2.8 The Derivative as a Function

Definition - Given a function $f(x)$, we define the derivative of $f$ to be the function $f'(x)$ where

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for any $x$ values in which this limit exists.

Definition - A function $f$ is differentiable at $a$ if $f'(a)$ exists. It is differentiable on an open interval $(a,b)$ or $(a,\infty)$ if it is differentiable at every number in that interval.

Example: $f(x) = x^2 + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h \to 0} \frac{2xh}{h} = \lim_{h \to 0} 2x = 2x$$

Thus, $f'(x) = 2x$. Note: $f$ is differentiable on $(-\infty, \infty)$.
$f'(x)$ is a function. It is related to $f(x)$ in the following way:

Given $x = a$, if $a$ is in the domain of $f(x)$, then there is a point $(a, f(a))$ on the graph $y = f(x)$.

If there is a tangent line to the graph of $y = f(x)$ at $(a, f(a))$, then the slope of the tangent line is $f'(a)$.

Ex/ Use the graph of $y = f(x)$ to find $f'(2)$.

Ex/ Given the graph of $y = f(x)$, sketch a graph of $f'(x)$.
Ex: Use the definition of derivative to find \( f'(x) \).
\[
f(x) = ax + b
\]
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[a(x+h) + b] - [ax + b]}{h}
\]
\[
= \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} = \lim_{h \to 0} \frac{ah}{h} = \lim_{h \to 0} a = a
\]
So, \( f'(x) = a \)

Alternate definition of \( f'(x) \)
\[
f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}
\]

Using this definition we can find the derivative of \( f(x) = ax + b \) as well...
\[
f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{[at + b] - [ax + b]}{t - x}
\]
\[
= \lim_{t \to x} \frac{at + b - ax - b}{t - x} = \lim_{t \to x} \frac{at - ax}{t - x} = \lim_{t \to x} \frac{a(t-x)}{t-x} = \lim_{t \to x} a = a
\]
\[
f'(x) = a
\]

Note: \( f(x) = ax + b \) is a nonvertical straight line.

Does it make sense that the derivative is the constant \( f'(x) = a \)?

HINT: What is the slope of the line \( y = ax + b \)?

Is that slope the same for any point \((x, f(x))\) on the line?
NOTATION

Given \( y = f(x) \)

\( x \) is the independent variable

\( f'(x) \) is the derivative of \( f \) with respect to \( x \).

Alternate notation

\[
\frac{dy}{dx} \quad \frac{df}{dx}
\]

\[
\frac{d}{dx} f(x) \quad Df(x) \quad D_x f(x)
\]

\( \frac{d}{dx} \) is a differentiation operator

\( \frac{d}{dx} f(x) \) means “take the derivative with respect to \( x \), of \( f(x) \)”

When you take the derivative (if it exists) you get

\[
\frac{d}{dx} f(x) = \frac{df}{dx} \quad \text{the derivative}
\]

Similarly,

\[
\frac{d}{dx} y = \frac{dy}{dx}
\]

\[
\frac{d}{dx} y = y' \quad \frac{d}{dx} f(x) = f'(x)
\]

\( \frac{dy}{dx} \) is the name of a function, “the derivative of \( y \) with respect to \( x \)”

To represent an input into this function, we use the following notation:

\[
\left. \frac{dy}{dx} \right|_{x=a}
\]

“input a for \( x \) in \( \frac{dy}{dx} \)"
Ex/ \( f(x) = |x-2| \)

Note: \( f(x) = \begin{cases} x-2, & \text{if } x \geq 2 \\ -(x-2), & \text{if } x < 2 \end{cases} \)

If \( x > 2 \), then we can choose \( h \) small enough (while \( h \to 0 \)) so that \( x + h > 2 \)

Then for \( x > 2 \)

\[
P'(x) = \lim_{h \to 0} \frac{|(x+h)-2| - |x-2|}{h} = \lim_{h \to 0} \frac{x+h-2-x+2}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1
\]

If \( x < 2 \), then we can choose \( h \) small enough so that \( x + h < 2 \)

Then for \( x < 2 \)

\[
P'(x) = \lim_{h \to 0} \frac{|(x+h)-2| - |x-2|}{h} = \lim_{h \to 0} \frac{-(x+h-2) - -(x-2)}{h} = \lim_{h \to 0} \frac{-x-h+2+x-2}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1
\]

However, if \( x = 2 \), what is \( P'(2) \)?

\[
P'(2) = \lim_{h \to 0} \frac{|(2+h)-2| - |2-2|}{h} = \lim_{h \to 0} \frac{|h|}{h} \quad \text{DNE} \]

Consider two cases

\[
\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} h = \lim_{h \to 0^-} h = \lim_{h \to 0^-} h = 0
\]

So, \( \lim_{h \to 0^+} \frac{|h|}{h} = 1 \) and \( \lim_{h \to 0^-} \frac{|h|}{h} = -1 \)

Also, \( \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} -1 = -1 \)

So, \( \lim_{h \to 0} \frac{|h|}{h} \) DNE, so \( P'(2) \) does not exist

So, \( f \) is differentiable at all \( x \) except \( 2 \).
Ex: Given \( f(x) = \frac{3}{x^2} \) find \( \frac{df}{dx} \).

\[
P'(x) = \lim_{h \to 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} = \lim_{h \to 0} \frac{\frac{3x^2 - 3(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h \to 0} \frac{3x^2 - (3x^2 + 6xh + 3h^2)}{hx^2(x+h)^2}
\]

\[
= \lim_{h \to 0} \frac{3x^2 - 3x^2 - 6xh - 3h^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-6xh - 3h^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-6x - 3h}{x^2(x+h)^2}
\]

\[
= \lim_{h \to 0} \frac{-6x}{x^2} = \lim_{h \to 0} \frac{-6}{x^2} = -\frac{6}{x^2}
\]

So \( f'(x) = -\frac{6}{x^2} \) \[ \text{or } \frac{df}{dx} = -\frac{6}{x^2}, \frac{dy}{dx} = -\frac{6}{x^2} \]

Note: \( f \) is differentiable at all \( x \) except 0.

Compute \( f'(-3) \)

\[
f'(-3) = -\frac{6}{(-3)^2} = -\frac{6}{9} = -\frac{2}{3}
\]

in other notation, \( \left. \frac{df}{dx} \right|_{x=-3} = -\frac{6}{9} = \frac{2}{3} \)

or \( \left. \frac{dy}{dx} \right|_{x=-3} = -\frac{6}{9} = \frac{2}{3} \)

THEOREM - If \( f \) is differentiable at \( a \) then \( f \) is continuous at \( a \).

[Equivalently: If \( f \) is not continuous at \( a \) then \( f \) is not differentiable at \( a \).]

Ex/ \( f(x) = |x-2| \) is not differentiable at 2 but it is continuous at 2. [The theorem does not apply here, why?]

Ex/ \( f(x) = \frac{3}{x^2} \) is differentiable at all \( x \) except 0, thus (by theorem above) \( f \) is continuous at all \( x \) except 0.
Three important ways a graph can fail to be differentiable at \( x = a \),

1. \( f \) is discontinuous at \( a \) (thus \( f \) is not differentiable at \( a \))
   
   \[ \text{Example: } f(x) = \frac{1}{x-1}, \text{ at } x = 1 \]

2. \( f \) has a “sharp corner” or “kink” at \( x = a \)
   
   \[ \text{Example: } f(x) = |x-2| \text{ at } x = 2 \]

3. \( f \) has a vertical tangent at \( x = a \), that is
   
   \[ \lim_{x \to a} |f'(x)| = \infty \]
   
   \[ \text{Example: } f(x) = \sqrt[3]{x} \text{ at } x = 0 \]

**Ex.** Assume \( \lim_{x \to \frac{\pi}{3}} \frac{\sin x - \sin \frac{\pi}{3}}{x - \frac{\pi}{3}} \) represents the derivative of some function \( f \) at some number \( a \). Find \( f \) at \( a = \frac{\pi}{3} \)

Notice: It looks like the definition \( f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \) with \( a = \frac{\pi}{3} \)

and \( f(x) = \sin x \), since \( \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \).

So, \( f(x) = \sin x \) and \( f'(\frac{\pi}{3}) = \lim_{x \to \frac{\pi}{3}} \frac{\sin x - \sin \frac{\pi}{3}}{x - \frac{\pi}{3}} \).
Higher order derivatives.

If \( f \) is differentiable, then its derivative is \( f' \).
Then, \( f' \) is a function and \( f' \) might be differentiable.
If so, its derivative is \( (f')' \) or simply \( f'' \).
\( f'' \) is called the second derivative of \( f \).

Recall \( \frac{dy}{dx} = \frac{dy}{dx} \) 1st derivative

If we take the derivative twice:

\[
\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} 2nd \ derivative
\]

Three times yields:

\[
\frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3} 3rd \ derivative
\]

In general, the notations are as follows:

\[
\begin{array}{c|c|c|c}
\hline
y & f & \text{Function} \\
\hline
y' & \frac{dy}{dx} & f' & \frac{df}{dx} & 1st \ der. \\
\hline
y'' & \frac{d^2 y}{dx^2} & f'' & \frac{d^2 f}{dx^2} & 2nd \ der. \\
\hline
y''' & \frac{d^3 y}{dx^3} & f''' & \frac{d^3 f}{dx^3} & 3rd \ der. \\
\hline
y^{(n)} & \frac{d^n y}{dx^n} & f^{(n)} & \frac{d^n f}{dx^n} & n^{th} \ der. \\
\hline
\end{array}
\]

Note: \( \frac{d^2 y}{dx^2} \) means \( \frac{dy}{dx} \) was applied twice. It does not mean “squared”. \( \frac{d^2 y}{dx^2} \) is NOT \( \left( \frac{dy}{dx} \right)^2 \).
Example: \( f(x) = 2x^3 - 5x + 7 \)

Find \( f'(x) \).

[You try it...]

\[ f'(x) = 6x^2 - 5 \]

Find \( f''(x) \).

We take the derivative of \( f' \)

\[ f''(x) = (f'(x))' = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[6(x+h)^2 - 5] - [6x^2 - 5]}{h} \]

= \ldots \text{[You do it]} \ldots = 12x \]

So, \( f''(x) = 12x \)

Find \( f'''(x) \).

\[ \frac{d}{dx} f''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{12(x+h) - 12x}{h} = \ldots = 12 \]

So, \( f'''(x) = 12 \)

What is \( f^{(4)}(x) \)? \( f^{(5)}(x) \)?

Note: Given \( s(t) \) is a position function. Then \( v(t) = s'(t) \) is the velocity function.
Moreover, \( a(t) = v'(t) = s''(t) \) is the acceleration function. It is the rate of change of velocity with respect to time.