The radius, \( r \), of a sphere is changing in size over time according to
\[
r(t) = t^3 + 2t + 1
\]
where \( r \) is in inches and \( t \) is in seconds.

Find
1. Volume at \( t = 2 \)
2. How fast is the radius growing at \( t = 2 \)?
3. How fast is the volume growing at \( t = 2 \)?
4. How fast is the surface area growing at \( t = 2 \)?

(1) Volume of a sphere: \( V = \frac{4}{3} \pi r^3 \) \[\text{[known this?]}\]

Need \( r \) at \( t = 2 \):
\[ r(2) = (2)^3 + 2(2) + 1 = 13 \text{ in.} \]

So,
\[
V = \frac{4}{3} \pi (13)^3 = \frac{8788}{3} \pi \text{ in.}^3 = 9,203 \text{ in.}^3
\]

(2) Rate in change of \( r \) with respect to \( t \)
\[
\frac{dr}{dt} = \frac{d}{dt}(t^3 + 2t + 1) = 3t^2 + 2
\]
Then
\[
\frac{dr}{dt} \bigg|_{t=2} = 3(2)^2 + 2 = 14 \text{ in./sec}, \quad \text{[units?]}
\]

(3) Rate in change of \( V \) with respect to \( t \)
\[
\frac{dV}{dt} \quad ?
\]
\[
V = \frac{4}{3} \pi r^3
\]
So,
\[
\frac{dV}{dt} = \frac{d}{dt} \left[ \frac{4}{3} \pi r^3 \right] = \frac{4}{3} \pi 3r^2 \frac{dr}{dt}
\]
\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

[\text{Ok...}]
\[
\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = \left( \frac{d}{dr} \frac{4}{3}\pi r^3 \right) \left( \frac{dr}{dt} \right)
\]
\[
= 4\pi r^2 \frac{dr}{dt}
\]

Then, at \( t = 2 \)

\[
\left. \frac{dV}{dt} \right|_{t=2} = 4\pi (13)^2 \frac{dr}{dt} = 676\pi \cdot 14 = 9464\pi\text{ in}^3/\text{sec.}
\]

\( r(2) = 13 \) found in previous part

4. Surface area of a sphere: \( S = 4\pi r^2 \)

Then \( \frac{dS}{dt} = \frac{d}{dt} [4\pi r^2] = 4\pi \frac{d}{dt} r^2 = 4\pi 2r \frac{dr}{dt} \) (chain rule)

At \( t = 2 \)

\[
\left. \frac{dS}{dt} \right|_{t=2} = 4\pi \cdot 2(13) \frac{dr}{dt} = 4\pi \cdot 26 \cdot 14 = 1456\pi\text{ in}^2/\text{sec.}\]
A particle is traveling along the graph of $y = x^2 + 1$.

Note: The distance from the particle to the origin is

$$w = \sqrt{(x-0)^2 + (y-0)^2}$$

or simply

$$w = \sqrt{x^2 + y^2}$$

Note: The angle $\theta$ measured counterclockwise from the positive $x$-axis to the particle's position is given by

$$\tan \theta = \frac{y}{x}$$

Given: At $t = -1$, $x(-1) = 2$ and $\frac{dx}{dt} \bigg|_{t=-1} = -5$ (t in seconds)

Find
1) How fast is the $y$-coordinate changing at $t = -1$?
2) How fast is the particle moving away from the origin at $t = -1$?
3) How fast is $\theta$ changing at $t = -1$?

Find $\frac{dy}{dt}$.
\[
\frac{dy}{dt} = \frac{1}{x^2 + 1} \cdot \frac{dx}{dt}
\]

Then, at \( t = -1 \)

\[
\left. \frac{dy}{dt} \right|_{t=-1} = 2 \cdot \left. \frac{dx}{dt} \right|_{t=-1} = 2 \cdot 2 \cdot (-5) = -20 \text{ units/sec}
\]

\( y \) is decreasing at a rate of 20 units/sec.

2) Find \( \frac{dw}{dt} \)

\[
w = (x^2 + y^2)^{1/2}
\]

So,

\[
\frac{d}{dt}[w] = \frac{d}{dt}[(x^2 + y^2)^{1/2}] = \frac{1}{2} (x^2 + y^2)^{-1/2} \frac{d}{dt}(x^2 + y^2)
\]

\[
\frac{dw}{dt} = \frac{1}{2} (x^2 + y^2)^{-1/2} \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right)
\]

So,

\[
\frac{dw}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}}
\]

Then at \( t = -1 \), we will need \( x, y, \frac{dx}{dt}, \frac{dy}{dt} \)

\( x(-1) = 2 \) given, \[ \frac{dx}{dt} \bigg|_{t=-1} = -5 \) given

\[ \frac{dy}{dt} \bigg|_{t=-1} = -20 \) found in previous part
well, \( y = x^2 + 1 \)

so \( y = (2)^2 + 1 = 5 \)

Then, at \( t = -1 \)

\[
\frac{dw}{dt}\bigg|_{t=-1} = \frac{(2)(-5) + (5)(-20)}{\sqrt{(2)^2 + (5)^2}}
\]

\[
= \frac{-110}{\sqrt{29}} \text{ units/sec}
\]

the particle is moving "away" from the origin at \( -\frac{110}{\sqrt{29}} \text{ units/sec} \)

or...

the particle is moving TOWARDS the origin at \( \frac{110}{\sqrt{29}} \text{ units/sec} \).

3. Find \( \frac{d\theta}{dt} \).

\[
\tan \theta = \frac{y}{x}
\]

so,

\[
\frac{d}{dt}[\tan \theta] = \frac{d}{dt} \left[ \frac{y}{x} \right]
\]

\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2}
\]

\[
\frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2} \cos^2 \theta
\]
Then, at $t = -1$

we need $x, y, \frac{dx}{dt}, \frac{dy}{dt}$, and $\theta$

$x(-1) = 2$, $y(-1) = 5$, $\frac{dx}{dt} \bigg|_{t=-1} = -5$, $\frac{dy}{dt} \bigg|_{t=-1} = -20$

$\theta$?

$\tan \theta = \frac{y}{x}$ so at $t = -1$, $\tan \theta = \frac{5}{2}$

so $\theta = \tan^{-1} \left( \frac{5}{2} \right)$

Thus,

$\cos \theta = \frac{2}{\sqrt{29}}$

So,

$\frac{d\theta}{dt} \bigg|_{t=-1} = \frac{(2)(-20) - (5)(-5)}{(2)^2} \left( \frac{2}{\sqrt{29}} \right)^2$

$= \frac{-40 + 25}{4 \cdot 29} = \frac{-15}{29}$ radians/sec.

$\theta$ is DECREASING at a rate of $\frac{15}{29}$ rad/sec.
Conical Tank

Water is leaking out through the bottom at a rate of 6 m³/hour.

1) How fast is the water level falling when the level is at 5 m?
2) How fast is the water level falling when the volume of water in the tank is 2 m³?

Volume? Cone → \( V = \frac{1}{3} \pi r^2 h \)

We need \( \frac{dh}{dt} \).

NOTE: We are given \( \frac{dV}{dt} \). The volume of water is decreasing at 6 m³/hour, so \( \frac{dV}{dt} = -6 \) m³/hr

To find \( \frac{dh}{dt} \) we can use \( V = \frac{1}{3} \pi r^2 h \)

\[
\frac{1}{dt}[V] = \frac{1}{dt}[\frac{1}{3} \pi r^2 h]
\]

... both are functions of \( t \) ...
\[ \frac{dv}{dt} = \frac{1}{3} \pi \left[ 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right] \] but what is \( r' \)?

Use similar triangles

\[ \frac{r}{h} = \frac{1}{24} \]

So, \( r = \frac{1}{4} h \)

So, we could rewrite the volume first.

\[ V = \frac{1}{3} \pi \left( \frac{1}{4} h \right)^2 h \]

\[ V = \frac{1}{48} \pi h^3 \]

Then

\[ \frac{dv}{dt} = \frac{d}{dt} \left[ \frac{1}{48} \pi h^3 \right] \]

\[ \frac{dv}{dt} = \frac{\pi h^2}{16} \frac{dh}{dt} \]

[try getting this from * and **]

So,

\[ \frac{dh}{dt} = \frac{16}{\pi h^2} \frac{dv}{dt} \]

\[ \frac{dh}{dt} \text{ when } h = 5? \]

\[ \frac{dh}{dt} \bigg|_{h=5} = \frac{16}{\pi (5)^2} (-6) = -\frac{96}{25\pi} \text{ m/hr} \]

“Falling” at \( \frac{96}{25\pi} \text{ m/hr} \)
2. \( \frac{dh}{dt} \) when \( V = 2 \)?

\[ h = \frac{1}{18} \pi h^3 \]

So,
\[ Q = \frac{\pi}{18} h^3 \quad h^3 = \frac{96}{\pi} \]

\[ h = 3 \sqrt{\frac{96}{\pi}} \]

Then

\[ \frac{dh}{dt} = \frac{16}{\pi} \left( \frac{\pi}{3} \frac{\sqrt{96}}{\sqrt{\pi}} \right) (-6) = \frac{-96}{\pi} \frac{\sqrt{96}}{\pi} = \frac{-96}{\pi^2} = \frac{-3 \sqrt{96}}{\pi} \text{ m/hr} \]

"Falling" at \( \frac{3 \sqrt{96}}{\pi} \text{ m/hr} \)

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Example:

Sphere: \( V_s = \frac{4}{3} \pi R^3 \)

Cone (of water): \( V_c = \frac{1}{3} \pi r^2 h \)

Given Sphere: \( \frac{dR}{dt} = -3 \text{ cm/s} \)

Cone: \( r = \frac{1}{2} h \)

Liquid (water) volume is (a constant) \( 16 \pi \text{ cm}^3 \)

1. Find how fast the level of water in the cone is rising when \( R = 2 \) cm.

So, find \( \frac{dh}{dt} \) when \( R = 2 \).

**Note:** \( V_s + V_c \) is the total water volume

So,
\[ V_s + V_c = 16 \pi \text{ cm}^3 \]
Thus, 
\[ \frac{4}{3} \pi R^3 + \frac{1}{12} \pi (\frac{1}{2}h)^2 h = 16 \pi \]

\[ \text{given} \]

\[ \frac{4}{3} \pi R^3 + \frac{1}{12} \pi h^3 = 16 \pi \]

Implicit differentiation yields

\[ \frac{d}{dt} \left[ \frac{4}{3} \pi R^3 + \frac{1}{12} \pi h^3 \right] = \frac{d}{dt} [16 \pi] \]

\[ 4 \pi R^2 \frac{dR}{dt} + \frac{\pi}{4} h^2 \frac{dh}{dt} = 0 \]

So,

\[ \frac{dh}{dt} = - \frac{4 \pi R^2 \frac{dR}{dt}}{\frac{\pi}{4} h^2} \]

or

\[ \frac{dh}{dt} = - \frac{16 R^2}{h^2} \frac{dR}{dt} \]

At \( R=2 \), we need \( h \) and \( \frac{dR}{dt} \).

Given: \( \frac{dR}{dt} = -3 \text{ cm/s} \)

\( h? \) well, \( \frac{1}{3} \pi (2)^3 + \frac{1}{12} \pi h^3 = 16 \pi \)

so \( \frac{1}{12} \pi h^3 = 16 \pi - \frac{3^2}{3} \pi \)

\[ h^3 = 64 \]

\[ h = 4 \]

Then

\[ \frac{dh}{dt} \bigg|_{R=2, \ h=4} = - \frac{16 (2)^2}{(4)^2} (-3) = 12 \text{ cm/s} \]

\[ \bigg( + \bigg), \text{ so } h \text{ is increasing}. \]