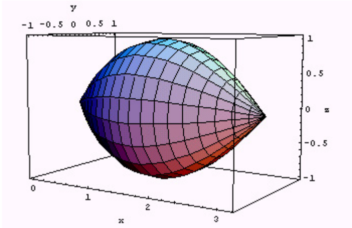
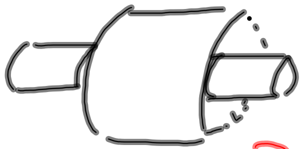
Math 152 Calculus and Analytic Geometry II

Sec 6.2 Volumes

We can use definite integrals to find volumes of three dimentional solids by breaking them in to many slices (or disks or washers)





approximate by 3 cylinders

Definition of Volume: Let S be a solid that lies between x=a and x=b. If the cross-sectional area (perpendicular to the x-axis) is A(x) (a continuous function) then the Volume of S is

$$V = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n}}_{\text{add up lots of cross sectns}} A(x_i)\Delta x = \int_a^b A(x)dx$$

add up lots of cross sectns

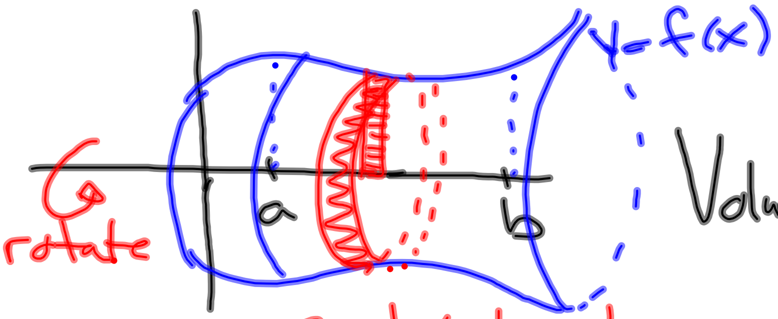
area of cross sectn

thickness of cross sectn

Solids of Revolution:

Consider the region bounded by y=f(x), the line y=0 and x=a and x=b

Make a 3-D solid by rotating that region around the x-axis. How can you find the volume of that?



$$\text{Volume} = \int_a^b \pi(f(x))^2 dx$$

each interval is a disk when rotated

http://higheredbcs.wiley.com/legacy/college/anton/0470183454/applets/ch6/figure6_2_13/washer_ex4.htm

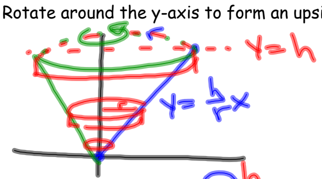
Volume of disk $= A(x)\cdot dx$

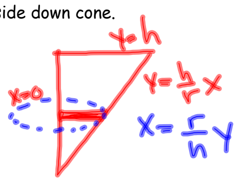
$\pi(f(x))^2 \cdot dx$

radius

Here's geometric volume that we know:

Consider the Triangle bounded by y=h, x=0 and y= (h/r)x
Rotate around the y-axis to form an upside down cone.





$$\text{Volume}=\int_0^h \pi\left(\frac{r}{h}y\right)^2 dy$$

radius in terms of y

$$\frac{\pi r^2}{h^2}\int_0^h y^2\,dy$$

$$\frac{\pi r^2}{h^2}\left.\frac{y^3}{3}\right]_0^h$$

$$\frac{\pi r^2}{h^2}\left(\frac{h^3}{3}-\frac{0}{3}\right)$$

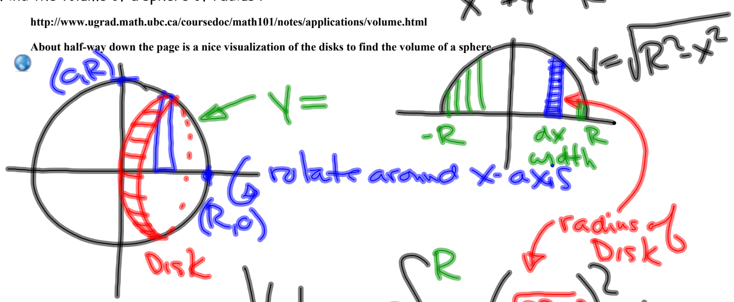
Volume of Cone $\rightarrow = \frac{1}{3}\pi r^2 h$

Find the volume of a sphere of radius r

http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/applications/volume.html

About half-way down the page is a nice visualization of the disks to find the volume of a sphere

$x^2+y^2=R^2$



$$\text{Volume}=2\int_0^R \pi\left(\sqrt{R^2-x^2}\right)^2\cdot dx$$

$$=2\pi\int_0^R R^2-x^2\,dx$$

constant

$$=2\pi\left.\left(R^2x-\frac{x^3}{3}\right)\right]_0^R$$

$$=2\pi\left(R^3-\frac{R^3}{3}\right)$$

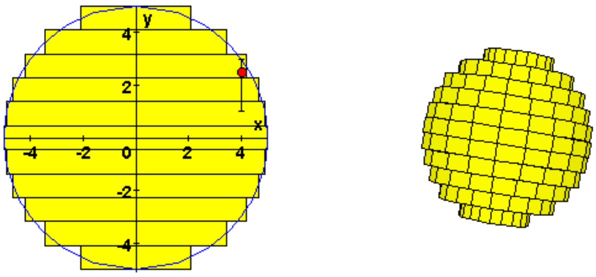
Volume of Sphere $\rightarrow = \frac{4}{3}\pi R^3$

http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/a… — Calculating Simple Volumes

Thus the volume contained within all n disks is:

$$V_n = \sum_{k=1}^{n} \pi \left(R^2 - y_k^2\right) \Delta y$$

It should come as no surprise that when we increase the number of disks used to approximate the volume of the sphere, we get a closer and closer approximation. You can see this yourself by moving the red dot in the demo below. (The red dot controls the number of slices used in the approximation.) To see what the figure looks like in 3D, you might want to point your mouse at the 3D shape and move or rotate it a bit.



Returning to our calculation, and taking a limit as $n \to \infty$, we find that

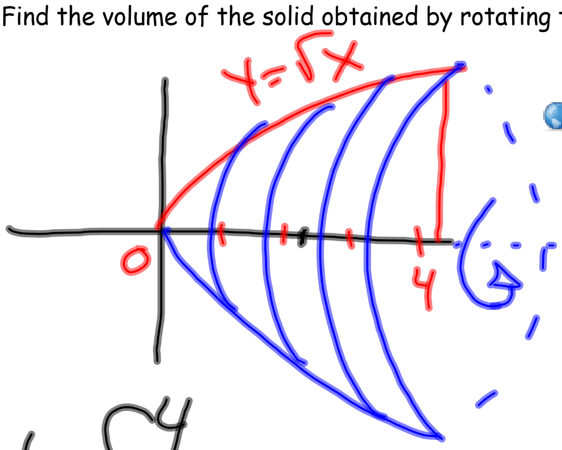
$$V = \int_{-R}^{R} \pi(R^2 - y^2)\, dy$$

Here R is a constant. We can compute this integral, and we find that

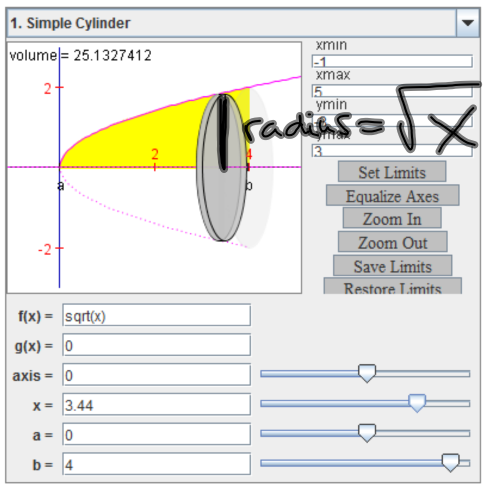
$$V = \pi[R^2 y - \frac{y^3}{3}]|_{-R}^{R}$$

Consider the region bounded by $y = \sqrt{x}$ and the x-axis from 0 to 4

Find the volume of the solid obtained by rotating the region about the x-axis.



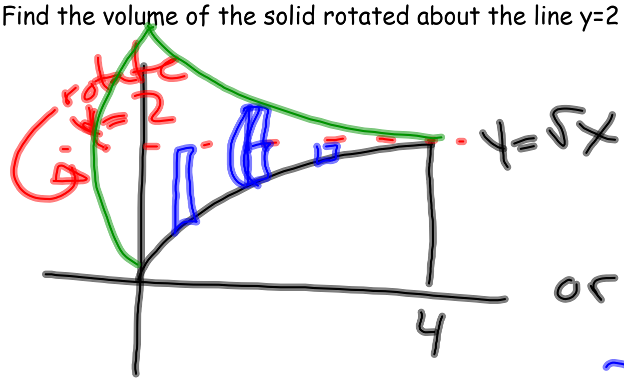
http://www.calculusapplets.com/revolution.html (allows you to change functions and the axis of revolution)



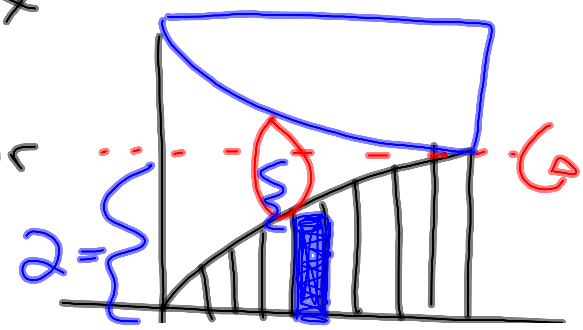
$$V = \int_0^4 \pi(\sqrt{x})^2 dx$$

Consider the region bounded by $y = \sqrt{x}$ and $\boxed{\text{the x-axis}}$ from 0 to 4

Find the volume of the solid rotated about the line y=2



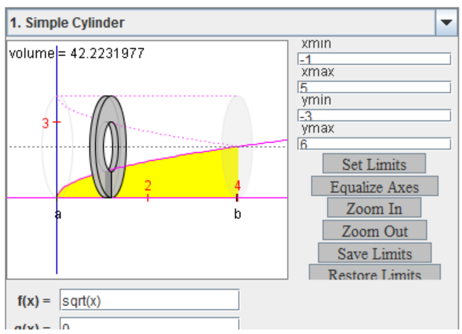
or



outside − inside

$$\text{Volume} = \int_0^4 \pi(2)^2 - \pi(2-\sqrt{x})^2\,dx$$

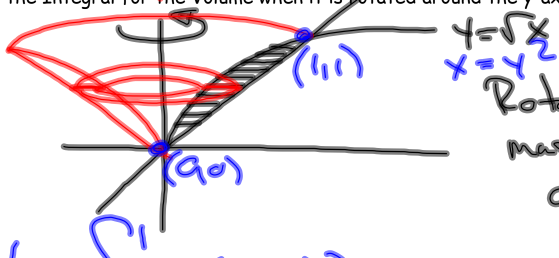
Area of ⊙ washer



Consider the region bounded by:

$y = \sqrt{x}$ and $y = x$

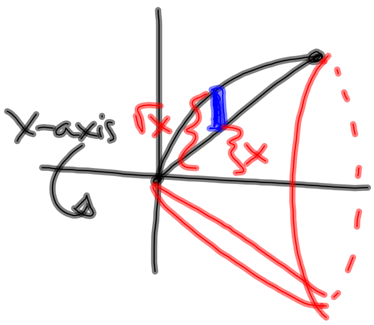
Find the Integral for the Volume when it is rotated around the y-axis.



Rotate around y-axis must be horizontal disks in terms of y

$$V = \int_0^1 \underbrace{\pi(y-0)^2}_{\text{outside}} - \underbrace{\pi(y^2-0)^2}_{\text{inside}}\, dy$$

Find the Integral for the Volume when it is rotated around the x-axis.



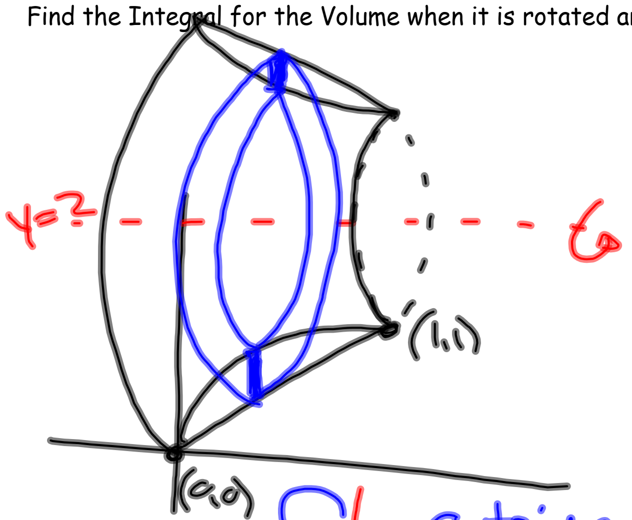
Area of washer

$$\text{Volume} = \int_0^1 \pi(\sqrt{x})^2 - \pi(x)^2\, dx$$

Consider the region bounded by:

$y = \sqrt{x}$ and $y = x$

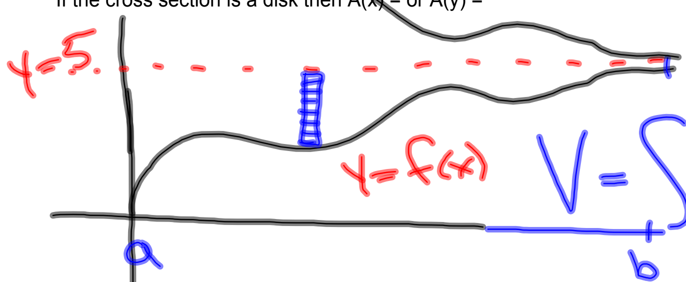
Find the Integral for the Volume when it is rotated around the line y=2.



$$V = \int_0^1 \overbrace{\pi(2-x)^2}^{\text{outside}} - \overbrace{\pi(2-\sqrt{x})^2}^{\text{inside}}\,dx$$

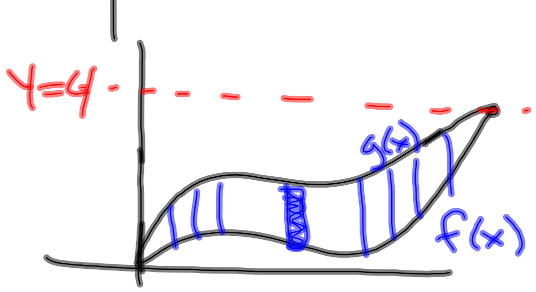
Solids of Revolution

If the cross section is a disk then A(x) = or A(y) =



$$V = \int_a^b \pi(\text{axis} - f(x))^2\,dx$$

If the cross-section is a washer, then



$$V = \int_a^b \pi(\text{axis} - f(x))^2 - \pi(\text{axis} - g(x))^2\,dx$$

Practice Problems

Try the following problems: 1,3,5,7,13,17,21,23,49,51

http://higheredbcs.wiley.com/legacy/college/anton/0470183454/applets/ch6/figure6_3_7/shell.htm

7 $y=x^3$

$y=x$

rotate around

x-axis

17 $y=x^2$
$x=y^2$
about $x=-1$

6 $y=\ln(x)$
$y=1$
$y=2$
around y-axis

(14) $y = 1/x$

$y = 0$

$y = 3$

around $y = -3$

$$2\int \frac{e^{\sqrt{x}} \cdot \cos(e^{\sqrt{x}})}{\sin(e^{\sqrt{x}}) \cdot 2\sqrt{x}}\,dx$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$2 \cdot \int \frac{e^u \cdot \cos(e^u)}{\sin(e^u)}\,du$$

$$2\int \frac{\cos(w)}{\sin(w)}\,dw$$

$v = \sin(w)$

$dv = \cos(w)\,dw$

$=$

$\cancel{v = \cos(w)}$

$\cancel{dv = -\sin(w)\,dw}$