Differential Equations

A differential equation is an equation that involves an unknown function and derivatives of that function.

A solution to a differential equation is a function that makes the equation true.

Ex: \( \frac{dy}{dx} = 6x - 4 \) is a differential equation that involves a function \( y \) and its derivative \( y' \). (The independent variable is \( x \).)

Note: \( y = 3x^2 - 4x \) is a solution.

Verify \( \frac{dy}{dx} = \frac{d}{dx}(3x^2 - 4x) = 6x - 4 \)

\( \frac{dy}{dx} = 6x - 4 \)

Yes, true.

Also, \( y = 3x^2 - 4x + 9 \) is a solution.

Verify \( \frac{dy}{dx} = \frac{d}{dx}(3x^2 - 4x + 9) = 6x - 4 \)

\( \frac{dy}{dx} = 6x - 4 \)

Yes

In fact, for any constant \( C \), \( y = 3x^2 - 4x + C \) is a solution. (Verify \( \).)
Definition: The order of a differential equation is the order of the highest order derivative that appears in the equation.

Notation: We use \( P(t, y, y', y'', \ldots, y^{(n)}) \) to represent an expression that involves the independent variable \( t \), and the dependent variable \( y \) along with its derivatives \( y' \), \( y'' \), \ldots, \( y^{(n)} \).

Notice: It is implied here that \( y \) is a function of \( t \).

So, \( y' \) is \( \frac{dy}{dt} \), \( y'' \) is \( \frac{d^2y}{dt^2} \), \ldots

First-Order Differential Equation \( y' = P(t, y) \)
Second-Order \( y'' = P(t, y, y') \)
Third-Order \( y''' = P(t, y, y', y'') \)

Example: What is the order of the differential equation \( y'' - y' - 2y = 2t + 1 \)?

Order is Second-Order

Verify that \( y = e^{2t} - t \) is a solution.

\[ y' = 2e^{2t} - 1 \quad \text{and} \quad y'' = 4e^{2t} \]

"plug in"

\[ y'' - y' - 2y = (4e^{2t}) - (2e^{2t} - 1) - 2(e^{2t} - t) = 2t + 1 \]

\[ 4e^{2t} - 2e^{2t} + 1 - 2e^{2t} + 2t = 2t + 1 \]

\[ 2t + 1 = 2t + 1 \quad \checkmark \]
Linear vs. Nonlinear

Consider the differential equation

\[ y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}) \]

If the equation is linear in the variables \( y, y', y'', \ldots, y^{(n-1)}, y^{(n)} \), then we say the differential equation is linear.

Ex/ \[
\frac{d^2y}{dx^2} = 5x^2 \frac{dy}{dx} - y \cos x + \frac{1}{x} \quad \text{Linear}
\]

Second-Order

\[
\left(\frac{du}{dt}\right)^3 = y' + t \quad \text{Nonlinear}
\]

First-Order

\[
yx^3 = xsny + y'' \quad \text{Nonlinear}
\]

Third-Order

\[
y' = y(y - 1) \quad \text{Nonlinear}
\]

First-Order

Ex/ Order? Linear?

\[
(1 + y')(1 + y'') = (y')^2
\]

second-order, nonlinear

Verify that \( y = \sin x \) is a solution. [Independent variable is unspecified. I chose to use \( x \).]
"plug in"

\[
\begin{align*}
\frac{d}{dx} y^2 & \quad \frac{d^2}{dx^2} y^2 \quad \div \quad \frac{d}{dx} y^2 \\
(1 + (\sin x))(1 + (-\sin x)) & = (\cos x)^2 \\
1 - \sin^2 x & = \cos^2 x \quad \text{yes} \\
\end{align*}
\]

Is \( y = x^2 \) a solution?

\[
y' = 2x, \quad y'' = 2
\]

"plug in"

\[
\begin{align*}
\frac{d}{dx} y^2 & \quad \frac{d^2}{dx^2} y^2 \quad \div \quad \frac{d}{dx} y^2 \\
(1 + (x^2))(1 + (2)) & = (2x)^2 \\
(1 + x^2)3 & = 4x^2 \\
3x^2 + 3 & = 4x^2 \quad \text{NO.} \\
\end{align*}
\]

You may ask... \( 3x^2 + 3 = 4x^2 \)

\[
x^2 = 3 \quad x = \pm \sqrt{3} \quad \text{agree at 2 points?}
\]

We are looking for a function \( y \) that satisfies the differential equation ON AN INTERVAL on the real line.

The general first-order linear differential equation:

\[
\frac{dy}{dx} + p(x)y = g(x)
\]

Ex/ Consider \( y' = 2xy - 2x \)

Verify that \( y = Ce^{x^2} + 1 \) is a solution.
\[ y' = \frac{d}{dx}(Ce^{x^2} + 1) = C \frac{d}{dx}e^{x^2} + \frac{d}{dx}1 = 2Cxe^{x^2} \]

"plug in"

\[
\begin{align*}
    y' & > 2 \quad \Rightarrow \quad y' = 2x(Ce^{x^2} + 1) - 2x \\
    2Cxe^{x^2} & = 2xCe^{x^2} + 2x - 2x \quad \text{yes} \\
\end{align*}
\]

In fact, \( y = Ce^{x^2} + 1 \) is the most general family of solutions for this differential equation. Thus we would say that 
\( y = Ce^{x^2} + 1 \) is the \textit{general solution}.

Various solutions graphed together.

If we are after a particular solution, we would need to have more information, such as a point on the graph.

Say we know that the function \( y \) satisfies

1. \[ y' = 2xy - 2x \]
   \text{AND}
2. \[ y(0) = 3 \quad [\text{"initial conditions"}] \]

Then, by 1, we have seen that \( y = Ce^{x^2} + 1 \) for some \( C \).
Then we use the point given, \((0, 3)\) to find \(C\).

\[
\begin{align*}
(3) &= Ce^{0} + 1 \\
3 &= Ce^{0} + 1 \\
3 &= C + 1 \\
C &= 2
\end{align*}
\]

Thus, \(y = 2e^{x^2} + 1\) is the particular solution to the initial value problem.

Ex/ Solve the initial value problem.

\[y'(t) = 1 - \sin t, \quad y\left(\frac{\pi}{2}\right) = 0\]

Note: \(y'(t) = f(t)\) \\
\(\text{NO}\ y's\)

So, \(y(t) = \int f(t)\,dt\)

\[
y(t) = \int (1 - \sin t)\,dt = t + \cos t + C \quad \text{(general solution)}
\]

Then find \(C\), use \(\left(\frac{\pi}{2}, 0\right)\)

\[
\begin{align*}
y\left(\frac{\pi}{2}\right) &= C + \cos\left(\frac{\pi}{2}\right) + C \\
(0) &= C + \cos\left(\frac{\pi}{2}\right) + C \\
C &= 2
\end{align*}
\]

so, \(y = t + \cos t - \frac{\pi}{2}\) (particular solution)
Ex/ Solve the initial value problem.

\[ u''(t) = 9e^{3t} - 2e^{-t}, \quad u'(0) = 4, \quad u(0) = 5 \]

\[ \begin{align*}
\text{All in } t, \text{ no } u's? \\
\text{So}, \quad u'(t) &= \int u''(t) \, dt = \int (9e^{3t} - 2e^{-t}) \, dt \\
&= 3e^{3t} + 2e^{-t} + C_1
\end{align*} \]

Thus \( u' = 3e^{3t} + 2e^{-t} + C_1 \). We can use \( u'(0) = 4 \) to find \( C_1 \).

\[ \begin{align*}
(4) &= 3e^{3(0)} + 2e^{-0} + C_1 \\
4 &= 3 + 2 + C_1, \quad C_1 = -1
\end{align*} \]

So, \( u'(t) = 3e^{3t} + 2e^{-t} - 1 \).

We solve the new diff. eqn.

\[ u(t) = \int u'(t) \, dt = \int (3e^{3t} + 2e^{-t} - 1) \, dt \]

\[ = e^{3t} - 2e^{-t} - t + C_2 \quad \text{Use } u(0) = 5 \text{ to find } C_2 \]

So, \( u = e^{3t} - 2e^{-t} - t + C_2 \).

\[ \begin{align*}
(5) &= e^{3(0)} - 2e^{-0} - (0) + C_2 \\
5 &= 1 - 2 + C_2, \quad C_2 = 6
\end{align*} \]

Finally, \( u(t) = e^{3t} - 2e^{-t} - t + 6 \) check it :)
Motion in a gravitational field.

\[ s''(t) = v(t) \quad s''(t) = a(t) \]

Initial position \( s(0) \) or \( s_0 \)

Initial velocity \( v(0) \) or \( v_0 \) or \( s'(0) \)

\[ s''(t) = -9.8 \quad \text{(second-order, linear)} \]

\[ v(t) = s'(t) = \int s''(t) \, dt = \int -9.8 \, dt = -9.8t + C_1 \]

\( C_1 \) ? use \( s'(0) = v_0 \)

\( \therefore v_0 = -9.8(0) + C_1 \Rightarrow C_1 = v_0 \)

Thus, \( v(t) = s'(t) = -9.8t + v_0 \)

\[ s(t) = \int s'(t) \, dt = \int (-9.8t + v_0) \, dt = -4.9t^2 + v_0 t + C_2 \]

\( C_2 \) ? use \( s(0) = s_0 \)

\( \therefore s_0 = -4.9(0)^2 + v_0(0) + C_2 \Rightarrow C_2 = s_0 \)

Thus, \[ s(t) = -4.9t^2 + v_0 t + s_0 \quad \text{check it!} \]

Ex/ Find the general solution.

\[ y' = \frac{t(t-1)}{ \quad \text{all ln} \, t \, , \, \text{no} \, y' \, ! } \]

\( \therefore y = \int \frac{1}{t(t-1)} \, dt = \int \left( \frac{1}{t-1} - \frac{1}{t} \right) \, dt \)

\[ = \ln|t-1| - \ln|t| + C \quad \text{or} \quad \ln \left| \frac{t-1}{t} \right| + C \quad \text{check it!} \]
Ex/ Consider \( y' = \frac{4y^2}{y-1} \)

\[
y = \int dy = \int y' dt = \int 4y^2 dt
\]

\[\text{I \&? Cannot simply integrate as it is \& We need more techniques.}\]

Verify that \( y = \frac{1}{c-4t} \) is a solution.

\[
y' = \frac{d}{dt}(c-4t)^{-1} = (-1)(c-4t)^{-2}(-4) = \frac{4}{(c-4t)^2}
\]

\[\text{"plug in"}\]

\[
\left(\frac{4}{(c-4t)^2}\right) = 4 \left(\frac{1}{c-4t}\right)^2 \quad \text{yes} \checkmark
\]

Note: Verifying a solution is generally easier than finding a solution.

Existence and Uniqueness Theorems.

The solution of a general first-order initial value problem

\[
y'(t) = f(t, y), \quad y(t_0) = y_0
\]

exists and is unique in some region that contains the point \((t_0, y_0)\) provided \(f\) is "well behaved."
Ex/ Consider \( y'' + 2y' + y = 0 \)

Order?

Linear?

Verify that \( y = C_1 e^{-x} + C_2 xe^{-x} \) is a solution.

\[
y' = -C_1 e^{-x} + C_2 e^{-x} - C_2 xe^{-x} = (C_2 - C_1) e^{-x} - C_2 xe^{-x}
\]

\[
y'' = C_1 e^{-x} - C_2 e^{-x} - C_2 e^{-x} + C_2 xe^{-x} = (C_1 - 2C_2) e^{-x} + C_2 xe^{-x}
\]

"Plug in"

\[
(c_1 - 2c_2)e^{-x} + 2(c_2 - c_1)e^{-x} + (c_1 e^{-x} + c_2 xe^{-x}) = 0
\]

\[
(c_1 - 2c_2 + 2c_2 - 2c_1 + c_1)e^{-x} + (c_2 - 2c_2 + c_2)xe^{-x} = 0
\]

\[
0e^{-x} + 0xe^{-x} = 0 \quad \text{yes} \quad \checkmark
\]

Ex/ Consider \( y' = \cos^2 y \)

Order?

Linear?

Verify that \( y = \tan^{-1}(x + c) \) is a solution. You try...