1) (20 points) The base of a certain solid is the region between the graphs of

\[ x = 6y^3 + 5y^2 \quad \text{and} \quad x = 3y^2, \quad 0 \leq y \leq 4 \]

Cross-sections perpendicular to the y-axis are semicircles with diameter along the base. Find the volume of the solid. [Note: This is not a solid of revolution nor half of one.]

Intersections
\( (0,0) \quad \left(\frac{1}{3}, -\frac{1}{3}\right) \)

The radius of the semicircle is given by

\[ r(y) = \frac{1}{2} [6y^3 + 5y^2 - (3y^2)] = 3y^3 + y^2 \]

The area of the semicircle is

\[ A(y) = \pi \left(3y^3 + y^2\right)^2 \]

The volume is given by

\[ V = \int_0^4 A(y) \, dy = \int_0^4 \pi \left(3y^3 + y^2\right)^2 \, dy = \int_0^4 \pi \left(9y^6 + 6y^5 + y^4\right) \, dy \]

\[ = \left(\frac{9\pi}{7}y^7 + \pi y^6 + \frac{\pi}{5}y^5\right) \bigg|_0^4 = \frac{9\pi}{7}4^7 + \pi 4^6 + \frac{\pi}{5}4^5 \]
2) (20 points) A hemispherical tank (bottom hemisphere) whose radius is 11 ft is filled to half its depth with a liquid whose density is 55 lb/ft³.

(a) What is the volume of the liquid in the tank?

(b) Find the work done in pumping the liquid from the tank out over its top.
3) (20 points) Find the average value of the function \( f(x) = x \sin(4x) \sin(5x) \) over the interval \([0, \pi]\).
(Hint: the trig identities on the attached formula sheet may be helpful.)

\[
\frac{1}{\pi} \int_{0}^{\pi} x \sin(4x) \sin(5x) \, dx
\]

\[
= \frac{1}{\pi} \left[ x \left( \frac{1}{8} \sin(-x) - \frac{1}{16} \sin(9x) \right) \right]_{0}^{\pi} - \int_{0}^{\pi} \left( \frac{1}{8} \sin(-x) - \frac{1}{16} \sin(9x) \right) \, dx
\]

\[
= \frac{1}{\pi} \left[ \pi(0 - 0) - 0(0 - 0) - \left( -\frac{1}{2} \cos(-x) + \frac{1}{162} \cos(9x) \right) \right]_{0}^{\pi}
\]

\[
= \frac{1}{\pi} \left[ -\left( -\frac{1}{2} \cos(-\pi) + \frac{1}{162} \cos(9\pi) \right) + \left( -\frac{1}{2} \cos(0) + \frac{1}{162} \cos(0) \right) \right]
\]

\[
= \frac{1}{\pi} \left[ -\left( \frac{1}{2} - \frac{1}{162} \right) + \left( -\frac{1}{2} + \frac{1}{162} \right) \right] = -\frac{80}{81 \pi}
\]
4) (20 points) Find the following integral
\[
\int \sin^4(2x) \cos^3(2x) \, dx
\]

Let \( u = \sin(2x) \)
\[
du = 2\cos(2x) \, dx
\]
\[
dx = \frac{du}{2\cos(2x)}
\]

Then the integral becomes
\[
\int u^4 \cos^2(2x) \cos(2x) \frac{du}{2\cos(2x)}
\]

\[
= \int u^4 (1 - \sin^2(2x)) \frac{1}{2} \, du
\]

\[
= \int u^4 (1 - u^2) \frac{1}{2} \, du
\]

\[
= \frac{1}{2} \int u^4 - \frac{1}{2} u^6 \, du
\]

\[
= \frac{1}{10} u^5 - \frac{1}{14} u^7 + C
\]

\[
= \frac{1}{10} \sin^5(2x) - \frac{1}{14} \sin^7(2x) + C
\]
5) (20 points) Find an appropriate trigonometric substitution of one of the forms $x = C\sin(\theta)$, $x = C\sec(\theta)$, or $x = C\tan(\theta)$ to simplify the following integral: $\int \frac{x^4}{\sqrt{2 + x^2}} \, dx$. Leave your answer as an INDEFINITE integral of trig. functions of $\theta$. (Simplify this integral so as to eliminate the square root but do NOT evaluate this integral.)

\[
1 + \tan^2 \theta = \sec^2 \theta \quad \Rightarrow \quad 2 + 2\tan^2 \theta = 2\sec^2 \theta
\]

want $x^2 = 2\tan^2 \theta$

\[
x = \sqrt{2}\tan \theta
\]

\[
dx = \sqrt{2} \sec^2 \theta \, d\theta
\]

\[
\int \frac{x^4}{\sqrt{2 + x^2}} \, dx = \int \frac{(\sqrt{2}\tan \theta)^4}{\sqrt{2 + 2\tan^2 \theta}} \sqrt{2} \sec^2 \theta \, d\theta
\]

\[
= \int \frac{4\tan^4 \theta \sqrt{2} \sec^2 \theta}{\sqrt{2} \sec \theta} \, d\theta = \int 4\tan^4 \theta \sec \theta \, d\theta
\]
Possibly Helpful Trigonometric Identities

1) \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \)

2) \( \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \)

3) \( \sin(A) \cos(B) = \frac{1}{2} \left[ \sin(A + B) + \sin(A - B) \right] \) with special case \( \sin(A) \cos(A) = \frac{1}{2} \sin(2A) \)

4) \( \sin(A) \sin(B) = \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right] \)

5) \( \cos(A) \cos(B) = \frac{1}{2} \left[ \cos(A + B) + \cos(A - B) \right] \)

6) \( \cos^2(\theta) + \sin^2(\theta) = 1 \)

7) \( \sec^2(\theta) - \tan^2(\theta) = 1 \)