\left(\tfrac{8}{n}\right)-i\left(\tfrac{8}{n}\right)\left(\tfrac{8}{n}\right)5+2\cdot\tfrac{8}{n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{8^3}{n^3}\sum_{i=1}^{n}i^2\right)-\left(5\frac{8^2}{n^2}\sum_{i=1}^{n}i\right)+2\frac{8}{n}\cdot\sum_{i=1}^{n}1$$

$$\lim_{n\to\infty}\underbrace{\frac{8^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}}_{\to 2}-\underbrace{\frac{5\cdot 8^2}{n^2}\cdot\frac{n(n+1)}{2}}_{\to 1}+2\cdot\frac{8}{\cancel{n}}\cdot\cancel{n}$$

$$=8^3\cdot\frac{2}{6}\qquad-\frac{5\cdot 8^2}{2}\cdot 1\qquad+16$$

$=$ a number $26.666....$

$$f(x)=x^2-5x+2$$

$$\int_0^8 x^2-5x+2\,dx$$

$$v(t)=t^2-5t+2$$

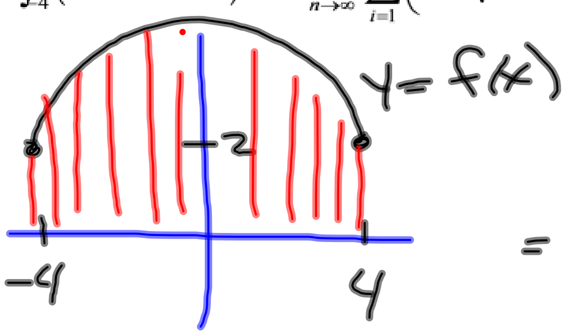
$$s(t)=\frac{t^3}{3}-\frac{5t^2}{2}+2t$$

$$s(8)=\frac{8^3}{3}-\frac{5\cdot 8^2}{2}+16$$

$=26.666$ Same answer.

Evaluate the following definite integral. (Think before you try some ugly limits...)

$$\int_{-4}^{4}\left(2+\sqrt{16-x^2}\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\sqrt{16-x_i^{\,2}}\right)\frac{b-a}{n}$$



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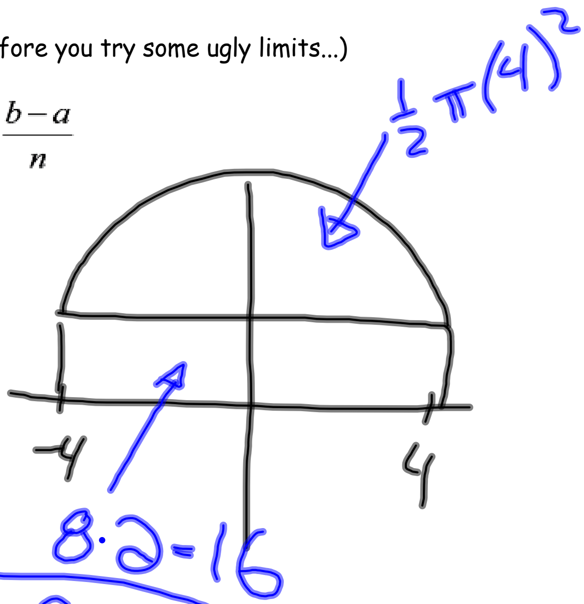
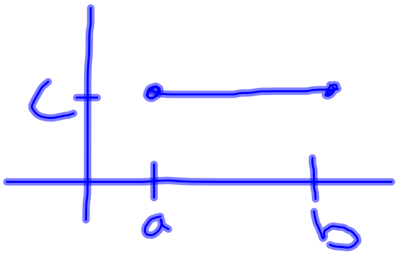


Total Area $= 8\pi + 16$

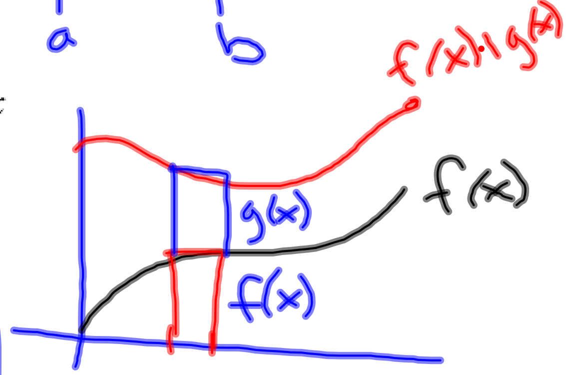
Another Riemann Sum Plotter
http://mathworld.wolfram.com/RiemannSum.html

Properties of a Definite Integral

$$\int_a^b c\,dx = c(b-a)$$



$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

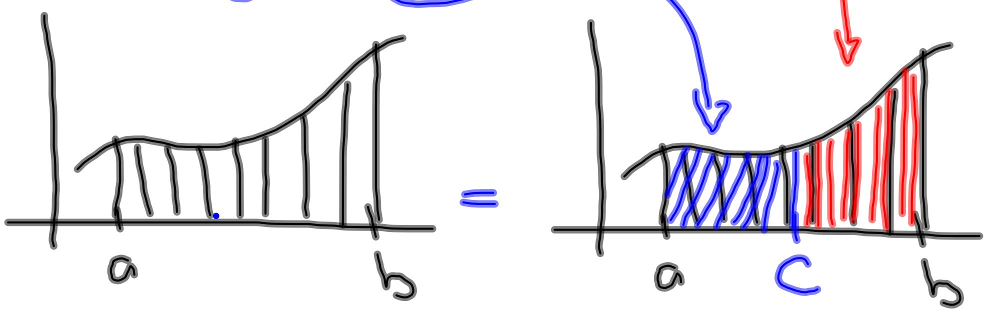


$$\int_a^b c(f(x))dx = c\int_a^b f(x)dx$$

factor out a constant

integrate term by term.

$$\int_a^b (f(x))dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



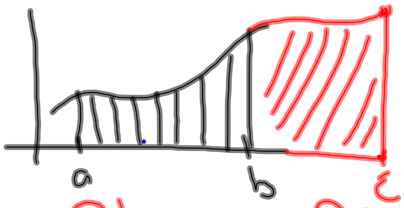
Notice that these properties are true if a<b.

But also if a=b   or if a<b

What does that mean?

$$\int_a^a f(x)\,dx = 0$$

$$\int_5^2 v(t)\,dt = -\int_2^5 v(t)\,dt$$



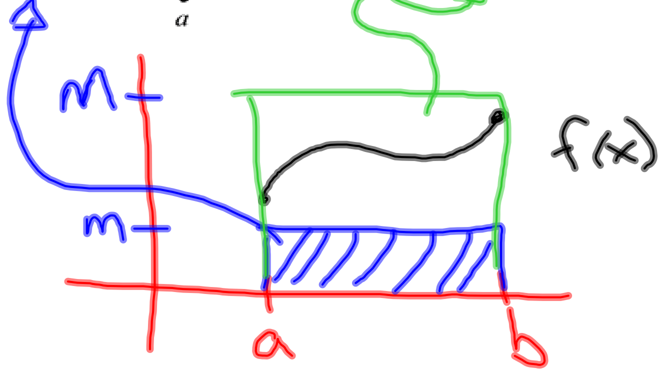
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

Comparison Theorems for Definite Integrals

If $m \le f(x) \le M$ for all $a \le x \le b$ then,

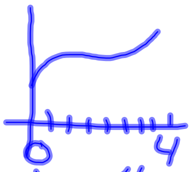
$$m(b-a) \le \int_a^b (f(x))dx \le M(b-a)$$



Evaluate the following using rules for summations. Use the interval [0,4]

$$\int_0^4 (x^3 - 6x^2)dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\left(\frac{4}{n}i\right)^3 - 6\left(\frac{4}{n}i\right)^2\right)}_{f(x_i)} \cdot \underbrace{\left(\frac{4}{n}\right)}_{\Delta x}$$



$\Delta x = \frac{4}{n}$

$x_i = i\left(\frac{4}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right)^4 i^3 - \sum_{i=1}^{n} 6\left(\frac{4}{n}\right)^3 \cdot i^2$$

$$= \lim_{n\to\infty} \frac{4^4}{n^4}\sum_{i=1}^{n} i^3 - \frac{6\cdot 4^3}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{4^4}{n^4}\left(\left(\frac{n(n+1)}{2}\right)^2\right) - \frac{6\,4^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

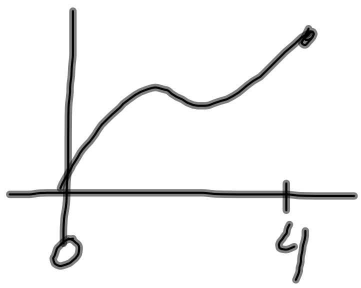
$$= \frac{4^4}{2^2}\cdot 1 - \frac{6\cdot 4^3}{6}\cdot 2$$

$$= 64 - 128 = -64$$

$\int_0^4 x^3 - 6x^2 dx$ = Distance Travelled.

$v(t) = x^3 - 6x^2$

$v(t) = t^3 - 6t^2$



$s(t) = \frac{t^4}{4} - \frac{6t^3}{3} + C$ (positn is antiderivative of velocity)

$s(4) = \frac{256}{4} - 2\cdot 64 = \boxed{-64}$

$-s(0) = 0$

Evaluate the following using rules for summations. Use the interval [-3,5]

$\Delta x = 8/n$

$x_i = -3 + \frac{8}{n} i$

$$\int_{-3}^{5} 3x^2 - 4x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{8}{n}i - 3\right)^2 - 4\left(\frac{8}{n}i - 3\right) \right] \cdot \left(\frac{8}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(\frac{64}{n^2}i^2 - \frac{48}{n}i + 9\right) - \frac{32}{n}i + 12\right)\frac{8}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{192}{n^2}i^2 - \frac{144}{n}i + 27 - \frac{32}{n}i + 12\right)\frac{8}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} i^2\left(\frac{1536}{n^3}\right) - \sum_{i=1}^{n} i\,\frac{1408}{n^2} + \sum_{i=1}^{n} 1\,\frac{312}{n}$$

$$= \lim_{n\to\infty} \frac{1536}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1408}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{312}{n}(n)$$

$$= 1536 \cdot \frac{2}{6} - 1408 \cdot \frac{1}{2} + 312$$

$$= 512 - 704 + 312$$

$$= 120$$