Counting embeddings

Fedor Manin

University of Toronto

manin@math.toronto.edu

June 10, 2016

Encoding homotopical information

Let X and Y be finite simplicial complexes...

Fact

The number of simplicial maps from X to Y is bounded above by $|Y|^{|X|}$ (exponential in $|X|$.)

Theorem (Gromov)

Fix a simply connected Y and the homotopy type of X. Then the number of **homotopy classes** *of simplicial maps $X \to Y$ is a* **polynomial** $P(|X|)$.

(Contrast growth of fundamental groups)

???

I lied!

Here's Gromov's actual theorem*:

Theorem

Let X and Y be compact Riemannian manifolds with boundary, Y simply connected. Then the number of homotopy classes of L-Lipschitz maps $X \to Y$ is $O(L^\alpha)$, where α depends only on the rational homotopy of X and Y.

But the first formulation I gave is closely related via...

The quantitative simplicial approximation theorem

Any L-Lipschitz map between simplicial complexes X and Y is close to one which is simplicial on a subdivision of X at scale L.

*may not actually be a theorem of Gromov

A sketch of the proof of Gromov's theorem

Let $f : X \to Y$ be a map. Two key observations:

- Rational homotopy theory gives invariants classifying maps $f : X \to Y$ up to some finite torsion part. These are forms built from $f^*\omega_i$, for some finite set $\{\omega_i\}$, by repeated multiplication and antidifferentiation.
- A **coisoperimetric inequality**: every exact $\omega \in \Omega^n(Y)$ has an antidifferential $\alpha \in \Omega^{n-1}(Y)$ such that $\|\alpha\|_\infty \leq C_{n,Y} \|\omega\|_\infty$.

Therefore the obstructions classifying an L-Lipschitz map can't be more than polynomial in size.

Isoperimetric duality

- The **coisoperimetric inequality** we want: every exact $\omega \in \Omega^n(Y)$ has an antidifferential $\alpha \in \Omega^{n-1}(Y)$ such that

$$\|\alpha\|_\infty \le C_{n,Y}\|\omega\|_\infty.$$

- This has a dual **isoperimetric inequality**: every boundary $T \in \mathbf{N}_{n-1}(Y)$ has a filling $S \in \mathbf{N}_n(Y)$ such that

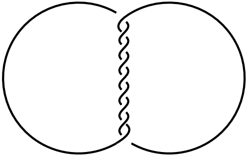
$$\mathrm{mass}(S) \le C_{n,Y}\mathrm{mass}(T).$$

  - This follows from the Federer–Fleming deformation theorem.
- **Isoperimetric duality** says the two constants are equal.
  - This is an application of the Hahn–Banach theorem.

What about embeddings?

Consider embeddings of a manifold M in a manifold N. Such an embedding can be complicated even if its Lipschitz constant is small. E.g.:

- A tiny but complicated knot in S^3
- Two linked S^n's in S^{2n+1}:

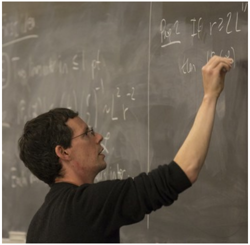


What geometric quantities might encapsulate the "complexity" of embeddings?

Thick embeddings

If we force our embeddings to have tubular (or regular) neighborhoods of radius $1/L$, that seems to limit the amount of information. This has not been studied much as far as I know, except for:

- Ropelength and physical knot theory (many authors)
- "Combinatorially" thick embeddings of simplicial complexes in $\mathbb{R}^n$ (Gromov–Guth)



C^2 conditions have a similar effect.

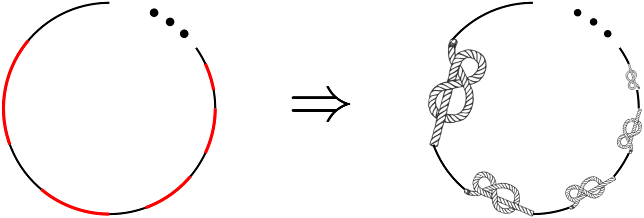
The bilipschitz constant: a weaker bound

We will say $f : X \to Y$ is *L-bilipschitz* if

$$\frac{1}{L}d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2).$$

(Some might call this L^2-bilipschitz.) What happens if we restrict embeddings to be L-bilipschitz?

- Links can't get too close
- With knots, though, you can do this:



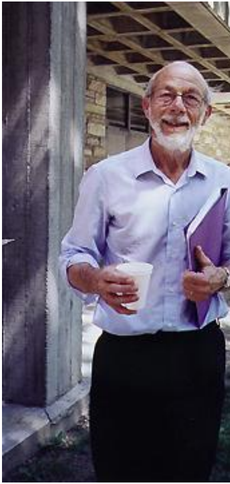
What can we say about bilipschitz embeddings $M^m \to N^n$?

It depends on codimension and category:

- When $n - m = 2$, there are always knots.
- When $n/3 \gtrsim n - m \geq 3$, there are smoothly knotted spheres in $\mathbb{R}^n$ (Haefliger); however, these are PL unknotted.

So in these two situations, there's an infinite number of isotopy classes with the same bilipschitz constant. On the other hand:

- When $n - m \neq 2$, the number of topological isotopy classes of embeddings with bilipschitz constant $\leq L$ is finite (Joshua Maher, unpublished thesis)



Reducing embedding theory to homotopy theory

Theorem (Haefliger)

When $2m > 3(n+1)$, isotopy classes of embeddings $M^m \to N^n$ correspond to homotopy classes of maps $F : M \times M \times [0,1] \to N \times N$ preserving the following structure:

(E1) *$F|_t$ is equivariant with respect to the involution $(x,y) \mapsto (y,x)$*

(E2) *$F|_{t=1}$ is isovariant* with respect to this involution*

(E3) *$F|_{t=0} = f \times f$ for some map $f : M \to N$.*

When $N = \mathbb{R}^n$, this reduces to $\mathbb{Z}/2\mathbb{Z}$-equivariant homotopy classes of maps

$$M \times M \setminus \Delta \to S^{n-1}.$$

*__isovariant__: preimages of fixed points are fixed points

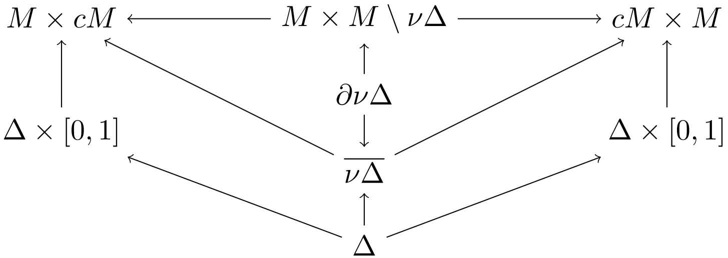
Homotopy theory of diagrams

These are not homotopy classes of maps between spaces! So Gromov's theorem doesn't directly apply.

Definition

Let $\mathcal{D}$ be a small category. A $\mathcal{D}-$*diagram of spaces* is a functor $\mathcal{D} \to \text{Top}$. These map to each other in the obvious way.

E.g., here's (part of) what's preserved in Haefliger's theorem:



Gromov's theorem for diagrams

Theorem (M.–Weinberger)

Let $\underline{X}$ and $\underline{Y}$ be free diagrams of simply connected spaces over a finite EI** category $\mathcal{D}$ such that tensor products of injective $\mathbb{Q}\mathcal{D}$-modules are injective. Then the number of homotopy classes of diagram maps $\underline{X} \to \underline{Y}$ which are objectwise L-Lipschitz is polynomial in L.*

Applications:

- Equivariant maps (here $\mathcal{D}$ is the **orbit category**)
- L-bilipschitz isovariant maps
- L-bilipschitz embeddings in the metastable range (as asserted by Gromov)

*the sort you can do homotopy theory with
in an **EI category, all endomorphisms are automorphisms

What's special about the metastable range?

Two things (apparently coincidentally.)

1. Generically, there are no triple intersections.
2. Homotopy classes of isovariant maps $TM \to TN$ are the same as homotopy classes of bundle monomorphisms $TM \to TN$.

The second of these is harder to deal with than the first...

The calculus of manifolds

Theorem (Goodwillie–Klein–Weiss)

There is a sequence of functors T^k from manifolds to diagrams(-ish) such that, if $n - m \geq 3$, then for every r there is a large enough $k = k(r, m, n)$ such that the map $\mathrm{Emb}(M, N) \to \mathrm{Map}(T^k M, T^k N)$ is r-connected.

This would be just what we need, but the "-ish" includes some tangential information.

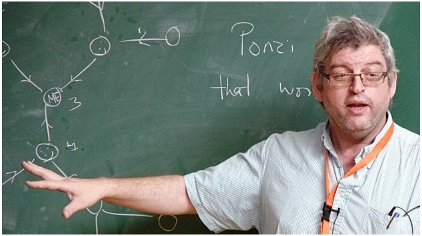


A conjecture

Conjecture (Ferry–Weinberger, 2013)

If M^m and N^n are (topological or PL) manifolds and $n - m \geq 3$, then the number of isotopy classes of L-bilipschitz embeddings $M \to N$ is polynomial in L.

Perhaps this can be proven by harnessing the calculus of manifolds in a clever way?



Thank you!