MAXIMUM AND MINIMUM VALUES - EXTREME VALUES

Given a function $f$ with domain $D$.

Definition - $f$ has an absolute (or global) maximum at $c$ if $f(c) \geq f(x)$ for all $x$ in $D$. The number $f(c)$ is called the maximum value of $f$ on $D$.

Definition - $f$ has an absolute (or global) minimum at $c$ if $f(c) \leq f(x)$ for all $x$ in $D$. The number $f(c)$ is called the minimum value of $f$ on $D$.

Note: Maximum and minimum values of $f$ on $D$ are called extreme values of $f$ on $D$.

Note: The domain $D$ is important when considering extreme values.

Ex/ $f(x) = x^2$  

- Domain is assumed to be $(-\infty, \infty)$

- No Absolute Maximum (why?)

- Absolute minimum at $x=0$, $f(0) = 0$

- The minimum value of $f$ is 0

Ex/ $f(x) = x^2 + 1$, $[-1, 4]$  

- Domain given.

- The maximum value of $f$ is $f(4) = 4^2 + 1 = 17$

- The minimum value of $f$ is $f(-1) = (-1)^2 + 1 = 1$

Ex/ $f(x) = x^2 + 1$, $(-1, 4)$  

- Domain given

- No Absolute maximum (why?)

- The minimum value of $f$ is $f(0) = 0^2 + 1 = 1$
Definition - \( f \) has a local (or relative) maximum at \( c \) if \( f(c) \geq f(x) \) when \( x \) is "near" \( c \).

[More formally: "if there exists some open interval such that \( f(c) \geq f(x) \) for all \( x \) in that open interval."

Definition - \( f \) has a local (or relative) minimum at \( c \) if \( f(c) \leq f(x) \) when \( x \) is "near" \( c \).

Ex/ Given the graph of \( y = f(x) \) with domain \([a, b]\):

\[ \begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\end{array} \]

\( f \) has NO absolute minimum.

\( f \) has NO relative minimum at \( a \).

\( f \) has a relative maximum at \( c \), the value is \( f(c) \).

\( f \) has a relative minimum at \( d \), the value is \( f(d) \).

\( f \) has a relative minimum at \( g \), the value is \( f(g) \).

\( f \) has a relative maximum at \( h \), the value is \( f(h) \).

\( f \) has a relative minimum at \( i \), the value is \( f(i) \).

\( f \) has no relative extrema at \( j \).

\( f \) has a relative (and more so, an absolute) maximum at \( k \), the maximum value of \( f \) is \( f(k) \).

\( f \) has a relative minimum at \( b \), value is \( f(b) \).
THE EXTREME VALUE THEOREM

If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

**Example:**

- If \( f \) is continuous and \( f \) is an "unbroken piece of string," then "there is a highest point on the string and a lowest point." If \( f \) is not continuous on a closed interval \([a, b]\), then No Abs Max.
  
  Though, there is an Abs Min at \( c \).

- If \( f \) is not continuous on a closed interval \([a, b]\), then No Abs Max, No Abs Min.
Ex/ $P$ continuous
Not a closed interval, $(a, b)$

No Abs Min.
Though Abs Max at $c$

Ex/
$P$ continuous
Not a closed interval, $[a, b)$

No Abs Max.
Though Abs Min at $c$

**FERMAT'S THEOREM**
If $P$ has a relative max or min at $c$ and $P'(c)$ exists,
then $P'(c) = 0$.

NOTE: The following is **not** true in general.

If $P'(c) = 0$ then $P$ has a relative max or min.
**NO.**

Ex/

$P'(c) = 0$ but no relative max or min

Relative max at $b$ and $P'(b)$ exists, thus $P'(b) = 0$
Relative min at $c$ but $P'(c)$ DNE.
NOTE: If there is a relative extreme, then either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

So, if we wish to narrow our search for places on \( f(x) \) where a relative extreme occurs, then we need only look for places where \( f'(x) = 0 \) or \( f'(x) \DNE \).

**Definition:** A critical number of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \DNE \).

Hence,

If \( f \) has a relative max or min at \( c \), then \( c \) is a critical number.

**Ex:** Find the critical numbers of the function

1) \( f(x) = x^2 - 6x \)

\[ f'(x) = 2x - 6 \]

When is \( f'(x) = 0 \)?

When is \( f'(x) \DNE \)?

\[ f'(x) = 0 \]

Solve \( 2x - 6 = 0 \)

\[ x = 3 \]

\[ f'(x) \DNE \]

\[ 2x - 6 \] is always defined
2) \( f(x) = \frac{1}{x} \) or \( x^{-1} \)

\[ f'(x) = -x^{-2} \]

\[ \frac{f'(x) = 0}{\text{solve } -\frac{1}{x^2} = 0} \]

- \( x = 0 \) never

\[ \frac{f'(x) \text{ DNE}}{-\frac{1}{x^2} \text{ is undefined when } x = 0} \]

**But**, \( x = 0 \) is **not** in the **domain** of \( f \), so it is **not** a critical number.

3) \( f(x) = |x| \)

\[ f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \]

\[ \frac{f'(x) = 0}{-1 = 0} \]

- \( 1 = 0 \) never

\[ \frac{f'(x) \text{ DNE}}{\text{at } x = 0, \text{ and } x = 0 \text{ is in the domain of } f, \text{ so } x = 0 \text{ is a critical number}} \]

4) \( f(x) = 3\sqrt{x} \) or \( x^{\frac{1}{3}} \)

\[ f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \]

\[ \frac{f'(x) = 0}{\text{solve } \frac{1}{3x^{\frac{2}{3}}} = 0} \]

- \( 1 = 0 \) **no**

\[ \frac{f'(x) \text{ DNE}}{\frac{1}{3x^{\frac{2}{3}}} \text{ is undefined when } x = 0 \text{ (in domain)}} \]
5) \( f(x) = ax^2 + bx + c \quad (a \neq 0) \) Quadratic

\[
P(x) = 2ax + b
\]

\[
P'(x) = 0 \quad \text{Solve} \quad 2ax + b = 0 \quad x = -\frac{b}{2a}
\]

\[
P'(x) \, \text{DNE} \quad \text{NEVER}
\]

6) \( f(x) = \tan x - x \)

\[
P'(x) = \sec^2 x - 1
\]

\[
P'(x) = 0 \quad \text{solve} \quad \sec^2 x - 1 = 0 \quad \frac{1}{\cos^2 x} - 1 = 0
\]

\[
1 - \cos^2 x = 0 \quad \cos^2 x = 1
\]

\[
\cos x = \pm 1
\]

\[
\text{when} \quad \cos^2 x = 0, \quad \text{solve}
\]

\[
\cos x = 0 \quad \rightarrow \quad x = \frac{\pi}{2} + k\pi
\]

\[
\text{but} \quad \frac{\tan x - x}{\cos x - x} \quad \text{so} \quad x = \frac{\pi}{2} + k\pi \quad \text{are not in domain of} \quad f
\]

\[
\text{Not critical points}
\]

7) \( f(x) = \ln |x^2 + 1| \)

\[
P'(x) = \frac{1}{x^2 + 1} \cdot 2x
\]

\[
P'(x) = 0 \quad \frac{2x}{x^2 + 1} = 0 \quad 2x = 0 \quad x = 0
\]

\[
P'(x) \, \text{DNE} \quad \frac{2x}{x^2 + 1} \, \text{DNE when} \quad x^2 + 1 = 0, \quad \text{never}
\]
8) \( f(x) = x^2 - 6x, \quad 0 \leq x \leq 2 \)  
   \[ \text{Domain:} \quad [0, 2] \]  
   \[ f'(x) = 2x - 6 \]

   \[ f'(x) = 0 \]  
   \[ 2x - 6 = 0 \quad x = 3 \quad \text{NOT IN DOMAIN} \]

   \[ f'(x) \text{ undefined} \]  
   \[ 2x - 6 \text{ always defined.} \]

   However, due to the domain, \( f \) is  
   NOT differentiable at \( x = 0 \) and \( x = 2 \).

THE CLOSED INTERVAL METHOD

To find the Absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\)

1. Find the critical numbers of \( f \) THAT ARE IN THE INTERVAL \((a, b)\)

2. Evaluate \( f \) at the critical numbers inside \((a, b)\) AND at the ENDPOINTS \( a \) and \( b \).

3. Compare the values of \( f \) found. The largest is the Abs Max value, the least value is the Abs Min value.
Ex/ Find the Abs Max and Min of \( f(x) = 3x - 2 \) on \([-1, 4]\)

1st \( f \) is cont on \([-1, 4]\)? \( \checkmark \)
2nd \([-1, 4]\) is a closed interval? \( \checkmark \)

Critical numbers of \( f \):

\[
\begin{array}{c|c|c|c}
\theta & f(\theta) & \text{abs max} & \text{abs min} \\
--- & --- & --- & --- \\
-\pi/6 & -\pi/6 & -\pi/6 & -\pi/6 \\
& & & \\
0 & 0 & 0 & 0 \\
& & & \\
\pi/6 & \pi/6 & \pi/6 & \pi/6 \\
\end{array}
\]

The Abs Max value is 10. It occurs at \( x = 4 \)
The Abs Min value is -5. It occurs at \( x = -1 \)

Ex/ Find the Abs Max and Min of \( f(\theta) = \sin^2 \theta \) on \([-\pi/2, \pi/6]\)

1st \( f \) is cont on \([-\pi/2, \pi/6]\)? \( \checkmark \)
2nd closed interval? \( \checkmark \)

Critical numbers \( f'(\theta) = 2 \sin \theta \cos \theta \)

\[
\begin{align*}
f'(\theta) &= 0 & \text{solve } 2 \sin \theta \cos \theta &= 0 \\
\sin \theta &= 0 & \Rightarrow & \theta = k \pi & k \text{ any integer} \\
\cos \theta &= 0 & \Rightarrow & \theta = \pm \pi/2 + k \pi & \text{integer} \\
\end{align*}
\]

which ones are INSIDE \((-\pi/4, \pi/6)\)

only \( \theta = 0 \)

2nd \( f'(\theta) \) is always defined

\[
\begin{array}{c|c|c|c}
\theta & f(\theta) & \text{abs max} & \text{abs min} \\
--- & --- & --- & --- \\
-\pi/4 & -\pi/4 & -\pi/4 & -\pi/4 \\
& & & \\
-\pi/6 & -\pi/6 & -\pi/6 & -\pi/6 \\
& & & \\
0 & 0 & 0 & 0 \\
& & & \\
\pi/6 & \pi/6 & \pi/6 & \pi/6 \\
\end{array}
\]

Abs Max Abs Min
Ex/ Abs Max/Min of $f(x) = (x+1)e^x$, $-3 \leq x \leq 0$

$f$ cont. ✓ closed interval ✓

Critical Numbers: $f'(x) = e^x + (x+1)e^x = (x+2)e^x$

$f'(x) = 0$ Solve $(x+2)e^x = 0$

$e^x = 0$ or $x + 2 = 0$

$\downarrow$ no

$x = -2$

$f'(x)$ DNE never

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>-3</td>
<td>$f(-3) = -2e^{-3}$</td>
</tr>
<tr>
<td>0</td>
<td>$f(0) = 1$ ← Abs Max</td>
</tr>
<tr>
<td>-2</td>
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Ex/ Abs Max/Min of $f(x) = \frac{1}{x^2 + 1}$ on $[0, 1]$

continuous ✓ closed ✓

$f'(x) = -(x^2+1)^{-2}(2x) = \frac{-2x}{(x^2+1)^2}$

$f'(x) = 0$ solve $\frac{-2x}{(x^2+1)^2} = 0$

$-2x = 0$ $x = 0$

NOT "inside" $(0, 1)$

$f'(x)$ DNE never

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