Properties of Power Series

Definition - Power Series
A power series has the general form

\[ \sum_{k=0}^{\infty} c_k(x-a)^k \]

where \( a \) and \( c_k \) are real numbers, and \( x \) is a variable. The \( c_k \)'s are the coefficients of the power series and \( a \) is the center of the power series. The set of values of \( x \) for which the series converges is its interval of convergence. The radius of convergence of the power series, denoted \( R \), is the distance from the center of the series to the boundary of the interval of convergence.

Note: The terms of a power series may positive or negative. When finding the radius of convergence, we usually test for absolute convergence.

Note: \( \sum_{k=0}^{\infty} |c_k(x-a)^k| \) Root or Ratio Test are useful here.

Radio:
\[ \lim_{k \to \infty} \frac{|c_{k+1}(x-a)^{k+1}|}{|c_k(x-a)^k|} = \lim_{k \to \infty} \frac{|c_{k+1}| |x-a|}{|c_k|} = |x-a| \lim_{k \to \infty} \frac{|c_{k+1}|}{|c_k|} = 1 \]

Root:
\[ \lim_{k \to \infty} \sqrt[k]{|c_k(x-a)^k|} = \lim_{k \to \infty} \sqrt[k]{|x-a|} = |x-a| \lim_{k \to \infty} \sqrt[k]{|c_k|} = 1 \]

NOTE: \( |x| < M \Rightarrow -M < x < M \) (for \( M > 0 \))
Ex/ Find the interval and radius of convergence.

1. \(\sum_{k=0}^{\infty} x^k\) center is 0.

Consider \(\sum |x|^k\)

Ratio Test: \(\lim_{k \to \infty} \frac{|x|^{k+1}}{|x|^k} = \lim_{k \to \infty} |x| = |x| \lim_{k \to \infty} 1 = |x| \cdot 1\)

Thus, \(\lim_{k \to \infty} \frac{|x|^{k+1}}{|x|^k} < 1\) when \(|x| < 1\)

\(-1 < x < 1\)

Thus the series converges for \(x\) in \((-1, 1)\)

Radius of convergence \(R = 1\)

Convergence at the endpoints: \(x = -1, x = 1\)

\(x = 1\) \(\sum_{k=0}^{\infty} k\) Diverges (why?)

\(x = -1\) \(\sum_{k=0}^{\infty} (-1)^k\) Diverges (why?)

Thus, the Interval of Convergence is \((-1, 1)\)

2. \(\sum_{k=0}^{\infty} (x+3)^k\) center is -3 \((x - (-3))^k\)

Ratio Test: \(\lim_{k \to \infty} \frac{|(x+3)^{k+1}|}{|(x+3)^k|} = \lim_{k \to \infty} |x+3| = |x+3|\)

Converges when \(|x+3| < 1\)

\(-1 < |x+3| < 1\)
\(-4 < x < -2\)

Radius, \(R = | -2 - (-3) | = 1\)

Endpoints \(x = -4, x = -2\)

Diverges at both (why?)

Interval of Convergence \((-4, -2)\)

\[\sum_{k=0}^{\infty} (3x - 4)^k\]

Center is \(\frac{4}{3}\)

Ratio Test: \(\lim_{k \to \infty} \left| \frac{(3x - 4)^{k+1}}{(3x - 4)^k} \right| = \lim_{k \to \infty} |3x - 4| = |3x - 4|\)

Converges when \(|3x - 4| < 1\)

\(-1 < 3x - 4 < 1\)

\(3 < 3x < 5\)

\(1 < x < \frac{5}{3}\)

Radius, \(R = \left| \frac{4}{3} - 4\right| = \frac{1}{3}\)

Endpoints: \(x = 1, x = \frac{5}{3}\) Both diverge (why?)

I.O.C. \(1, \frac{5}{3}\)

\[\sum_{k=1}^{\infty} \frac{k^x}{k}\]

Center is 0

Ratio Test: \(\lim_{k \to \infty} \left| \frac{\frac{k^{x+1}}{x+1}}{\frac{k^x}{x}} \right| = \lim_{k \to \infty} \frac{x+1}{x} = 1\)

Converges when \(|x| < 1\) \(-1 < x < 1\)
Radius, $R = 1 - 0 = 1$

Endpoints:

$x = 1$, $x = -1$

$x = 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

Diverges

$x = -1$:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

Converges (why?)

$\sum_0^\infty (C \Gamma(-1, 1)) - \frac{\Gamma(0, 0)}{\Gamma(1)}$

Root Test:

$$\lim_{k \to \infty} \sqrt[k]{|k^x|^k} = \lim_{k \to \infty} k^{\frac{x}{k}} \left(\frac{1}{k}\right)^k$$

$$= |x| \lim_{k \to \infty} k = \infty > 1 \text{ for all } x \neq 0$$

$\Rightarrow$ Except when $x$ is at the center ($x = 0$)

$$\sum_{k=0}^{\infty} k^0 = \sum_{0}^{\infty} = 0$$

Radius, $R = 0$

$I.O.C. \ 0$

$\sum_{k=0}^{\infty} \Gamma(-1, 1)$

Center is $0$

Radius Test:

$$\lim_{k \to \infty} \frac{|(-1)^{k+1} x^{k+1}|}{(2k+1)!} \cdot \frac{|x^k|}{(2k)!}$$

$$= |x| \lim_{k \to \infty} \frac{1}{\frac{1 \cdot 2 \cdot 3 \ldots (2k+1)}{(2k+1)}} = |x| \lim_{k \to \infty} \frac{1}{(2k+2)} = 0$$
\[
\lim_{k \to \infty} \frac{1}{k^2 + x^2} = 0 < 1 \quad \text{for all } x.
\]

Converges for all \( x \)

\[ R = \infty \]

I.O.C. \((-\infty, \infty)\)

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**Theorem - Convergence of a Power Series**

A power series \( \sum_{k=0}^{\infty} c_k (x-a)^k \) centered at \( a \) converges in one of three ways.

1) The series converges absolutely for all \( x \).
   
   \( \text{I.O.C. } (-\infty, \infty) \quad R = \infty \)

2) There is a real number \( R > 0 \) such that the series converges absolutely for \( |x-a| < R \) and diverges for \( |x-a| > R \), in which case the radius of convergence is \( R \).

3) The series converges only at \( a \). \( R = 0 \)
Theorem - Combining Power Series

Suppose the power series \( \sum c_k x^k \) and \( \sum d_k x^k \) converge absolutely to \( f(x) \) and \( g(x) \), respectively, on an interval \( I \).

1) Sum and Difference: The power series \( \sum (c_k + d_k) x^k \) converges absolutely to \( f(x) + g(x) \) on \( I \).

2) Multiplication by a power: The power series \( x^m \sum c_k x^k = \sum c_k x^{k+m} \) converges absolutely to \( x^m f(x) \) on \( I \) (provided \( m \) is an integer such that \( k+m \geq 0 \) for all terms of the series.)

3) Composition: If \( h(x) = bx^m \), where \( m \) is a positive integer and \( b \) is a real number, the power series \( \sum c_k (h(x))^k \) converges absolutely to the composite function \( f(h(x)) \), for all \( x \) such that \( h(x) \) is in \( I \).

Ex/ Recall the Geometric Series.

\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots \quad \text{for } |x| < 1
\]

Power series centered at 0.

I.O.C. \((-1, 1)\)

Use this to find the power series and interval of convergence for the following functions.
1. \[ \frac{3^x}{1-x} = 3 \times 3 \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} 3 \times 3^k x^k = \sum_{k=0}^{\infty} 3^k x^{k+1} \]

**Ratio Test:** \[ \lim_{k \to \infty} \left| \frac{3^k x^{k+1}}{3^k x^k} \right| = |x| < 1 \]

Converges when \(-1 < x < 1\) \[ \text{I.O.C.} (-1, 1) \]

Diverges at \(-1\) and \(1\) (why?)

2. \[ \frac{1}{2 - 3x} \]

\[ 2 - 3x \leftrightarrow 1 - (\_\_\_\_) \]

\[ = \frac{1}{1 - (3x - 1)} \]

\[ = 1 + (3x - 1) + (3x - 1)^2 + \ldots = \sum_{k=0}^{\infty} (3x - 1)^k \]

**Ratio Test:** \[ \lim_{k \to \infty} \left| \frac{(3x - 1)^{k+1}}{(3x - 1)^k} \right| = |3x - 1| < 1 \]

\(-1 < 3x - 1 < 1\)

\(0 < 3x < 2\)

\(0 < x < \frac{2}{3}\)

Diverges at \(0\) and \(\frac{2}{3}\). \[ \text{I.O.C.} \left(0, \frac{2}{3}\right) \]

3. \[ \frac{1}{1 + x^4} \]

\[ 1 + x^4 \leftrightarrow 1 - (\_\_\_\_) \]

\[ = \frac{1}{1 - (-x^4)} = \sum_{k=0}^{\infty} (-x^4)^k = \sum_{k=0}^{\infty} (-1)^k x^{4k} \]

**Ratio Test:** \[ \lim_{k \to \infty} \left| \frac{(-1)^{k+1} x^{4(k+1)}}{(-1)^k x^{4k}} \right| = |x^4| < 1 \]

\[ \frac{1}{1} < 1 \]

\[-1 < x < 1\]

Diverges at \(-1\) and \(1\) \[ \text{I.O.C.} (-1, 1) \]
Theorem - Differentiating and Integrating Power Series

Let the function \( f \) be defined by the power series \( \sum_{k=0}^{\infty} c_k(x-a)^k \) on its interval of convergence \( I \).

1) \( f \) is a continuous function on \( I \)

2) The power series may be differentiated or integrated term by term, and the resulting power series converges to \( f'(x) \) or \( \int f(x) \, dx + C \), respectively, at all points in the interior of \( I \) (\( C \) is an arbitrary constant.)

**Ex.** Let \( f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots \) for \( |x| < 1 \)

Then,

\[
\frac{d}{dx} \frac{1}{1-x} = -\frac{1}{(1-x)^2} = \frac{1}{1-x^2}
\]

And

\[
\frac{d}{dx} \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (k+1)x^k
\]

or

\[
\frac{d}{dx} (1 + x + x^2 + x^3 + \ldots) = 0 + 1 + 2x + 3x^2
\]

Then,

\[
\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k
\]

Converges for \( |x| < 1 \)

Diverges at \( x = -1, 1 \)
Also, \( \int \frac{1}{1-x} \, dx = -\ln |1-x| + C_1 \)

And \( \int x \, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C_2 \)

Thus,

\(-\ln |1-x| = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C\)

when \( x=0 \)

\(-\ln |1-0| = 0 = C_0 + C\)

\( C = 0 \)

So,

\(-\ln |1-x| = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \)

OR \( \ln (1-x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{k+1}}{k+1} \) converges for \(|x|<1\)

Endpoints?

\( x = 1 \rightarrow \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{k+1} = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots \)

Diverges

\( x = -1 \rightarrow \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \)

Converges \(?\) (why?)

I.O.C \([-1, 1]\)

Note: At \( x = -1 \), \( \ln (1-(-1)) = \ln 2 \), Thus

\( \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \)
Ex: Find a power series representation for \( f(x) = x^2 \tan^{-1}(3x) \)
centered at \( x = 0 \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

\[ \frac{x^2 \tan^{-1}(3x)}{3} \]

**easy part** \[ \text{tricky} \]

Note: \( \frac{d}{du} \tan^{-1} u = \frac{1}{1+u^2} = \frac{1}{1-(u^2)} = 1 + (u^2) + (u^2)^2 + \ldots \)
for \( |u^2| < 1 \)

So, \( \frac{d}{du} \tan^{-1} u = \sum_{k=0}^{\infty} (-1)^k u^{2k} \)

Then, \( \tan^{-1} u = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k}}{2k+1} \)

\[ C \text{ when } u = 0 \quad \tan^{-1} 0 = 0 = C \quad \rightarrow \quad C = 0 \]

Thus, \( \tan^{-1} u = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{2k+1} \quad \text{for } |u| < 1 \)

So, \( \tan^{-1}(3x) = \sum_{k=0}^{\infty} (-1)^k \frac{(3x)^{2k+1}}{2k+1} \quad \text{for } |3x| < 1 \)

Finally,

\[ x^2 \tan^{-1}(3x) = x^2 \sum_{k=0}^{\infty} (-1)^k \frac{(3x)^{2k+1}}{2k+1} \]

\[ = \sum_{k=0}^{\infty} (-1)^k \frac{3^{2k+1}}{2k+1} x^{2k+3} \quad \text{for } |x| < \frac{1}{3} \]

\[ \text{T.O.C } \left[-\frac{1}{3}, \frac{1}{3}\right] \]

Converges at \( x = -\frac{1}{3}, \frac{1}{3} \) (why?)