Rolle's Theorem

Let \( f \) be a function that satisfies the following three hypotheses:
1. \( f \) is continuous on the closed interval \([a, b]\)
2. \( f \) is differentiable on the open interval \((a, b)\)
3. \( f(a) = f(b) \)

Then there is at least one number \( c \) in \((a, b)\) such that \( f'(c) = 0 \)

Proof: Let \( f \) be a function that satisfies 1, 2, and 3.

Case 1: \( f(x) = k \) a constant on \([a, b]\)

Then \( f'(x) = 0 \) so \( c \) can be taken as any number in \((a, b)\).

Case 2: \( f(x) > f(a) \) for some \( x \) in \((a, b)\).

By the Extreme Value Theorem, \( f \) has a maximum value somewhere in \([a, b]\). Since \( f(a) = f(b) \), the max cannot occur at \( a \) or \( b \). So \( f \) attains the maximum value at some \( c \) in \((a, b)\). Moreover \( f \) is differentiable at \( c \), hence \( f'(c) = 0 \).

Case 3: \( f(x) < f(a) \) for some \( x \) in \((a, b)\).

[Similar argument to case 2 except we use the minimum value of \( f \) to find \( c \).]

Ex/ Consider \( y \)

\[
\begin{align*}
\text{Graph:} & \quad f(x) = 2 \\
\text{Points:} & \quad f(-1) = f(4) \\
& \quad f \text{ cont on } [-1, 4] \\
& \quad f \text{ diff on } (-1, 4)
\end{align*}
\]

Thus \( f'(c) = 0 \) for some \( c \) in \((-1, 4)\).

In fact, for this example \( f'(c) = 0 \) for all \( c \) in \((-1, 4)\).
Ex/ \( f(x) = \cos(3x) \) on \([0, \frac{\pi}{3}]\)

\[
\begin{align*}
\phi(0) &= 1 = \phi\left(\frac{\pi}{3}\right) \\
\phi &\text{ is cont on } [0, \frac{\pi}{3}] \\
\phi &\text{ is diff on } (0, \frac{\pi}{3})
\end{align*}
\]
Thus, \( \phi'(c) = 0 \) for some \( c \) in \( (0, \frac{\pi}{3}) \).

In fact, \( \phi'(\frac{\pi}{6}) = 0 \), \( \phi'(\frac{2\pi}{3}) = 0 \), and \( \phi'(\pi) = 0 \).

Ex/ Prove that \( p(x) = x^3 - 5 \) has exactly one root in the interval \([1, 2]\).

1st - \( p(1) = 1^3 - 5 = -4 < 0 \)
and \( p(2) = 2^3 - 5 = 3 > 0 \)
Now, \( p \) is continuous on \([1, 2]\), so by the Intermediate Value Theorem, there is some \( c \) in \((1, 2)\) where \( p(c) = 0 \).
Thus, there is AT LEAST one root of \( x^3 - 5 \) in \([1, 2]\).

2nd - We need to prove that there is no other root.

\( \Rightarrow \) Suppose \( c_2 \) is another root, \( c_1 \neq c_2 \), in \([1, 2]\).

Then \( p(c_2) = 0 \) and so \( p(c) = p(c_2) \)
and \( p \) is continuous on \([c, c_2]\) (or \([c_2, c]\))
and \( p \) is differentiable on \((c, c_2)\) (or \((c_2, c)\)).

Thus, by Rolle's Theorem, \( p'(d) = 0 \) for some \( d \) in the interval (which is inside \((1, 2)\)).

Note \( p'(x) = 3x^2 \)
so, \( 3d^2 = 0 \). But then \( d = 0 \) which is NOT in \((1, 2)\).

So \( \Rightarrow \) cannot be true.
Example: \( f(x) = 1 \times x \) on \([-2, 2]\)

Note: \( f(-2) = 1 \times (-2) = -2 \)
and \( f(2) = 1 \times 2 = 2 \) and \( f \) is continuous on \([-2, 2]\)

But, there is no \( c \) in \((-2, 2)\) where \( f'(c) = 0 \).

This does not contradict Rolle's Theorem because \( f \) is not differentiable everywhere in \((-2, 2)\).

\( f'(c) \neq 0 \) due to:

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

(or equivalently, \( f(b) - f(a) = f'(c)(b - a) \))
proof - Assume $f$ satisfies the hypotheses.

Define the function $h$ on $[a, b]$ as follows:

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} (x-a)$$

[constants $\%$]

Then, $h$ is continuous on $[a, b]$ (because $f$ and $x$ are continuous on $[a, b]$)

Also, $h$ is differentiable on $(a, b)$ and

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

[we called this $M_{AB}$]

Next,

$$h(a) = f(a) - f(a) - M_{AB}(a-a) = 0 - M_{AB}(0) = 0$$

$$h(b) = f(b) - f(a) - M_{AB}(b-a) = f(b) - f(a) - [f(b) - f(a)] = 0$$

Thus $h(a) = h(b)$, so $h$ satisfies the three hypotheses for Rolle’s Thm.

Hence, $h'(c) = 0$ for some $c$ in $(a, b)$.

And so,

$$f'(c) - \frac{f(b) - f(a)}{b-a} = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

for some $c$ in $(a, b)$.

Ex/ Given $f$ is differentiable on $(-1, 4)$ and continuous on $[-1, 4]$ with $f(-1) = 6$ and $f(4) = -24$.

Then we can apply the MVT to claim

$$f'(c) = \frac{f(4) - f(-1)}{4 - (-1)} = \frac{-24 - 6}{5} = -6$$

for some $c$ in $(-1, 4)$.  

//
Ex/ Given \( f(x) = x^2 + x - 2 \) on \([-1, 3]\)

Find the average (or mean) slope of the function on this interval.

\[
\begin{align*}
  f(-1) &= (-1)^2 + (-1) - 2 = -2 & A & (-1, -2) \\
  f(3) &= (3)^2 + (3) - 2 = 10 & B & (3, 10)
\end{align*}
\]

Then \( M_{AB} = \frac{f(3) - f(-1)}{3 - (-1)} = \frac{10 - (-2)}{4} = 3 \)

The MVT applies to our function on this interval (why?) so there is some \( c \) in \((-1, 3)\) where \( f'(c) = 3 \).

Find all such \( c \).

\[ f'(x) = 2x + 1 \]

solve \( f'(c) = 3 \)

\[
\begin{align*}
  2c + 1 &= 3 \\
  2c &= 2 \\
  c &= 1
\end{align*}
\]

check that the value(s) are in the interval \((-1, 3)\).

\[
\begin{align*}
  f'\left(\frac{\pi}{2}\right) &= \sin t \\
  \text{solve } &\cos c = \frac{2}{\pi} \\
  c &= \cos^{-1}\left(\frac{2}{\pi}\right) + 2k\pi \quad \text{and} \quad -\cos^{-1}\left(\frac{2}{\pi}\right) + 2k\pi \quad (k \text{ any integer})
\end{align*}
\]

We want \( c \) in \((0, \frac{\pi}{2})\). Only one value works

\[ c = \cos^{-1}\left(\frac{2}{\pi}\right) \]
Ex/ \( \frac{f(4) - f(1)}{4 - 1} = \frac{\frac{1}{4} - 1}{3} = \frac{-\frac{3}{4}}{3} = -\frac{1}{4} \) 

By MVT there is some \( c \) in \((-1, 4)\) such that \( f'(c) = -\frac{1}{4} \).

Find all such \( c \).

\( f'(x) = \frac{d}{dx}x^{-1} = -x^{-2} \)

solve \( f'(c) = -\frac{1}{4} \)

\(-\frac{1}{c^2} = -\frac{1}{4} \)

\(-4 = -c^2 \quad c^2 = 4 \)

\( c = \pm 2 \)

But only \( c = 2 \) is in \((1, 4)\).

Ex/ Given \( f(0) = 4 \) and \( f'(x) \leq 2 \) for all \( x \). How large can \( f(7) \) be at most?

Note: We are given \( f \) is diff (and so cont.) for all \( x \), in particular, for all \( x \) in \([0, 7]\).

So, we can apply MVT and say that

\( f'(c) = \frac{f(7) - f(0)}{7 - 0} = \frac{f(7) - 4}{7} \) for some \( c \) in \((0, 7)\)

So, \( 7f'(c) = f(7) - 4 \)

\( f(7) = 7f'(c) + 4 \)

Since \( f'(c) \leq 2 \), we see that

\( f(7) = 7 \cdot f'(c) + 4 \leq 7 \cdot 2 + 4 = 18 \)
THEOREM

If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\),

then \( f \) is constant on \((a, b)\).

Think about it.
Take ANY \( x_1, x_2 \) in \((a, b)\).

By MVT (why?)

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)
\]

so \( 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = 0 \)

\( f(x_1) = f(x_2) \) \( \forall x_1, x_2 \) in \((a, b)\).

COROLLARY

If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a, b)\),

then \( f - g \) is constant on \((a, b)\); that is

\( f(x_1) = g(x_1) + C \) for some constant \( C \).

Think about it.
Consider \( F(x_1) = f(x_1) - g(x_1) \).

Then \( F'(x) = f'(x_1) - g'(x_1) = 0 \) by hypotheses.

So...?
Ex/ A "proof" that \( \cos^2 \theta + \sin^2 \theta = 1 \)

Let \( f(\theta) = \cos^2 \theta + \sin^2 \theta \)

Then \( f'(\theta) = 2 \cos \theta (-\sin \theta) + 2 \sin \theta \cos \theta \)
\[ = -2 \sin \theta \cos \theta + 2 \sin \theta \cos \theta \]
\[ = 0 \]

Thus \( f(\theta) \) is a constant.

But what constant?

Choose \( \theta = 0 \) to compute

\[ f(0) = \cos^2(0) + \sin^2(0) = 1^2 + 0^2 = 1 \]

So, the constant is 1.

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

Ex/ If \( f'(x) = 2x \) for all \( x \), what can we say about \( f(x) \)?

Notice: Given \( g(x) = x^2 \)

We know \( g'(x) = 2x \) for all \( x \)

So \( P'(x) = g'(x) \) for all \( x \)

Thus by the corollary, \( P(x) = g(x) + C \)

\( f(x) = x^2 + C \) for some constant \( C \).