RELATED RATES

Instantaneous rates of change take the form of derivatives.

For example, \( \frac{dy}{dx}, \frac{dy}{dt}, \frac{dx}{dt}, \frac{dx}{dy}, \frac{dt}{dy}, \text{etc...} \)

When we are dealing with only 2 quantities, say \( x \) and \( y \), which are related by some equation, we can try to find \( \frac{dy}{dx} \) (or \( \frac{dx}{dy} \)) by our rules of differentiation and, if needed, using implicit differentiation.

Ex/ \( y = \cos(x^2) \)

\[ \frac{dy}{dx} = -3x^2 \sin(x^2) \]

Ex/ \( x^2 + y^4 + y = 1 \)

\[ \frac{d}{dx} [x^2 + y^4 + y] = \frac{d}{dx} [1] \]

\[ 2x + 4y^3 \frac{dy}{dx} + \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{2x}{4y^3 + 1} \]

When we have 3 or more quantities (say \( x, y, \) and \( t \)), then there are several "rates" \( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dt}{dx}, \frac{dx}{dy}, \frac{dy}{dt}, \frac{dt}{dy} \), that might be found. These rates can be said to be "related" via the chain rule.
NOTE:
\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\quad \frac{dx}{dt} = \frac{dy}{dx} \frac{dy}{dt}
\]
\[
\frac{d}{dt} = \frac{d}{dx} \frac{dx}{dt}
\quad \frac{d}{dt} = \frac{d}{dy} \frac{dy}{dt}
\]
\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}
\quad \frac{dt}{dx} = \frac{dt}{dy} \frac{dy}{dx}
\]

Possible "related rates"

Ex/
\[y = \cos x, \quad x = t^3 - \sin t\]

Find \(\frac{dy}{dt}\). ← we need \(y\) vs. \(t\)

1. 
   or
   
   we need \(\frac{dx}{dt}\) and \(\frac{dx}{dt}\)

2. 
   or
   
   \(\frac{dx}{dt}\) and \(\frac{dy}{dx}\)

3. 
   or
   
   \(\frac{dy}{dx}\) and \(\frac{dt}{dx}\)

4. 
   or sometimes try implicit differentiation

5. 
   
   or substitute \(x = t^3 - \sin t\)

\[y = \cos (t^3 - \sin t)\]

\[
\frac{dy}{dt} = \frac{d}{dt} \cos (t^3 - \sin t) = -\sin (t^3 - \sin t) \frac{d}{dt} (t^3 - \sin t)
\]

\[
= -\sin (t^3 - \sin t) (3t^2 - \cos t)
\]

6. 
   
   Note:
   
   \[\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}\]

\[
\frac{dy}{dt} = \frac{dx}{dt} \cos x = -\sin x
\]

\[
\frac{dt}{dx} = \frac{dt}{t^3 - \sin t} = 3t^2 - \cos t
\]

So, \[\frac{dy}{dt} = (-\sin x) (3t^2 - \cos t)\]
\( \frac{dx}{dt} = 3t^2 - \cos t \)

\( \frac{dy}{dx} \) ?

\( y = \cos x \rightarrow x = \cos^{-1} y \)

\( \frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}} \)

\( \frac{dy}{dt} = \left(3t^2 - \cos t\right) \left(-\frac{1}{\sqrt{1-y^2}}\right) \)

\( \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \)

\( \frac{dt}{dx} \) ?

\( x = t^2 - \sin t \)

\( \frac{d}{dx} [x] = \frac{d}{dx} [t^2 - \sin t] \)

\( \frac{dt}{dx} = \frac{1}{3t^2 - \cos t} \)

\( \frac{dy}{dt} = -\sin x \left(3t^2 - \cos t\right) \)

\( y = \cos x \) \( \frac{dy}{dt} \) ?

\( \frac{d}{dt} [y] = \frac{d}{dt} [\cos x] \quad \text{chain rule?} \)

\[ \frac{dy}{dt} = -\sin x \frac{dx}{dt} \]

though we still need \( \frac{dx}{dt} \)

Compare the results of 1-5. Show they all agree.
Ex/ Given \( z^2 + x^2 + 3y^2 = 10 \)

where \( x, y, z \) are differentiable functions of \( t \).
Write \( \frac{dx}{dt} \) in terms of \( x, y, z, \frac{dx}{dt}, \) and \( \frac{dy}{dt}, \) use implicit diff.

\[
\frac{dt}{d}[z^2 + x^2 + 3y^2] = \frac{dt}{d}[10]
\]

\[
2z \frac{dz}{dt} + 2x \frac{dx}{dt} + 6y \frac{dy}{dt} = 0
\]

\[
2z \frac{dz}{dt} = -2x \frac{dx}{dt} - 6y \frac{dy}{dt}
\]

\[
\frac{dz}{dt} = -\frac{2x \frac{dx}{dt} + 6y \frac{dy}{dt}}{2z}
\]

Ex/ Given \( z^2 + x^2 + 3y^2 = 10 \)

where \( y \) and \( z \) are differentiable functions of \( x \).
Write \( \frac{dz}{dx} \) in terms of \( x, y, z, \frac{dy}{dx} \), use implicit diff.

\[
\frac{dx}{d}[z^2 + x^2 + 3y^2] = \frac{dx}{d}[10]
\]

\[
2z \frac{dz}{dx} + 2x + 6y \frac{dy}{dx} = 0
\]

\[
2z \frac{dz}{dx} = -2x - 6y \frac{dy}{dx}
\]

\[
\frac{dz}{dx} = \frac{-2x - 6y \frac{dy}{dx}}{2z}
\]
Ex! Given a rectangle that is changing in size where
\[ \frac{dw}{dt} = 5 \text{ in/ sec} \quad \text{and} \quad \frac{dl}{dt} = 2 \text{ in/ sec}. \]

At time \( t = 4 \) sec, we are given width \( w = 6 \) in and length \( l = 10 \) in.

Find the rate at which the AREA is changing at time \( t = 4 \).

Find \( \frac{dA}{dt} \) when \( t = 4 \).

Can we write the area, \( A \), in terms of \( t \)?

No matter, we want \( \frac{dA}{dt} \).

We know \( \frac{dw}{dt}, \frac{dl}{dt} \). Also, \( A = l \cdot w \).

So, using implicit differentiation,
\[ \frac{dl}{dt} [A] = \frac{d}{dt} [l \cdot w] \]

product rule?

\[ \frac{dA}{dt} = l \frac{dw}{dt} + w \frac{dl}{dt} \]

GENERAL expression

Then, find \( \frac{dA}{dt} \) when \( t = 4 \)

\[ \left. \frac{dA}{dt} \right|_{t=4} = (10 \text{ in}) (5 \text{ in/sec}) + (6 \text{ in}) (2 \text{ in/sec}) = 62 \text{ in}^2/\text{sec} \]
A particle is moving along the graph of \(x^2 + y^2 = 25\) when the particle is at the point \((3, 4)\), the \(y\)-coordinate is decreasing at a rate of 6 units/sec. How fast is the \(x\)-coordinate changing at that instant?

Find \(\frac{dx}{dt}\) when the particle is at \((3, 4)\).

**Note:** we are given \(\frac{dy}{dt}\) when the particle is at \((3, 4)\).

**1st** Find the GENERAL expression for \(\frac{dx}{dt}\).

\[
x^2 + y^2 = 25
\]

\[
\frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}[25]
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

\[
\frac{dx}{dt} = -\frac{y}{x}, \quad \frac{dy}{dt}
\]

**2nd** Find the value of \(\frac{dx}{dt}\) at the SPECIFIC moment when \((x, y)\) is \((3, 4)\).

\[
\left.\frac{dx}{dt}\right|_{x=3, y=4} = \left(-\frac{4}{3}\right) \cdot (-6) = 8 \text{ units/sec}
\]
A street lamp is mounted at the top of a 10 ft. tall pole. A dog 2 ft tall walks directly at the pole with a speed of 3 ft/s. How fast is the tip of his shadow moving when he is 8 ft. from the pole?

**TIP** - Draw diagrams

```
10 ft.

right angle

x

2 ft.
```

**TIP** - Introduce notation, variables.

"speed of dog (moving towards pole) is 3 ft/s"  
rate of change in [ ] with respect to time

let $x =$ distance from dog to pole (feet)

$t =$ time (seconds)

Then $\frac{dx}{dt} = -3 \text{ ft/s} \quad [\text{why negative?!}]$

"tip of shadow"

options $h =$ distance from dog to tip of shadow (ft)

$y =$ distance from pole to tip of shadow (ft)

[Note: $y = x + h$]
However, \( \frac{dy}{dt} \) is how fast the tip of the shadow is moving relative to the dog’s position. We want \( \frac{dx}{dt} \), how fast the tip is moving relative to the “stationary” pole.

\[ y = x + h \]

so, \( \frac{dy}{dt} = \frac{dx}{dt} + \frac{dh}{dt} \)

\[ \frac{dx}{dt} = \frac{1}{4} \cdot \frac{dx}{dt} + \frac{1}{4} \cdot \frac{dy}{dt} \]

\[ \frac{dy}{dt} = \frac{5}{4} \cdot \frac{dx}{dt} \]

\[ \frac{dx}{dt} = \frac{1}{4} \cdot \frac{dy}{dt} \]

Either option leads us to

\[ \frac{dy}{dt} = \frac{5}{4} \cdot \frac{dx}{dt} \quad [\text{GENERAL EXPRESSION}] \]
Then, find \( \frac{dy}{dt} \) when "the dog is 8 ft from pole", when \( x = 8 \).

\[
\left. \frac{dy}{dt} \right|_{x=8} = \frac{8}{4} \left( -3 \right) = -\frac{15}{4} \text{ ft/s}
\]

\[
\left. \frac{dx}{dt} \right|_{x=8}
\]

Negative? It means \( y \) is decreasing, so the tip is moving TOWARDS the pole.

The tip of the shadow is moving at \( \frac{15}{4} \text{ ft/s} \) TOWARDS the pole.

EX/ A plane flies horizontally at an altitude of 6 miles passing directly over an observer on the ground.

when the angle of elevation is \( 45^\circ \), the angle is decreasing at a rate of \( 36^\circ \) per minute.

How fast is the plane traveling at that time?

![Diagram of plane and observer](image)

We want \( \frac{dx}{dt} \) (in GENERAL first)

**Note:**

\[
\tan \theta = \frac{6}{x} \quad \text{or} \quad \tan \theta = 6x^{-1} \\
\theta \text{ in RADIANS?}
\]

Then,

\[
\frac{d}{dt} \left[ \tan \theta \right] = \frac{d}{dt} \left[ 6x^{-1} \right] \quad \theta \text{ in RADIANS ?}
\]

\[
\sec^2 \theta \frac{d\theta}{dt} = -6x^{-2} \frac{dx}{dt}
\]

So,

\[
\frac{dx}{dt} = -\frac{x^2 \sec^2 \theta}{6} \frac{d\theta}{dt}
\]
Then, find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$ [45° in Radians] 

$$\left. \frac{dx}{dt} \right|_{\theta = \frac{\pi}{4}} = \frac{(-6)^2 \sec^2\left(\frac{\pi}{4}\right) \left(-\frac{\pi}{5}\right)}{6}$$

$\theta = \frac{\pi}{4}$ \hspace{1cm} 36° per min.

$x!$ use, $\tan \theta = \frac{6}{x}$ so $x = \frac{6}{\tan \theta}$

When $\theta = \frac{\pi}{4}$, $x = \frac{6}{\tan \frac{\pi}{4}} = \frac{6}{1} = 6$

So,

$$\left. \frac{dx}{dt} \right|_{\theta = \frac{\pi}{4}, \ x = 6} = \frac{(-6)^2 \sec^2\left(\frac{\pi}{4}\right) \left(-\frac{\pi}{5}\right)}{6} \approx 452 \text{ mph}$$