Bounding Anti-Ramsey numbers via Turán numbers

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Abstract

Let n be a positive integer and $\mathcal{H}$ a family of graphs. The *Turán number* $ex(n, \mathcal{H})$ is the maximum number of edges of an n-vertex graph that does not contain any member of $\mathcal{H}$ as a subgraph. The *Anti-Ramsey number* $AR(n, \mathcal{H})$ is the maximum number of colors in an edge-coloring of K_n that does not contain any member of $\mathcal{H}$ all of whose edges are colored differently. These two functions are closely related.

Given a graph H, let $\mathcal{H}^-$ denote the family of graphs obtainable from H by removing a single edge. Erdős, Simonovits, and Sós showed that always $1 + ex(n, \mathcal{H}^-) \leq AR(n, H) \leq ex(n, \mathcal{H}^-) + o(n^2)$, where the lower bound is achieved when $H = K_p$. Later results by various authors indicated that the lower bound is within $O(n)$ from the actual value of $AR(n, H)$ when H is a tree, a cyle, or the complete bipartite graph $K_{2,m}$.

Here, we sharpen the upper bound above for some special classes of graphs to show that the lower bound is tight or asymptotically tight for those classes of graphs. In the first part, we show that if H is a subdivided graph (a graph obtained from another graph by subdividing each edge once) then $AR(n, H) \leq ex(n, \mathcal{H}^-)+O(n)$. In the second part, we show that if H is a graph such that $\chi(H - e) \geq p + 1$ for each edge $e \in E(H)$ and there exist two edges e_1, e_2 of H for which $\chi(H - e_1 - e_2) = p$, then for sufficiently large n, $AR(n, H) = 1 + ex(n, \mathcal{H}^-) = t(n, p) + 1$ or $t(n, p) + p$, where $t(n, p)$ denotes the number of edges in the Turán graph $T_{n,p}$. This yields one of the very few exact results on Anti-Ramsey numbers. The proof uses the first stability theorem of Simonovits for the Turán problem.

The work in the second part of the talk is joint with Oleg Pikhurko.