Problem 1  Compute the following limits

a) [12 pts] \[ \lim_{x \to -1} \left( \frac{2}{x+1} - \frac{8}{x^2 - 2x - 3} \right) \]

rewrite?

\[ \lim_{x \to -1} \left( \frac{2}{x+1} - \frac{8}{(x+1)(x-3)} \right) \]

\[ \lim_{x \to -1} \left( \frac{2(x-3)}{(x+1)(x-3)} - \frac{8}{(x+1)(x-3)} \right) \]

\[ \lim_{x \to -1} \left( \frac{2x - 6 - 8}{(x+1)(x-3)} \right) = \lim_{x \to -1} \frac{2x - 14}{(x+1)(x-3)} \]

\[ \frac{2(-1) - 14}{0 \cdot (-4)} = \frac{-16}{0} \]

DNE

Note: \( \lim_{x \to -1} \left( \frac{2x - 14}{(x+1)(x-3)} \right) = +\infty \); \( \lim_{x \to -1} \frac{2x - 14}{(x+1)(x-3)} = -\infty \)

Notice, if we have this instead

\[ \lim_{x \to -1} \left( \frac{2}{x+1} + \frac{8}{x^2 - 2x - 3} \right) \]

\[ = \quad \text{(similar steps)} \]

\[ = \lim_{x \to -1} \left( \frac{2(x-3) + 8}{(x+1)(x-3)} \right) \]

\[ = \lim_{x \to -1} \left( \frac{2x + 2}{(x+1)(x-3)} \right) \]

\[ = \lim_{x \to -1} \frac{2(x+1)}{(x+1)(x-3)} \]

\[ = \lim_{x \to -1} \frac{2}{x-3} = \frac{2}{-1-3} = \frac{1}{2} \]

b) [12 pts] \[ \lim_{t \to \infty} \frac{3 - 4t^2}{\sqrt{9t^4 - 2t + 6}} \]

rewrite

\[ \lim_{t \to \infty} \frac{3 - 4t^2}{\sqrt{9t^4 - 2t + 6}} \]

\[ \lim_{t \to -\infty} \frac{3 - 4t^2}{\sqrt{9t^4 - 2t + 6}} \]

\[ \frac{1}{t^2} > 0 \text{ so use } \sqrt{\frac{1}{t^4}} = \frac{1}{t^2} \]

\[ \lim_{t \to \infty} \frac{3}{t^2} - \frac{4t^2}{t^2} \]

\[ = \frac{0 - 4}{\sqrt{9 - 0 + 6}} = \frac{-4}{3} = \left( \frac{4}{3} \right) \]
Problem 2. Let \( f \) be given by \( f(x) = 2x^2 - 3x \) on the interval \([-2, 5]\).

a) [8 pts] Use the definition of the derivative as a limit to find a general expression for \( f'(a) \) in terms of \( a \). What is the domain of the function \( f'(x) \)?

\[
P'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[2(a+h)^2 - 3(a+h)] - [2a^2 - 3a]}{h}
\]

\[
= \lim_{h \to 0} \frac{2(a^2 + 2ah + h^2) - 3a - 3h - 2a^2 + 3a}{h}
\]

\[
= \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 - 3a - 3h - 2a^2 + 3a}{h}
\]

\[
= \lim_{h \to 0} \frac{2h^2 + 4ah - 3h}{h} = \lim_{h \to 0} \frac{h(2h + 4a - 3)}{h} = \lim_{h \to 0} (2h + 4a - 3) = 4a - 3
\]

b) [9 pts] Use the expression in part a) to find the equation of the tangent line at \( x = 1 \).

slope \( m = f'(1) = 4(1) - 3 = 1 \)

point \( (x, f(x)) \rightarrow (1, f(1)) = (1, 2(1)^2 - 3(1)) = (-1) \), so \((1, -1)\)

\[
y - y_1 = m(x - x_1)
\]

\[
y - (-1) = 1(x - 1) \]

\[
y + 1 = x - 1 \]

\[ y = x - 2 \]

So, \( P'(x) = 4x - 3 \)

C) [8 pts] Use part a) to find an expression for \( f''(a) \).

\[
f''(a) = (P'(a))' = \lim_{h \to 0} \frac{[h(2a+h-3) - 4a - 3]}{h}
\]

\[
= \lim_{h \to 0} \frac{4a + 4h - 3 - 4a + 3}{h} = \lim_{h \to 0} \frac{4h}{h} = \lim_{h \to 0} 4 = 4 \]

So, \( f''(x) = 4 \)
Problem 3  On the grid provided below sketch the graph of a function $f$, with $\text{Dom}(f) = \mathbb{R}$, satisfying the following properties:

- [6 pts] $f$ is continuous on $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$

- [5 pts] $\lim_{x \to -\infty} f(x) = -3$;  $\lim_{x \to \infty} f(x) = 4$  H.A.  $y = -3$, and $y = 4$

- [5 pts] $\lim_{x \to -3^-} f(x) = f(-3) = 2$;  $\lim_{x \to -3^+} f(x) = -\infty$  v.A.  $x = -3$

- [5 pts] $\lim_{x \to 2^-} f(x) = f(2) = 4$;  $\lim_{x \to 2^+} f(x) = \infty$  v.A.  $x = 2$

- [5 pts] $f$ is differentiable everywhere it is continuous except at $x = -1$.

"Make a sharp turn at $x = -1$"

Note: Your graph should be completely sketched on the interval $[-10, 10]$, and have all asymptotes clearly indicated with dotted lines.

**Note:** There are many possible correct graphs.
Problem 4  let \( f \) be given by

\[
f(x) = \begin{cases} 
    \frac{x^2 - 9}{x - 3} & \text{if } x < 3 \\
    \frac{(x+3)(x-3)}{x-3} & \text{if } 3 \leq x < 4 \\
    ax^2 - bx + 6 & \text{if } 3 \leq x < 4 \\
    x + 2a + b & \text{if } x \geq 4 
\end{cases}
\]

a) [8 pts] Using directional limits, find an equation that \( a \) and \( b \) must satisfy in order that \( f \) be continuous at \( x = 3 \).

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3^-} (x+3) = 6
\]

\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (ax^2 - bx + 6) = a(3)^2 - b(3) + 6
\]

Need \( \lim_{x \to 3} f(x) \) to exist, so need \( 9a - 3b + 6 = 6 \)

so, \( 3a = b \)

b) [8 pts] Using directional limits, find an equation that \( a \) and \( b \) must satisfy in order that \( f \) be continuous at \( x = 4 \).

\[
\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (ax^2 - bx + 6) = a(4)^2 - b(4) + 6 = 16a - 4b + 6
\]

\[
\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (x + 2a + b) = 4 + 2a + b
\]

Need \( \lim_{x \to 4} f(x) \) to exist, so need \( 4 + 2a + b = 16a - 4b + 6 \)

\[
14a - 5b + 2 = 0
\]

c) [9 pts] Using parts a) and b), determine all values of \( a \) and \( b \) (if any) for which \( f \) is continuous everywhere.

Solve the system

\[
\begin{align*}
3a &= b \\
14a - 5b + 2 &= 0
\end{align*}
\]

Substitute \( b \) = 2

\[
14a - 5(3a) + 2 = 0 \\
14a - 15a + 2 = 0 \\
a = 2
\]

Then \( b = 3(2) = 6 \)

\( b = 6 \)