$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx$$

$$\int x^2 \cdot \underbrace{\frac{x}{\sqrt{4+x^2}}\,dx}_{dv}$$

$u = x^2$

$$V = \int \frac{x}{\sqrt{4+x^2}}\,dx =$$

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$$

Inverse Trig Substitutions

$x = 2\tan(\theta)$

$dx = 2\sec^2\theta\, d\theta$ $\quad \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$\int \frac{1}{(4\tan^2\theta)\underbrace{\sqrt{4+4\tan^2\theta}}}\cdot 2\sec^2\theta \cdot d\theta$$

$\sqrt{4}\sqrt{1+\tan^2\theta}$

$2\cdot\sec\theta$

$$\int \frac{\sec\theta}{4\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

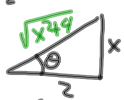
$u = \sin\theta$

$du = \cos\theta \cdot d\theta$

$$\boxed{\frac{1}{4}\int \frac{1}{u^2}\,du}$$

$$\frac{1}{4} - u^{-1} + C$$

$\frac{x}{2} = \tan\theta$



$\sin(\theta) = \frac{x}{\sqrt{x^2+4}}$

$$-\frac{1}{4}\,\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{4}\,\frac{1}{\sin\left(\tan^{-1}\left(\frac{x}{2}\right)\right)} + C$$

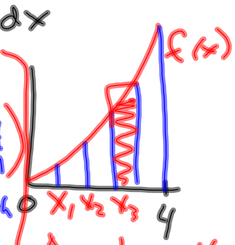
$$= -\frac{1}{4}\,\frac{1}{\left(\frac{x}{\sqrt{x^2+4}}\right)} + C$$

$$= \boxed{\frac{-\sqrt{x^2+4}}{4x} + C}$$

Riemann Sum

$$\int_0^4 x^2+7x+3 \, dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\left(\frac{4}{n}i\right)^2+7\left(\frac{4}{n}i\right)+3\right)}_{\text{height}}\underbrace{\left(\frac{4}{n}\right)}_{\text{width}}$$



$$\Delta x = \frac{b-a}{n} = \frac{4}{n}$$

$$x_i = \left(\frac{4}{n}\right)i$$

$$\text{height} = f\left(\frac{4}{n}i\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4^3}{n^3} i^2 + \sum_{i=1}^{n} \frac{7\cdot 4^2}{n^2} i + \sum_{i=1}^{n} \frac{12}{n}$$

$$\lim_{n\to\infty} \underbrace{\frac{4^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)}_{\to 2} + \underbrace{\frac{7\cdot 4^2}{n^2}\left(\frac{n(n+1)}{2}\right)}_{\to 1} + \underbrace{\frac{12}{n}(n)}_{\to 1}$$

$$= \frac{4^3\cdot 2}{6} + 7\cdot 4^2 \frac{1}{2} + 12$$

$$\int_0^{\pi/2} x\cdot \sin(x) \, dx$$

By Parts with Limits

$u = x \qquad dv = \sin(x)\,dx$

$du = dx \qquad v = -\cos(x)$

$$= -x\cos(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} -\cos(x) \, dx$$

$$= -x\cos(x)\Big]_0^{\pi/2} + \sin(x)\Big]_0^{\pi/2}$$

$$-x\cos(x) + \sin(x)\Big]_0^{\pi/2}$$

Integrate by Partial Fractions

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \frac{A(x^2+4)}{x(x^2+4)} + \frac{(Bx+C)x}{x^2+4\,(x)}$$

$x(x^2+4)$

$$\frac{A}{x} + \frac{Bx}{x^2+4} + \frac{C}{x^2+4}$$

$\ln(x)$ $u=x^2+4$ $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

Find A, B + C

$$2x^2-x+4 = A(x^2+4)+Bx^2+Cx$$

x^2 coefficient $\rightarrow$ $2 = A+B$

x's $\rightarrow$ $-1 = C$

constants $\rightarrow$ $4 = 4A$

$C=-1$

$A=1$

$B=1$

$$\int \text{original}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{-1}{x^2+4}\,dx$$

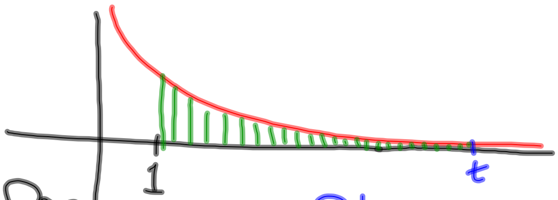
$$\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Math 152 Calculus and Analytic Geometry II

Sec. 7.8 Improper Integrals and Sec. 4.4 L'Hopital's Rule

Find the Area under the curve y=f(x) and above y=0 and to the right of x=1.

$y = 1/x^2$

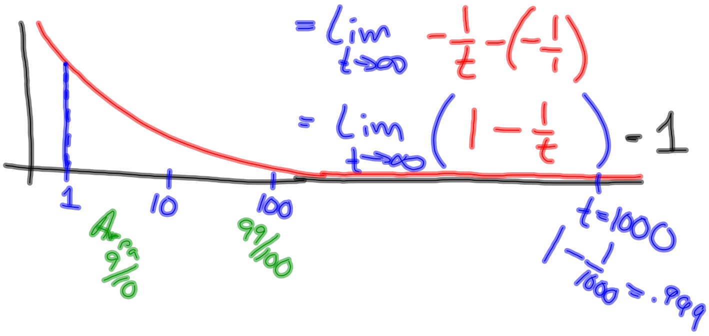


$$\text{Area} = \int_1^\infty 1/x^2\,dx = \lim_{t\to\infty}\int_1^t 1/x^2\,dx$$

$$= \lim_{t\to\infty} -x^{-1}\Big]_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{t} - \left(-\frac{1}{1}\right)$$

$$= \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1$$



Definition of Improper Integral I

If the integral from a to 't' exists for every 't', then

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^{t} f(x)dx$$

Similar for Limits to negative infinity...

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to -\infty} \int_t^{a} f(x)dx$$

The improper integral is <u>convergent</u> if

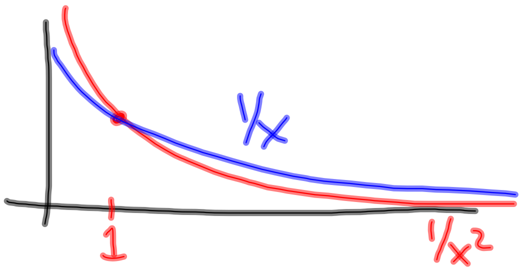
If limit exists and is finite

The improper integral is <u>divergent</u> if

If limit is <u>infinite</u>.

Find the Area under the curve y=f(x) and above y=0 and to the right of x=1.

$y = 1/x$

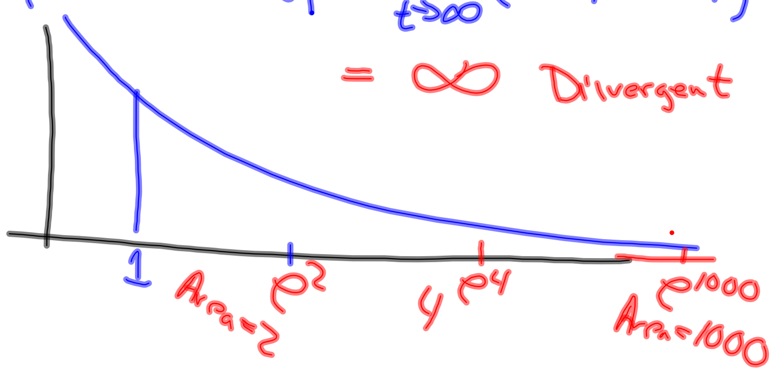


$$\int_1^{\infty} \frac{1}{x}dx$$

$$= \lim_{t\to\infty} \int_1^{t} 1/x\, dx$$

$$= \lim_{t\to\infty} \ln|x| \Big]_1^t = \lim_{t\to\infty} \left(\ln|t| - \ln|1|\right)$$

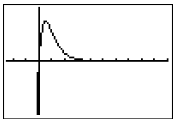
$$= \infty \text{ Divergent}$$



Compare the Graphs of the two functions

Evaluate the following: (Graph it first)

$$\int_0^\infty xe^{-2x}\,dx$$



$$= \lim_{t\to\infty} \int_0^t xe^{-2x}\,dx$$

$u = x \qquad dv = e^{-2x}\,dx$

$du = dx \qquad V = \frac{-e^{-2x}}{2}$

$$= \lim_{t\to\infty} \frac{-xe^{-2x}}{2} - \int \frac{-e^{-2x}}{2}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^t$$

$$= \lim_{t\to\infty} \left(-\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right) - \left(-\underbrace{\frac{0e^0}{2}}_{0} - \underbrace{\frac{e^0}{4}}_{1/4}\right)$$

$$= \lim_{t\to\infty} \frac{1}{4} - \frac{e^{-2t}}{4} - \frac{te^{-2t}}{4}$$

$$= 1/4 - 0 - \frac{(\text{Big})(\text{verysmall})}{4}$$

who wins?

Sec. 4.4 L'Hôpital's Rule

If Lim f(x)=0 and Lim g(x)=0 as x approaches 'a', then $\displaystyle \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

"$\frac{0}{0}$"

$$\lim_{x\to 1}\frac{x^2-5x+4}{x-1}$$

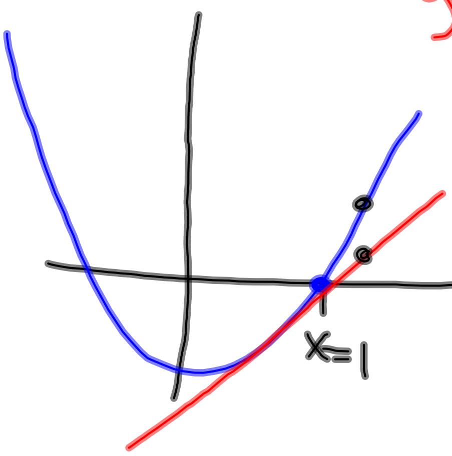
"$\frac{0}{0}$" form

$$\lim_{x\to 1}\frac{\cancel{(x-1)}(x-4)}{\cancel{(x-1)}} = \lim_{x\to 1} x-4 = -3$$

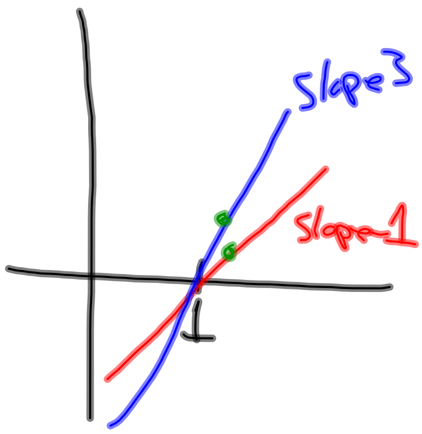
L'Hôpital's Rule

$$= \lim_{x\to 1}\frac{2x-5}{1}$$ plug in $= \frac{-3}{1}$

$$\lim_{x\to 1}\frac{x^2-5x+4}{x-1} = \frac{f(x)}{g(x)}$$



Approximate each func by line with same slope



Indeterminate Forms: All can be rewritten into form of 0/0 to use L'Hopital

$$\frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"} \qquad \frac{f(x)}{g(x)} = \text{"}\frac{\infty}{\infty}\text{"} \quad \frac{\frac{1}{g(x)}}{1/f(x)} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to\infty} \frac{x^2+3}{5x} = \lim_{x\to\infty} \frac{2x}{5} = \infty$$

$$f(x)\cdot g(x) = \text{"}\infty\cdot 0\text{"}$$

$$\underset{\text{"}\infty\cdot 0\text{"}}{x\cdot e^{-2x}} = \frac{x}{e^{2x}} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\lim_{t\to\infty} te^{-2t} = \lim_{t\to\infty} \frac{t}{e^{2t}} = \lim_{t\to\infty} \frac{1}{2e^{2t}} = \frac{1}{\infty} = 0$$

$$\lim_{t\to 0^+} t\ln(t)$$

(close to zero)(Big Negative)

$$\lim_{t\to 0^+} \frac{\ln(t)}{1/t} \quad \text{or} \quad \frac{t}{(1/\ln(t))}$$

harder

$$\lim_{t\to 0^+} \frac{(1/t)}{(-t^{-2})} = \lim_{t\to 0^+} \frac{1}{\cancel{t}}\cdot\frac{-\cancel{t^2}}{1} \quad -t \to 0$$

$$= \lim_{t\to 0^+} \frac{-t^2}{t}$$

$$= \lim_{t\to 0^+} \frac{-2t}{1} = 0$$

Sec. 4.4 L'Hopital's Rule and Indeterminate Forms

Suppose you are evaluating a limit (a fraction, product or exponent) by "plugging in" or finding the limits of each part separatly. The limit will have one of the following forms. Each "part" can approach either zero, infinity or a constant.

Exponential: Possible "Forms"

$$\lim_{x \to a} f(x)^{g(x)}$$

Results or "Indeterminate"

$$\lim_{x \to \infty}(x^7 - 6x^4 + 3x - 7)e^{(-3x)}$$

(Big) e^(Negative Big)

(Big)(very small)

$$\lim_{x \to \infty} \frac{x^7 - 6x^4 + 3x - 7}{e^{3x}}$$

L'H

$$\lim_{x \to \infty} \frac{7x^6 - 24x^3 + 3}{3e^{3x}} \quad \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{42x^5 - 72x^2}{9e^{3x}} \quad \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3^7 \cdot e^{3x}} = \frac{1}{\infty} = 0$$

one more time

$$\lim_{x \to \infty} \frac{0}{3^8 e^{3x}} = \frac{0}{\infty} = 0$$

$$\lim_{x\to\infty} \frac{135760600\,x^{375}}{e^{3x}} = 0$$

e^{3x} gets bigger faster

Indeterminate Forms: All can be rewritten into form of 0/0 to use L'Hopital

$$\lim_{t\to\infty} t e^{-2t}$$

$$\lim_{x \to 0} x^x$$

Back to Improper Integrals

Evaluate the following: (Graph it first)

$$\int_{-\infty}^{\infty} \frac{1}{\pi\left(1+x^2\right)}\,dx$$

Definition of Improper Integral II

If f(x) is continuous on [a,b) and discontinuous at x=b,

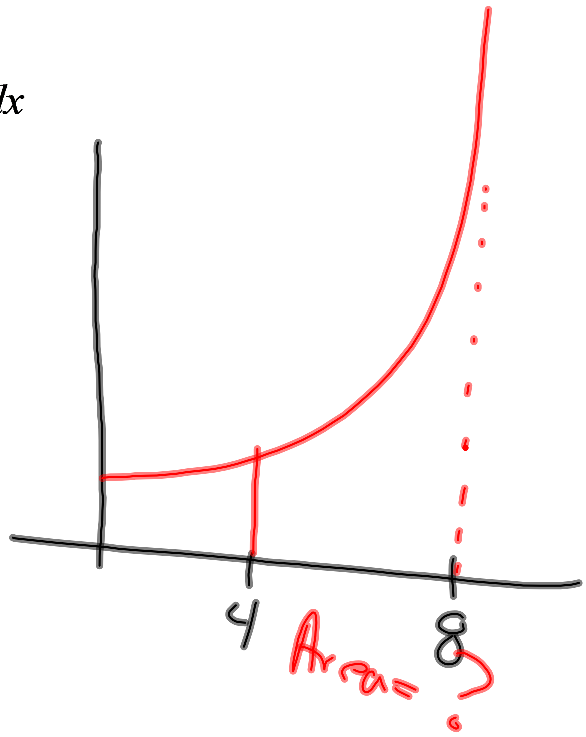
$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

Similar for continuous on (a,b].

The improper integral is <u>convergent</u> if

The improper integral is <u>divergent</u> if

$$\int_4^8 \frac{1}{\sqrt{8-x}}\,dx$$



Warning:

$$\int_4^8 \frac{1}{6-x}\,dx$$

$$\int_0^1 \ln(x)dx$$

$$\underset{x\to\infty}{Lim}\frac{x^7-1}{1000000x^6-5}$$