Problem 1  Compute the following limits

a) [12 pts]  \[ \lim_{x \to -2} \left( \frac{2}{x+2} + \frac{10}{x^2 - x - 6} \right) \]

\[ \frac{2}{0} + \frac{10}{0} \quad \text{indeterminant} \]

so rewrite

\[ \lim_{x \to -2} \left( \frac{2}{x+2} + \frac{10}{(x+2)(x-3)} \right) \]

\[ \lim_{x \to -2} \left( \frac{2}{x+2} \cdot \frac{x-3}{x-3} + \frac{10}{(x+2)(x-3)} \right) = \lim_{x \to -2} \left( \frac{2(x-3) + 10}{(x+2)(x-3)} \right) \]

\[ \lim_{x \to -2} \left( \frac{2x - 6 + 10}{(x+2)(x-3)} \right) = \lim_{x \to -2} \left( \frac{2x + 4}{(x+2)(x-3)} \right) = \lim_{x \to -2} \left( \frac{2(x+2)}{(x+2)(x-3)} \right) \]

\[ \lim_{x \to -2} \frac{2}{x-3} = -\frac{2}{5} = -\frac{2}{5} \]

b) [12 pts]  \[ \lim_{t \to \infty} \frac{6 - 2t^3}{\sqrt{16t^6 - 5t^4 + 10}} \]

\[ \frac{\infty}{\infty} \quad \text{indeterminant} \]

rewrite

\[ \lim_{t \to \infty} \frac{\frac{6}{t^3} - 2}{\sqrt{16} - \frac{5}{t^2} + \frac{10}{t^6}} \]

\[ t \to \infty, \quad \text{so} \quad \frac{1}{t^3} < 0, \quad \text{so} \quad \frac{1}{t^3} = -\sqrt{\frac{1}{t^6}} \]

\[ \lim_{t \to \infty} \frac{\frac{6}{t^3} - 2}{\sqrt{16} - \frac{5}{t^2} + \frac{10}{t^6}} \]

\[ = \lim_{t \to \infty} \frac{\frac{6}{t^3} - 2}{\sqrt{16} - \frac{5}{t^2} + \frac{10}{t^6}} \]

\[ = \frac{0 - 2}{-4} = \frac{2}{-4} = \frac{1}{2} \]
Problem 2  Let \( f \) be given by \( f(x) = 3x^2 - 5x \) on the interval \([-1, 4]\).

a) [8 pts] Use a definition of the derivative as a limit to find a general expression for \( f'(x) \). What is the domain of the function \( f'' \)?

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 5(x+h) - [3x^2 - 5x]}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h - 3x^2 + 5x}{h}
\]

\[
= \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h} = \lim_{h \to 0} (6x + 3h - 5) = (6x - 5)
\]

Domain \((-1, 4)\) EXCLUDE ENDPOINTS

b) [9 pts] Use the expression in part a) to find the equation of the tangent line at \( x = 2 \).

At \( x = 2 \), \( f(2) = 3(2)^2 - 5(2) = 3 \cdot 4 - 10 = 2 \), so \((2, 2)\) on \( f'(x) \).

Slope of tangent at \( x = 2 \), \( f'(2) = 6(2) - 5 = 7 \), \( \frac{dy}{dx} = 7 \), \( m \)

So, \( y - (2) = (7)(x - (2)) \)

\[ y - 2 = 7x - 14 \]

\[ y = 7x - 12 \]

c) [8 pts] Use a definition of the derivative and part a) to find an expression for the function \( f'' \) (there will be no credit for simply using a formula to find \( f'' \)).

\[
f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{6(x+h) - 5 - [6x - 5]}{h}
\]

\[
= \lim_{h \to 0} \frac{6x + 6h - 5 - 6x + 5}{h} = \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6.
\]
Problem 3  On the grid provided below sketch the graph of a function \( f \), with \( \text{Dom}(f) = \mathbb{R} \), satisfying the following properties:

- [6 pts] \( f \) is continuous on \( (-\infty, -4) \cup (-4, 3) \cup (3, \infty) \)

- [5 pts] \( \lim_{x \to -\infty} f(x) = -4; \quad \lim_{x \to \infty} f(x) = 2 \)
  
  \[ \text{H.A. } y = -4 \quad \text{H.A. } y = 2 \]

- [5 pts] \( \lim_{x \to (-4)^-} f(x) = f(-4) = 3; \quad \lim_{x \to (-4)^+} f(x) = \infty \)
  
  \[ \text{left cont. at } -4 \quad \text{V.A. } x = -4 \]

- [5 pts] \( \lim_{x \to 3^+} f(x) = f(3) = 4; \quad \lim_{x \to 3^-} f(x) = -\infty \)
  
  \[ \text{right cont. at } 3 \quad \text{V.A. } x = 3 \]

- [5 pts] \( f \) is differentiable everywhere it is continuous except at \( x = 1 \).
  
  \[ \text{sharp turn at } x = 1 \]

Note: Your graph should be completely sketched on the interval \([-10, 10]\), and have all asymptotes clearly indicated with dotted lines.
Problem 4  let $f$ be given by

$$f(x) = \begin{cases} 
\frac{x^2 - 9}{x + 3} & \text{if } x < 3 \\
ax^2 - bx & \text{if } 3 \leq x < 4 \\
2x + 5a - b & \text{if } x \geq 4
\end{cases}$$

a) [8 pts] Using one-sided limits, find an equation that $a$ and $b$ must satisfy in order that $f$ be continuous at $x = 3$.

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (ax^2 - bx) = a(3)^2 - b(3) = 9a - 3b \]

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x + 3} = \frac{3^2 - 9}{3 + 3} = 0 \]

so, for $\lim_{x \to 3} f(x)$ to exist, we need $9a - 3b = 0$

b) [8 pts] Using one-sided limits, find an equation that $a$ and $b$ must satisfy in order that $f$ be continuous at $x = 4$.

\[ \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (2x + Sa - b) = 2(4) + Sa - b = 8 + Sa - b \]

\[ \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (ax^2 - bx) = a(4)^2 - b(4) = 16a - 4b \]

so, for $\lim_{x \to 4} f(x)$ to exist, we need $8 + Sa - b = 16a - 4b$

or $11a - 3b = 8$

c) [9 pts] Using parts a) and b), determine all values of $a$ and $b$ (if any) for which $f$ is continuous everywhere.

solve $9a - 3b = 0 \quad \rightarrow \quad 9a = 3b \quad \rightarrow \quad b = 3a$

$11a - 3b = 8$

$a = 4, b = 12$