Problem 1 [20 points] A lighthouse is located on a small island 4 km away from the nearest point $P$ on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when the point where the beam meets the shoreline is 2 km from $P$?

\[
\frac{dw}{dt} = (4 \text{ rev/min})(2\pi \frac{\text{rad}}{\text{rev}}) = 8\pi \frac{\text{rad}}{\text{min}}
\]

\[
\tan w = \frac{x}{4}
\]

\[
\sec^2 w \cdot \frac{dw}{dt} = \frac{1}{4} \frac{dx}{dt}
\]

\[
\frac{dx}{dt} = 4 \sec^2 w \frac{dw}{dt} = 32\pi \sec^2 w
\]

When $x = 2$

\[
\tan w = \frac{2}{4} = \frac{1}{2}
\]

\[
w = \tan^{-1}(\frac{1}{2})
\]

\[
\sec^2 w = \tan^2 w + 1 \rightarrow \sec^2 w = (\frac{1}{2})^2 + 1 = \frac{5}{4}
\]

\[
\therefore \frac{dx}{dt} \bigg|_{x=2} = 4 \left( \frac{5}{4} \right)(8\pi) \frac{\text{km}}{\text{min}} = 40\pi \frac{\text{km}}{\text{min}}
\]
Problem 2  Let $f(x) = x^3(x - 1)^2$ on the interval $[0, \frac{3}{2}]$.

a) [10 points] Find all critical numbers of $f$ on the interval $(0, \frac{3}{2})$.

$$f'(x) = 3x^2(x-1)^2 + x^3 \cdot 2(x-1)$$
$$= x^2(x-1)[3(x-1) + 2x]$$
$$= x^2(x-1)(5x - 3)$$

Solve $f'(x) = 0$:

$$x^2(x-1)(5x - 3) = 0$$

$$x = 0, 1, \frac{3}{5}$$

Critical numbers on $(0, \frac{3}{2})$:

$$0, \frac{3}{5}$$

b) [6 points] Use the First Derivative Test to determine whether any of the points found in part a) are local max or local min.

$f'(x)$:

\[ x \quad 0 \quad \frac{3}{5} \quad 1 \quad \frac{3}{2} \]

$f(x)$:

\[ f(0) = 0, f\left(\frac{3}{5}\right) = 0, f(1) = 0, f\left(\frac{3}{2}\right) = \frac{27}{32} \]

$f$ is continuous at $x = \frac{3}{5}$ and $x = 1$.

So, Relative max at $x = \frac{3}{5}$, $\left(\frac{3}{5}, \frac{108}{3125}\right)$

Relative min at $x = 1$, $\left(1, 0\right)$

c) [4 points] Determine the absolute max and min of $f$ on $[0, \frac{3}{2}]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2} - 1\right)^2 = \frac{27}{32}$</td>
</tr>
<tr>
<td>$\frac{3}{5}$</td>
<td>$\left(\frac{3}{5}\right)^3 \cdot \left(\frac{3}{5} - 1\right)^2 = \frac{108}{3125}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Problem 3 [20 points] On the coordinate system below, sketch the graph of a function with the following properties. All of the required properties (except the domain) should be explicitly labeled on the graph.

- The domain of $f$ is \( \{x \neq \frac{1}{2}, -\frac{3}{2}\} \), and $f$ is differentiable on its domain;
- $f$ has a local min at $x = 1$ and local max at $x = 3$;
- $f$ has a horizontal asymptote at $y = 1$ as $x \to \infty$, and at $y = 0$ as $x \to -\infty$;
- $f$ has a vertical asymptotes at $x = \frac{1}{2}$ and $x = -\frac{3}{2}$;
- $f$ has points of inflection at $x = 0, 2, 4$; I.P.'s
- $f$ is concave up on \((-\infty, -\frac{3}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 2), (4, \infty)\);
- $f$ is concave down on \((-\frac{3}{2}, 0), (2, 4)\).
Problem 4 For each of the following statements, indicate if is true or false, and give a justification for your answer. Answers without justification, whether correct or not, will receive no credit.

a) [10 points] If \( f \) is a continuous function on the open interval \((a, b)\), it must achieve a maximum and minimum value on that interval.
   
   \[ \text{FALSE.} \]

b) [10 points] If \( f \) is continuous on \([a, b]\), then there exists a number \( c \in (a, b) \) where the derivative \( f'(c) = \frac{f(b) - f(a)}{b - a} \).
   
   \[ \text{FALSE.} \] Does not say \( f \) is differentiable!
Problem 5 [20 points] Find the dimensions of the rectangle with the largest area that has its base on the x-axis, its other two vertices above the x-axis, and which lies on the parabola \( y = 12 - x^2 \). You must label all quantities used in your work.

\[
A(x) = 2x(12-x^2) = 24x - 2x^3
\]

Domain \( 0 < x < \sqrt{12} \)

\[A'(x) = 24 - 6x^2\]

Solve \( A'(x) = 0 \)

\[24 - 6x^2 = 0\]

\[6x^2 = 24\]

\[x^2 = 4\]

\[x = \pm 2\]

\[x = 2\]

Relative max at \( x = 2 \)

In fact, there is an Absolute Max at \( x = 2 \) (why?)

So width is \( 2(2) = 4 \)

Height is \( 12 - (2)^2 = 8 \)

4 by 8