# Math 152 Calculus and Analytic Geometry II

## Sec 5.5 The Substitution Rule

One of the most common tools used to find antiderivatives.... The "anti-chain rule"

$$\int (3x^2-5x+7)^3(6x-5)\,dx$$

derivative of inside

$$F(x)=\frac{1}{4}(3x^2-5x+7)^4.$$

$$\frac{d}{dx}(F(g(x)))= F'(g(x))\cdot g'(x)$$

$$\int F'(g(x))g'(x)\,dx = F(g(x)).$$

So how do you recognize derivatives that come from the chain rule...especially when they are a bit disguised?

We will look for an appropriate "change of variable" or substitution to simplify the integral.

If you see an "inside function g(x)" with the derivative g'(x) (or something similar) in the function, then try a substitution. It may simplify things, or it may not...

$$\int (3x^2-5x+7)^3(6x-5)\,dx$$

guess inside function is

$$u=3x^2-5x+7$$

$$du=6x-5\ dx$$

$$\left(\frac{du}{dx}=6x-5\right)$$

derivatives

"differentials"

substitute to make integral in terms of u.

$$\int u^3\,du$$

$$\frac{1}{4}u^4+C \rightarrow \frac{1}{4}(3x^2-5x+7)^4+C$$

http://archives.math.utk.edu/visual.calculus/4/substitutions.3/

http://integrals.wolfram.com/index.jsp

$$\int \sqrt{(x^2+3)} \cdot 2x\,dx$$

$$u = x^2+3$$
$$du = 2x\,dx$$

$$\int \sqrt{u}\,du = \frac{2u^{3/2}}{3} + C$$

$$= \boxed{\frac{2}{3}(x^2+3)^{3/2} + C}$$

There are two ways to think about substitution: I'll try to do both. You can pick your favorite.

Let $u = g(x)$ then $\dfrac{du}{dx} = g'(x)$ or $du = g'(x)dx$

"derivatives" "differentials"

Translate the integral from in terms of 'x' to in terms of 'u'.

$$\int F'(g(x))\underbrace{g'(x)dx} =$$

$$\int F'(u)\,du$$

$$F(u)+C$$

$$F(g(x))+C$$

$$\int F'(g(x))g'(x)dx =$$

$$\int F'(u)\,\frac{du}{dx}\cdot dx$$

$$\int F'(u)\,du$$

$$F(g(x))+C$$

Two popular methods when g'(x) isn't exactly right in front of your nose...

$\int x\sqrt{3+4x^2}\,dx$

1) u=g(x), take derivative, then solve for dx and substitute.

$u=3+4x^2$

$\frac{du}{dx}=8x$

$\boxed{\frac{du}{8x}=dx}$

$\int x\sqrt{3+4x^2}\,dx$

$\int \cancel{x}\sqrt{u}\cdot\frac{du}{8\cancel{x}}=\frac{1}{8}\left(\frac{2}{3}u^{3/2}\right)+C$

All in terms of u.

$=\frac{1}{8}\frac{2}{3}(3+4x^2)^{3/2}+C$

2) u=g(x), take derivative, then manipulate the function so the whole derivative appears... then substitute.

$\int x\sqrt{3+4x^2}\,dx$

$u=3+4x^2$

$du=8x\,dx$

$\frac{1}{8}\int \underbrace{\sqrt{3+4x^2}}_{\sqrt{u}}\;\underbrace{8x\,dx}_{du}$

$\frac{1}{8}\int\sqrt{u}\,du$

Example:

$\int\frac{5x}{\sqrt{1-3x^2}}\,dx$

$=-\frac{1}{6}\int\frac{5(-6x\,dx)}{\sqrt{1-3x^2}}$

$u=1-3x^2$

$du=-6x\,dx$

$=-\frac{5}{6}\int\frac{1}{\sqrt{u}}\,du$

$\int\frac{1}{u}\,du=\ln|u|$

$\int\frac{1}{u^n}\,du$

$\int u^{-n}\,du$

$\frac{1}{-n+1}u^{-n+1}+C$

$=-\frac{5}{6}\cdot\frac{2}{1}u^{1/2}+C$

$=-\frac{5}{3}(1-3x^2)^{1/2}+C$

Try $\displaystyle\int \frac{5x^2}{\sqrt{1-3x^2}}\,dx = \frac{-1}{6}\int \frac{5x(-6x\,dx)}{\sqrt{1-3x^2}}$

$u = 1-3x^2$
$du = -6x\,dx$

x can't change easily into "u"
Hard to Integrate.

Ex $\displaystyle\int 9x e^{5-3x^2}\,dx$

$u = 5-3x^2$
$du = -6x\,dx$

$$\frac{-9}{6}\int e^{5-3x^2}(-6x\,dx)$$

$$-\frac{3}{2}\int e^{u}\,du$$

$$-\frac{3}{2}e^{u}+C$$

$$\boxed{-\frac{3}{2}e^{(5-3x^2)}+C}$$

Example:

$\int \tan(x)\,dx$ $\qquad \int \tan(u)\,du$

$u = x$

$du = dx$ $\qquad$ No Help

$$\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{\cos(x)}(-\sin(x)\,dx)$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\frac{d}{dx}\left(-\ln(\cos(x))\right) = -\frac{1}{\cos(x)} \cdot -\sin(x)$$

$$-\int \frac{1}{u}\,du$$

$$-\ln|u| + C$$

$$\boxed{-\ln|\cos(x)| + C}$$

Book $\rightarrow$ $\ln|\sec(x)| + C$

$$\sec(x) = \frac{1}{\cos(x)} = (\cos(x))^{-1}$$

Example:

$$2\int \frac{e^{Sin(\sqrt{x})}Cos(\sqrt{x})}{2\sqrt{x}}\,dx = 2\int e^{\sin(\sqrt{x})} \cdot \cos(\sqrt{x}) \frac{1}{2\sqrt{x}}\,dx$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x} \cdot du$

$$2\int e^{\sin(u)} \cos(u) \frac{1}{2u} 2\sqrt{x}\,du$$

$$= \boxed{2\int e^{\sin(u)} \cos(u)\,du}$$

substitute

$w = \sin(u)$

$dw = \cos(u)\,du$

$$= 2\int e^{w}\,dw$$

$$= 2e^{w} + C$$

$$= 2e^{\sin(u)} + C$$

$$= \boxed{2e^{\sin(\sqrt{x})} + C}$$

Definite Integrals: Again, two methods...

1) plug in limits after you use substitution to evaluate the integral.

$$\frac{1}{3}\int_1^7 \sqrt{3x+2}\,3\,dx = \frac{1}{3}\int_1^7 \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}$$

$u = 3x+2$

$du = 3\,dx$

$$= \frac{2}{9}(3x+2)^{3/2}\Big]_1^7$$

$$= \frac{2}{9}(23)^{3/2} - \frac{2}{9}(5)^{3/2}$$

2) Change the limits when you change variables....Plug the limit into u=g(x), then continue...

$$\int_1^7 \sqrt{3x+2}\,dx \qquad \frac{1}{3}\int_5^{23} \sqrt{u}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big]_5^{23}$$

$u = 3x+2$

$du = 3\,dx$

change limits

Example:

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_1^e \ln(x)\cdot\frac{1}{x}\,dx = \int_0^1 u\,du$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \frac{u^2}{2}\Big]_0^1$$

$$= \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

without limits

$$\int \frac{\ln(x)}{x}\,dx = \frac{(\ln(x))^2}{2} + C$$

Example:

$$\int_1^2 x\sqrt{x^2-1}\,dx = \frac{1}{2}\int \sqrt{x^2-1}\cdot 2x\,dx$$

$u = x^2-1$

$du = 2x\,dx$

$$= \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2}\frac{2}{3}u^{3/2}+C$$

$$= \frac{1}{3}(x^2-1)^{3/2}+C$$

$$\int_1^2 x\sqrt{x-1}\,dx$$

$u = x-1$

$du = dx$ $\underline{\underline{x = u+1}}$

$$= \int x\sqrt{u}\cdot du$$

$$= \int (u+1)\sqrt{u}\,du = \int \left(u^{3/2}+u^{1/2}\right)du$$

$$\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C$$

$$\boxed{= \frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C}$$