Math 152 Calculus and Analytic Geometry II

Sec. 7.3 Trigonometric Substitutions

This section has suggestions for how to solve integrals involving square roots by certain substitutions of trig functions.

For example. Integrate this by Substitution

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\cdot du$$

$u = x^2+1$

$du = 2x\,dx$

$$= \frac{1}{2}\int u^{-1/2}\,du$$

$$\frac{1}{2}\,\frac{2}{1}\,u^{1/2} + C$$

$$= \sqrt{x^2+1} + C$$

Compare that integral to the following: Normal Substitution won't work.

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$$ Try an 'inverse trig substitution' $x=\tan(\theta)$, $\theta=\tan^{-1}(x)$

substitution $u=x^2+1$ won't work.

$x=\tan(\theta)$

$dx=\sec^2(\theta)\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int \frac{1\cdot\sec^2\theta\cdot d\theta}{(\tan\theta)^2\sqrt{\tan^2\theta+1}}$$

$$=\int \frac{\sec^2\theta}{\tan^2\theta\cdot\sec(\theta)}\,d\theta$$

$$=\int \tan^{-2}(\theta)\cdot\sec(\theta)\,d\theta$$

$$\int \frac{\cos(\theta)^2}{(\sin(\theta))^2}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$=\int \frac{\cos(\theta)}{(\sin\theta)^2}\,d\theta$$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

$$=\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$=-u^{-1}+C$$

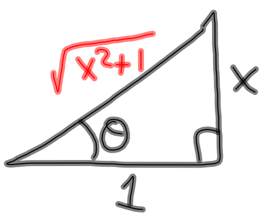
$$=-\frac{1}{\sin(\theta)}+C$$

Back in terms of x

$x=\tan(\theta)$

$\theta=\tan^{-1}(x)$

$$=-\frac{1}{\sin(\tan^{-1}(x))}+C$$



$$=-\frac{1}{\left(\frac{x}{\sqrt{x^2+1}}\right)}+C$$

$$=\boxed{\frac{-\sqrt{x^2+1}}{x}+C}$$