Math 152 Calculus and Analytic Geometry II

Sec. 7.4 Integration by Partial Fractions

This section has suggestions for how to solve integrals involving rational functions by breaking them into several fractions that are simpler to integrate.

For example.

$$\int \frac{x^3+x}{x-1}\,dx$$

If numerator has power equal or larger than denominator divide.

$$\begin{array}{r} x^2+x+2 \\ x-1\,\overline{)\,x^3+0x^2+1x+0} \\ -(x^3-x^2) \\ \hline x^2+x \\ -(x^2-x) \\ \hline 2x+0 \\ -(2x-2) \\ \hline 2 \end{array}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int x^2+x+2+\frac{2}{x-1}\,dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$$

$$\int \frac{1}{x^2-16}\,dx$$

$$\frac{1}{x^2-16} = \frac{1}{(x-4)(x+4)} = \frac{A}{x-4}\frac{(x+4)}{(x+4)} + \frac{B}{x+4}\frac{(x-4)}{(x-4)}$$

partial Fractions

get common denominator

So Set numerators equal

$$1 = A(x+4)+B(x-4)$$

$$0x+1 = (A+B)x+(4A-4B)$$

Equal Coefficients

$$\boxed{\begin{aligned} 0 &= A+B \\ 1 &= 4A-4B \end{aligned}}$$ Solve

elimination

$$\begin{aligned} 0 &= 4A+4B \\ \hline 1 &= 8A \end{aligned}$$

$$\boxed{A = \tfrac{1}{8}} \quad \boxed{B = \tfrac{-1}{8}}$$

$$\int \frac{1}{x^2-16}\,dx = \int \frac{1/8}{x+4}+\frac{-1/8}{x-4}\,dx$$

$$= \boxed{\frac{1}{8}\ln|x+4|-\frac{1}{8}\ln|x-4|+C}$$

other way to solve

Set numerators equal

$1 = A(x+4) + B(x-4)$

plug in "nice" values for x

$x=4 \Rightarrow 1 = A\cdot 8 + 0\cdot B$

$A = 1/8$

$x=-4 \Rightarrow 1 = 0A - 8B$

$B = -1/8$

Steps to integrate a rational function:

1) Long Division

2) Factor the denominators as much as possible

3) Write the rational functions in terms of their "Partial Fractions"

$$\frac{A}{(ax+b)^i} \qquad \frac{Ax+B}{(ax^2+bx+c)^i}$$

$$\frac{1}{(x-3)(x+2)(x+4)(x)} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{x+4} + \frac{D}{x}$$

$$\frac{1}{(x+3)^3} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

$$\frac{1}{x^2+4} = \frac{Ax+B}{x^2+4}$$

irreducible factor

$$\frac{1}{x^3+27} \qquad \frac{Ax^2+Bx+C}{x^3+27}$$

$(x+3)(x^2-3x+9)$

except that a cubic can always be factored

Quadratics are "worst" possible irreducible terms.

$$\frac{3x-7}{(x+2)^2(x-1)(x^2+9)^2}$$

$$= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x-1)} + \frac{Dx+E}{x^2+9} + \frac{Fx+G}{(x^2+9)^2}$$

$\ln|x+2|$

$u = x+2$
Substitution
Power rule

$$\int \frac{Dx+E}{x^2+9} = \int \frac{Dx}{x^2+9}\,dx + \int \frac{E}{x^2+9}\,dx$$

$u = x^2+9$
$du = 2x\,dx$

$E\frac{1}{3}\tan^{-1}(x/3)$

Example:

$$\int \frac{x-9}{(x+5)(x-2)}\,dx$$

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5}\frac{(x-2)}{(x-2)} + \frac{B}{x-2}\frac{(x+5)}{(x+5)}$$

$$x-9 = A(x-2) + B(x+5)$$

Coef of x $\quad 1 = A+B \qquad B = 1-A$

constant $\quad -9 = -2A+5B$

$$-9 = -2A + 5(1-A)$$
$$-9 = -2A + 5 - 5A$$
$$-14 = -7A$$
$$A = 2$$
$$B = -1$$

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2}{x+5} + \frac{-1}{x-2}$$

$$= 2\ln|x+5| - \ln|x-2| + C$$

Example:

$$\int \frac{2x+3}{(x+1)^2}$$

$$\frac{2x+3}{(x+1)^2} = \frac{A}{x+1}\frac{(x+1)}{(x+1)} \; \frac{B}{(x+1)^2}$$

$$2x+3 = A(x+1) + B$$

$$2x+3 = A\cdot x + (A+B)$$

$$\boxed{\begin{aligned} A &= 2 \\ B &= 1 \end{aligned}}$$

$$\int \frac{2}{x+1} + \frac{\quad}{(x+1)^2}\,dx$$

$$\boxed{2\ln|x+1| - (x+1)^{-1} + C}$$

Example:

$$\int \frac{x^3}{(x+1)^3}\,dx$$

Example:

$$\int \frac{1}{(x+5)^2(x-1)}\,dx$$

Irreducible Factor in the Denominator

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$$

Use $\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Find Partial Fractions Decomposition for the following

$$\int \frac{x}{(x+2)(x-3)^2(x^2+2x+4)}$$

Using this method you can always break down a rational function into terms that you can integrate.

1. Divide if numerator is same or higher degree as denominator.
2. Factor denominator as much as possible.
3. If term is A/((x-b)), ln|x-b| will work.
4. If term is A/((x-b)^n) , power rule will work.
5. If term is (Ax+B)/(x^2+c^2), then split into
    5a. If term is Ax/(x^2+c^2) then substitute u=x^2+c^2
    5b. If term is B/(x^2+c^2) then use B(1/c arctan(x/c))+C
6. If term is Ax+B/(x^2+bx+c), then complete the square in the denominator, make it A/((x+b/2)^2+(Something)^2) and see step 5.