Math 254 ~ Functions of Several Variables

13.1 – Introduction to Functions of Several Variables

Definition – Function of Two Variables

Let $D$ be a set of ordered pairs of real numbers. If to each ordered pair $(x, y)$ in $D$ there corresponds a unique real number $f(x, y)$, then $f$ is called a **function of $x$ and $y$**. The set $D$ is the **domain** of $f$, and the corresponding set of values for $f(x, y)$ is the **range** of $f$.

Note: Given $z = f(x, y)$, $x$ and $y$ are called the **independent variables** and $z$ is called the **dependent variable**.

Note: A function of $n$ variables has a domain, $D$, consisting of ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and for each $(x_1, x_2, \ldots, x_n)$ in $D$ there corresponds a unique real number $f(x_1, x_2, \ldots, x_n)$.

Note: Given an equation describing a function of several variables, unless otherwise restricted, we assume that the domain is the set of all points for which the equation is defined.

Note: Given two $n$-variable functions $f$ and $g$ we can form the sum, difference, product and quotient of two functions as follows:

$$
(f \pm g)(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) \pm g(x_1, x_2, \ldots, x_n)
$$

$$
(fg)(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_n)
$$

$$
\frac{f}{g}(x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n)}{g(x_1, x_2, \ldots, x_n)}, \quad g(x_1, x_2, \ldots, x_n) \neq 0
$$

Note: Given an $n$-variable function $h$ and a single variable function $g$ we can form the composite function as follows:

$$
(g \circ h)(x_1, x_2, \ldots, x_n) = g(h(x_1, x_2, \ldots, x_n))
$$
Note: A function that can be written as a sum of functions of the form $cx^m y^n$ (where $c$ is a real number and $m$ and $n$ are nonnegative integers) is called a **polynomial function** of two variables. A **rational function** is the quotient of two polynomial functions.

**Definition** – The **graph** of a function $f$ of two variables is the set of all points $(x, y, z)$ for which $z = f(x, y)$ and $(x, y)$ is in the domain of $f$.

Note: The graph of a function $f$ of $n$ variables is the set of all points $(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$ where $(x_1, x_2, \ldots, x_n)$ is in the domain of $f$.

Note: To sketch a surface in space, it is useful to use traces in planes parallel to the coordinate planes. Given $z = f(x, y)$, one can sketch the trace of the surface in the plane $z = c$ (graphed on an $xy$-coordinate system.) Also, one can sketch the trace of the surface in the plane $y = c$ or $x = c$ (graphed on an $xz$-coordinate system or $yz$-coordinate system respectively.)

Note: A scalar field assigns the scalar $z = f(x, y)$ to the point $(x, y)$. A scalar field can be characterized by level curves (or contour lines) along which the value of $f(x, y)$ is constant.

Note: If $f$ is a function of three variables and $c$ is a constant, the graph of the equation $f(x, y, z) = c$ is a level surface of the function $f$. 
13.2 – Limits and Continuity

**Definition** – The $\delta$-neighborhood about $(x_0, y_0)$ is the disk centered at $(x_0, y_0)$ with radius $\delta > 0$.

- **Open disk:** $\left\{ (x, y) \mid \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \right\}$
- **Closed disk:** $\left\{ (x, y) \mid \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq \delta \right\}$

**Definition** – A point $(x_0, y_0)$ in a plane region $R$ is an **interior point** of $R$ if there exists a $\delta$-neighborhood about $(x_0, y_0)$ that lies entirely in $R$. If every point in $R$ is an interior point, then $R$ is an **open region**. A point $(x_0, y_0)$ is a **boundary point** of $R$ if every open disk centered at $(x_0, y_0)$ contains points inside $R$ and points outside $R$. If a region contains all of its boundary points, the region is **closed**.

Note: A region that contains some but not all of its boundary points is neither open nor closed.

Note: Similar definitions are used in higher dimensions. The 3-dimensional analogy to an open disk is the open sphere.

**Definition – Limit of a Function of Two Variables**

Let $f$ be a function of two variables defined, except possibly at $(x_0, y_0)$, on an open disk centered at $(x_0, y_0)$, and let $L$ be a real number. Then

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$  

Note: The phrase “whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$” can be read as “for all points $(x,y) \neq (x_0,y_0)$ in the open disk of radius $\delta$ centered at $(x_0,y_0)$.”

Note: $(x,y) \to (x_0,y_0)$ means that the point $(x,y)$ is allowed to approach $(x_0,y_0)$ from any direction. If the value of $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ is not the same for all possible paths to $(x_0,y_0)$, the limit does not exist.
**Definition – Continuity of a Function of Two Variables**

A function $f$ of two variables is **continuous at a point** $(x_0, y_0)$ in an open region $R$ if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as $(x, y)$ approaches $(x_0, y_0)$. That is,

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0)$$

The function $f$ is **continuous in the open region** $R$ if it is continuous at every point in $R$.

**Note:** If one can remove a discontinuity at $(x_0, y_0)$ by redefining $f(x_0, y_0)$, then the discontinuity is called **removable**. Otherwise, the discontinuity is **nonremovable**.

**Theorem – Continuous Functions of Two Variables**

If $k$ is a real number and $f$ and $g$ are continuous at $(x_0, y_0)$, then the following functions are continuous at $(x_0, y_0)$.

1. Scalar multiple: $kf$
2. Sum and Difference: $f \pm g$
3. Product: $fg$
4. Quotient: $f/g$, if $g(x_0, y_0) \neq 0$

**Note:** It follows that polynomial and rational functions are continuous at every point in their domains.

**Theorem – Continuity of a Composite Function**

If $h$ is continuous at $(x_0, y_0)$ and $g$ is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at $(x_0, y_0)$. That is,

$$\lim_{(x,y) \to (x_0,y_0)} g(h(x,y)) = g(h(x_0,y_0)).$$

**Definition – Continuity of a Function of Three Variables**

A function $f$ of three variables is **continuous at a point** $(x_0, y_0, z_0)$ in an open region $R$ if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as $(x, y, z)$ approaches $(x_0, y_0, z_0)$. That is,

$$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0).$$

The function $f$ is **continuous in the open region** $R$ if it is continuous at every point in $R$.

**Note:** It is sometimes useful to use polar coordinates (or spherical coordinates) to evaluate a limit. For example, $(x, y) \to (0, 0)$ becomes $r \to 0$. 
13.3 – Partial Derivatives

Definition – Partial Derivatives of a Function of Two Variables

If \( z = f(x, y) \), then the first partial derivatives of \( f \) with respect to \( x \) and \( y \) are the functions \( f_x \) and \( f_y \) defined by

\[
\begin{align*}
 f_x(x, y) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\
 f_y(x, y) &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}
\end{align*}
\]

provided the limits exist.

Note: Given \( z = f(x, y) \), then to find \( f_x \) you consider \( y \) to be a constant and differentiate with respect to \( x \), and to find \( f_y \) you consider \( x \) to be a constant and differentiate with respect to \( y \).

Notation for First Partial Derivatives

For \( z = f(x, y) \), the partial derivatives \( f_x \) and \( f_y \) are denoted by

\[
\begin{align*}
 \frac{\partial}{\partial x} f(x, y) &= f_x(x, y) = z_x = \frac{\partial z}{\partial x} \\
 \frac{\partial}{\partial y} f(x, y) &= f_y(x, y) = z_y = \frac{\partial z}{\partial y}
\end{align*}
\]

and

The first partials evaluated at the point \((a, b)\) are denoted by

\[
\left. \frac{\partial z}{\partial x} \right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left. \frac{\partial z}{\partial y} \right|_{(a,b)} = f_y(a, b)
\]

Note: The values of \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) at the point \((x_0, y_0, f(x_0, y_0))\) denote the slopes of the surface in the \( x \)- and \( y \)-directions, respectively.
Note: The concept of partial derivative is extended to functions of three or more variables. Given \( w = f(x_1, x_2, \ldots, x_n) \), there are \( n \) first partial derivatives.

\[
\frac{\partial w}{\partial x_k} (x_1, x_2, \ldots, x_n) = \lim_{\Delta x_k \to 0} \frac{f(x_1, x_2, \ldots, x_k + \Delta x_k, \ldots, x_n) - f(x_1, x_2, \ldots, x_k, \ldots, x_n)}{\Delta x_k}
\]

for \( k = 1, 2, \ldots, n \).

To find the partial derivative with respect to one independent variable, consider the other independent variables constants and differentiate with respect to the given variable.

**Higher-Order Partial Derivatives**

Given \( z = f(x, y) \). There are four second partial derivatives (provided they exist.)

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \\
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} \\
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}
\]

The last two are called **mixed partials derivatives**.

Note: The order of differentiation is indicated differently in the two types of notation:

In this example, differentiation w.r.t \( x \) is first, then w.r.t \( y \)

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy}
\]

**Theorem – Equality of Mixed Partial Derivatives**

If \( f \) is a function of \( x \) and \( y \) such that \( f_{xy} \) and \( f_{yx} \) are continuous on an open disk \( R \), then, for every \((x, y)\) in \( R \),

\[
f_{xy}(x, y) = f_{yx}(x, y).
\]
13.4 – Differentials

Recall: Given \( y = f(x) \), the differential of \( y \) is defined as

\[
dy = f'(x)\,dx.
\]

Moreover, the change in \( y \), \( \Delta y = f(x + \Delta x) - f(x) \), can be approximated for small values of \( \Delta x = dx \). That is,

\[
f(x + dx) - f(x) \approx f'(x)dx,
\]

or more briefly, \( \Delta y \approx dy \).

Definition – Given \( z = f(x, y) \) then \( \Delta x \) and \( \Delta y \) are the increments of \( x \) and \( y \), and the increment of \( z \) is given by

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).
\]

Definition – Total Differential

If \( z = f(x, y) \) and \( \Delta x \) and \( \Delta y \) are increments of \( x \) and \( y \), then the differentials of the independent variables \( x \) and \( y \) are

\[
dx = \Delta x \quad \text{and} \quad dy = \Delta y
\]

And the total differential of the dependent variable \( z \) is

\[
dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y)dx + f_y(x, y)dy.
\]

Note: This is extended to functions of \( n \) variables. Given \( w = f(x_1, x_2, \ldots, x_n) \), the total differential of \( w \) is

\[
dw = \frac{\partial w}{\partial x_1} dx_1 + \frac{\partial w}{\partial x_2} dx_2 + \cdots + \frac{\partial w}{\partial x_n} dx_n = f_{x_1} dx_1 + f_{x_2} dx_2 + \cdots + f_{x_n} dx_n
\]
Definition – Differentiability

A function $f$ given by $z = f(x, y)$ is **differentiable** at $(x_0, y_0)$ if $\Delta z$ can be written in the form

$$\Delta z = f'_x(x_0, y_0) \Delta x + f'_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. The function $f$ is **differentiable in a region** $R$ if it is differentiable at each point in $R$.

Theorem – Sufficient Conditions for Differentiability

If $f$ is a function of $x$ and $y$, where $f'_x$ and $f'_y$ are continuous in an open region $R$, then $f$ is differentiable on $R$.

Note: The existence of the partials $f'_x$ and $f'_y$ does not guarantee that the function is differentiable.

Note: Differentiability is similarly defined for functions of more than three variables, and continuity of the first partials on an open region $R$ implies differentiability on $R$.

Approximation by Differentials

Provided that $z = f(x, y)$ is differentiable, then for small $\Delta x$ and $\Delta y$, we can approximate

$$\Delta z \approx dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Theorem – Differentiability Implies Continuity

If a function of $x$ and $y$ is differentiable at $(x_0, y_0)$, then it is continuous at $(x_0, y_0)$.
13.5 – Chain Rule for Functions of Several Variables

Theorem – Chain Rule: One Independent Variable

Let \( w = f(x, y) \), where \( f \) is a differentiable function of \( x \) and \( y \). If \( x = g(t) \) and \( y = h(t) \), where \( g \) and \( h \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \), and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.
\]

Note: The Chain Rule for one independent variable can be extended to any number of variables: Let \( w = f(x_1, \ldots, x_n) \), where \( f \) is a differentiable function of \( x_1, \ldots, x_n \) and if each \( x_i \) is a differentiable function of \( t \), then \( w \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial w}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n} \frac{dx_n}{dt}.
\]

Note: Given \( w = f(x, y) \), \( x = g(t) \), and \( y = h(t) \), we say that \( w \) is the dependent variable, \( t \) is the independent variable, and \( x \) and \( y \) are called intermediate variables.

Theorem – Chain Rule: Two Independent Variables

Let \( w = f(x, y) \), where \( f \) is a differentiable function of \( x \) and \( y \). If \( x = g(s, t) \) and \( y = h(s, t) \) such that the first partials \( \frac{\partial x}{\partial s} \), \( \frac{\partial x}{\partial t} \), \( \frac{\partial y}{\partial s} \), and \( \frac{\partial y}{\partial t} \) all exist, then \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) exist and are given by

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}.
\]
Note: The Chain Rule can be extended to any number of variables, intermediate or independent: Let \( w = f(x_1, \ldots, x_n) \), where \( f \) is a differentiable function of \( x_1, \ldots, x_n \) and where each \( x_j \) is a differentiable function of \( t_1, \ldots, t_m \). Then for each \( i = 1, 2, \ldots, m \) we have

\[
\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_i}.
\]

**Theorem – Chain Rule: Implicit Differentiation**

If the equation \( F(x, y) = 0 \) defines \( y \) implicitly as a differentiable function of \( x \), then

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \text{ where } F_y(x, y) \neq 0.
\]

If the equation \( F(x, y, z) = 0 \) defines \( z \) implicitly as a differentiable function of \( x \) and \( y \), then

\[
\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \text{ where } F_z(x, y, z) \neq 0.
\]
13.6 – Directional Derivatives and Gradients

Definition – Directional Derivatives

Let $f$ be a function of two variables $x$ and $y$ and let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ be a unit vector. Then the directional derivative of $f$ in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \to 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.

Theorem – Directional Derivative

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta .$$

Definition – Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of $x$ and $y$ such that $f_x$ and $f_y$ exist. Then the gradient of $f$, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} .$$

Note: $\nabla f$ is read “del $f$.” Another notation for the gradient is $\text{grad} f(x, y)$.

Note: For each point $(x, y)$, the gradient $\nabla f(x, y)$ is a vector in the $xy$-plane.

Theorem – Alternate Form of the Directional Derivative

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f = \nabla f(x, y) \cdot \mathbf{u} .$$
Theorem – Properties of the Gradient

Let $f$ be differentiable at the point $(x, y)$.

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_u f(x, y) = 0$ for all $u$.

2. The direction of maximum increase of $f$ is given by $\nabla f(x, y)$. The maximum value of $D_u f(x, y)$ is $\|\nabla f(x, y)\|$.

3. The direction of minimum increase of $f$ is given by $-\nabla f(x, y)$. The minimum value of $D_u f(x, y)$ is $-\|\nabla f(x, y)\|$.

Theorem – Gradient is Normal to Level Curves

If $f$ is differentiable at $(x_0, y_0)$ and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through $(x_0, y_0)$.

Directional Derivative and Gradient for Three Variables

Let $f$ be a function of $x$, $y$, and $z$, with continuous first partial derivatives. The directional derivative of $f$ in the direction of a unit vector $u = ai + bj + ck$ is given by

$$D_u f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The gradient of $f$ is defined to be

$$\nabla f(x, y, z) = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k.$$

Properties of the gradient are as follows:

1. $D_u f(x, y, z) = \nabla f(x, y, z)\cdot u$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_u f(x, y, z) = 0$ for all $u$.
3. The direction of maximum increase of $f$ is given by $\nabla f(x, y, z)$. The maximum value of $D_u f(x, y, z)$ is $\|\nabla f(x, y, z)\|$.
4. The direction of minimum increase of $f$ is given by $-\nabla f(x, y, z)$. The minimum value of $D_u f(x, y, z)$ is $-\|\nabla f(x, y, z)\|$.

Note: If $f$ is differentiable, then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface through $(x_0, y_0, z_0)$. 
13.7 – Tangent Planes and Normal Lines

Definition – Tangent Plane and Normal Line

Let $F$ be differentiable at the point $P(x_0, y_0, z_0)$ on the surface $S$ given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq 0$.

1. The plane through $P$ that is normal to $\nabla F(x_0, y_0, z_0)$ is called the tangent plane to $S$ at $P$.

2. The line through $P$ having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal line to $S$ at $P$.

Theorem – Equation of Tangent Plane

If $F$ is differentiable at $(x_0, y_0, z_0)$, then the equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at $(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0$$

Note: The equation can be written using the dot product as follows.

$$\nabla F(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

Note: Given $z = f(x, y)$ we can define $F(x, y, z) = f(x, y) - z$. Then the level surface $S$ given by $F(x, y, z) = 0$ is the graph of the surface $z = f(x, y)$. Moreover, if $f$ is differentiable at $(x_0, y_0)$ then $F$ is differentiable at $(x_0, y_0, z_0)$ and so an equation of the tangent plane to $S$ at the point $(x_0, y_0, z_0)$ is

$$f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) - (z-z_0) = 0$$

Theorem – Gradient is Normal to Level Surfaces

If $F$ is differentiable at $(x_0, y_0, z_0)$ and $\nabla F(x_0, y_0, z_0) \neq 0$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through $(x_0, y_0, z_0)$.
13.8 – Extrema of Functions of Two Variables

Recall: A region in the plane is closed if it contains all of its boundary points.

**Definition** – A region in the plane is called **bounded** if it is contained in some closed disk in the plane.

**Definition** – Let $f$ be a function of two variables defined in a region $R$. If there exists some point $(a,b)$ in $R$ such that $f(a,b) \leq f(x,y)$ for all $(x,y)$ in $R$ then $f(a,b)$ is called the **minimum** of $f$ in the region $R$. If there exists some point $(c,d)$ in $R$ such that $f(c,d) \geq f(x,y)$ for all $(x,y)$ in $R$ then $f(c,d)$ is called the **maximum** of $f$ in the region $R$.

Note: A minimum is also called an absolute minimum. Likewise, a maximum is also called an absolute maximum.

**Theorem – Extreme Value Theorem**

Let $f$ be a continuous function of two variables $x$ and $y$ defined on a closed bounded region $R$ in the $xy$-plane.

1. There is at least one point in $R$ where $f$ takes on a minimum value.
2. There is at least one point in $R$ where $f$ takes on a maximum value.

**Definition – Relative Extrema**

Let $f$ be a function defined on a region $R$ containing $(x_0,y_0)$.

1. The function $f$ has a **relative minimum** at $(x_0,y_0)$ if $f(x_0,y_0) \leq f(x,y)$ for all $(x,y)$ in an open disk containing $(x_0,y_0)$.

2. The function $f$ has a **relative maximum** at $(x_0,y_0)$ if $f(x_0,y_0) \geq f(x,y)$ for all $(x,y)$ in an open disk containing $(x_0,y_0)$. 
Definition – Critical Point

Let $f$ be defined on an open region $R$ containing $(x_0, y_0)$. The point $(x_0, y_0)$ is a **critical point** of $f$ if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Note: The first condition can be stated as $\nabla f(x_0, y_0) = \mathbf{0}$.

Theorem – Relative Extrema Occur Only at Critical Points

If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ is a critical point of $f$.

Definition – A critical point of $f$ of the form $\nabla f(x_0, y_0) = \mathbf{0}$ that is neither a relative minimum nor a relative maximum is called a **saddle point**.

Theorem – Second Partial Test

Let $f$ have continuous second partial derivatives on an open region containing the point $(a, b)$ for which

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$ 

To test for relative extrema of $f$, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$ 

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a relative minimum at $(a, b)$.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a relative maximum at $(a, b)$.
3. If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.
4. The test is inconclusive if $d = 0$. 
13.10 – Lagrange Multipliers

**Theorem – Lagrange’s Theorem**

Let $f$ and $g$ have continuous first partial derivatives such that $f$ has an extremum at a point $(x_0, y_0)$ on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq 0$, then there is a real number $\lambda$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Note: The scalar $\lambda$ (lambda) is called a **Lagrange Multiplier**.

**Method of LagrangeMultipliers**

Let $f$ and $g$ satisfy the hypothesis of Lagrange’s Theorem, and let $f$ have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of $f$, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$
\begin{align*}
  f_x(x, y) &= \lambda g_x(x, y) \\
  f_y(x, y) &= \lambda g_y(x, y) \\
  g(x, y) &= c
\end{align*}
$$

2. Evaluate $f$ at each solution point obtained in the first step. If $f$ attains a maximum it will be the largest of these values. If $f$ attains a minimum it will be the least of these values.

Note: A similar method can be used for functions of three or more variables. Also, if there are multiple constraints $g_1, g_2, \ldots, g_n$ then solve $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_n \nabla g_n$.

Note: The Extreme Value Theorem is useful here. If $f$ is continuous on the smooth constraint curve $g(x, y) = c$ and the curve is a closed, bounded region of the plane, then there is a maximum and a minimum value attained by $f$.

Note: Here is a method for finding absolute extrema for a continuous function $f$ on a closed, bounded region $R$. First, find all the critical points of $f$ that lie in the interior of $R$. Then, find all extrema on the boundary of $R$. This can be done using Lagrange Multipliers if the boundary is described by some $g(x, y) = c$ such that $\nabla g(x, y)$ exists. Or, one can rewrite $f$ as a function in one variable on a closed interval by substitution, then apply one variable calculus techniques to find the extrema. Lastly, compare the values of $f$ at all points found in $R$. The largest is the max; the smallest is the min.