Math 152 Calculus and Analytic Geometry II

Sec. 7.1 Ingtegration By Parts

More techniques to find antiderivatives:

$f(x) = x^n \quad f'(x) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$

Chain Rule leads to Substitution

$$\frac{d}{dx}\big(f(u(x))\big) = f'(u(x)) \cdot u'(x)$$

$$f(u(x)) = \int \big(f'(u(x)) \cdot u'(x)\big) dx = \int f'(u)\, du$$

$u = u(x)$

$du = u'(x)\, dx$

$= f(u) + C$

$= f(u(x)) + C$

Substitution

Product Rule Leads to "Integration by Parts"

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x) \cdot v'(x) + u'(x)v(x)$$

"Integrate both sides"

$$\int \left(\frac{d}{dx} u(x)v(x)\right) dx = \int u(x)v'(x)dx + \int u'(x)v(x)dx$$

$$u(x) \cdot v(x) = \int u(x)v'(x) + \int u'(x)v(x)dx$$

rearrange

$$\int u(x)\, \underbrace{v'(x)dx} = u(x)v(x) - \int v(x)\, \underbrace{u'(x)dx}$$

Simplified form

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Goal: Take an functn that we $\underline{\underline{\text{cant}}}$ integrate and rewrite it

as $(u\,v) - \int$(functn we can integrate)

Rearrange to get a useful identity.

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

or simply

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

We will use this technique to switch from a difficult integral to one that is "hopefully" easier to evaluate.

We will have to recognize the original function as a product.

Pick one term to be 'u' and the other to be 'dv'

Example:

$\int x\sin(x)dx$

Not $\cancel{\frac{x^2}{2}\cos(x)+C}$

let $u = x$ $\quad dv = \sin(x)dx$

$du = dx$ $\quad V = -\cos(x)$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$= x(-\cos(x)) - \int -\cos(x) \cdot dx$$

$$= -x\cos(x) + \int \cos(x)dx$$

easier to solve

$$= -x\cos(x) + \sin(x) + C$$

Double check → Take derivative

$$[(-1)(\cos(x)) + (-x)(-\sin(x))] + \cos(x)$$

$$= x \cdot \sin(x)$$

Example:

$$\int x^2 x^3 dx = \int x^2\, u \cdot \frac{du}{3x^2} \quad \text{or} \int x^5$$

$$\frac{x^6}{6} + C$$

$u = x^3$

$\underline{du} = 3x^2 dx$

$\frac{du}{3x^2} = dx$

$$\int \frac{1}{3} u\, du$$

$$= \frac{1}{3}\frac{u^2}{2} + C$$

$$= \frac{1}{6}(x^3)^2 + C$$

$$= \boxed{\frac{x^6}{6} + C}$$

$$\int \underbrace{x^2}_{u} \underbrace{x^3 dx}_{dv}$$

$u = x^2$ $\quad$ $dv = x^3 dx$

$du = 2x\,dx$ $\quad$ $V = \frac{x^4}{4}$

$$\int u \cdot dv = uv - \int v\, du$$

$$= x^2\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} 2x\, dx$$

$$= \frac{x^6}{4} - \int \frac{x^5}{2}\, dx$$

$$= \frac{3x^6}{3\cdot 4} - \frac{1}{2}\frac{x^6}{6} + C$$

$$= \frac{3x^6 - x^6}{12} + C$$

$$= \boxed{\frac{x^6}{6} + C}$$

$$\int \underbrace{x \cdot e^x \, dx} \qquad \int u \cdot dv = u \cdot v - \int v \cdot du$$

$u = x \qquad dv = e^x dx$

$du = 1 dx \qquad v = e^x$

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x \cdot 1 \, dx$$

$$= x \cdot e^x - e^x + C$$

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check: Take derivative of

$$\frac{dy}{dx} = (1 \cdot e^x + x \cdot e^x) - e^x$$

$$= x \cdot e^x$$

$$\int x \cdot e^x \, dx$$

What happens when you guess wrong?

$$\int \underbrace{e^x}_{u} \cdot \underbrace{x \, dx}_{dv}$$

$u = e^x \qquad dv = x \, dx$

$du = e^x dx \qquad v = \frac{x^2}{2}$

$$= e^x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} e^x dx$$

worse than it was.

Example:

$\int x^2 e^{-x} dx$

$u = x^2 \quad dv = e^{-x} dx$

$du = 2x\,dx \quad V = -e^{-x}$

$= x^2(-e^{-x}) - \int -e^{-x} \cdot 2x\,dx$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

(Do "parts" again)

$u = x \quad dv = e^{-x} dx$

$du = dx \quad V = -e^{-x}$

$= -x^2 e^{-x} + 2\left(x(-e^{-x}) - \int -e^{-x} dx \right)$

$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$\int x^{17} e^{-x} dx =$

Example:

$\int \ln(x) dx$

$u = \ln(x) \quad dv = dx$

$du = \frac{1}{x} dx \quad V = x$

$= \ln(x) \cdot x - \int x \cdot \frac{1}{x} dx$

$= x \cdot \ln(x) - \int 1\,dx$

$= \boxed{x \cdot \ln(x) - x + C}$

Derivative

$\left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) - 1$

$= \ln(x)$

Example:

$$\int \underbrace{\tan^{-1}(x)}_{u}\,\underbrace{dx}_{dv}$$

Know $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$

$u = \tan^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{x^2+1}dx$ $\quad V = X$

$$= X \cdot \tan^{-1}(x) - \frac{1}{2}\int 2X \cdot \frac{1}{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Find a "reduction" formula

$$\int x^n e^x dx$$

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$$

Evaluate

$$\int Sin^n(x)dx$$