Parametric Equations

Suppose \( x \) and \( y \) are both functions of a third variable \( t \) (called a parameter) by the equations

\[
x = g(t) \quad y = h(t) \quad \text{(called parametric equations)}
\]

Then, for each value of \( t \) (in the domain of \( g \) and \( h \)) we get values for \( x \) and \( y \), which we can consider as a point \((x, y)\).

As \( t \) varies, the point \((x, y) = (g(t), h(t))\) varies and traces out a curve, \( C \), which is called a parametric curve.

Note: \( t \) does not necessarily represent time, but this is a useful way to think of it.

A point (particle) is at \((g(t), h(t))\) at "time" \( t \).

Ex/ \( x = t + 1 \), \( y = 2t \); \(-\infty < t < \infty\)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>-2</td>
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<td>0</td>
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<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
We can eliminate the parameter (sometimes).

\[ \begin{align*}
  x &= t + 1 \\
  y &= 2t
\end{align*} \quad \Rightarrow \quad \begin{align*}
  t &= x - 1 \\
  y &= 2(x - 1)
\end{align*} \quad \text{substitute for } t \]

Ex/ \( x = t + 1, \ y = 2t \); \( 1 \leq t \leq 3 \)

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Definition: Forward or Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the forward, or positive, orientation of the curve.

[Note: A "graph" does not have an orientation.]
Ex/ \( x = \cos t, \ y = \sin t; \ 0 \leq t \leq 2\pi \)

The positive orientation is counterclockwise.

Note: \( \cos^2 t + \sin^2 t = 1 \)

Thus \( x^2 + y^2 = 1 \) (parameter eliminated)

Ex/ \( x = \sin t, \ y = \cos t; \ 0 \leq t \leq \pi \)

Clockwise?

\[(\sin t)^2 + (\cos t)^2 = 1 \rightarrow x^2 + y^2 = 1\]

But \( 0 \leq t \leq \pi, \ x = \sin t \geq 0 \) and

\[t = \cos y \rightarrow 0 \leq \cos y \leq \pi \]

\[\cos 0 \geq y \geq \cos \pi \]

\[1 \geq y \geq -1\]
Ex. \( x = 2\cos(3t), \ y = 2\sin(3t) \) \( 0 \leq t \leq 4\pi \)

The circle is traced multiple times here.

\[
\begin{align*}
(2\cos(3t))^2 + (2\sin(3t))^2 &= 4\left[\cos^2(3t) + \sin^2(3t)\right] = 4 \cdot 1 = 4 \\
So, \quad x^2 + y^2 &= 4
\end{align*}
\]

Circles

\[
(x-h)^2 + (y-k)^2 = r^2
\]

Curve: \( x = h + r\cos(wt), \ y = k + r\sin(wt) \)

yields \( x-h = r\cos(wt), \ y-k = r\sin(wt) \)

so, \( (x-h)^2 + (y-k)^2 = r^2\cos^2(wt) + r^2\sin^2(wt) = r^2 \)

If \( w > 0 \), the curve move in the counterclockwise direction.
Straight Lines

\[ x = x_0 + at, \quad y = y_0 + bt; \quad -\infty < t < \infty \]

If \( a \neq 0 \) then, \( t = \frac{x - x_0}{a} \)

\[ \begin{align*}
  y &= y_0 + b \left( \frac{x - x_0}{a} \right) \\
  \text{or,} \quad y - y_0 &= \frac{b}{a} (x - x_0) \\
  \text{line, slope } \frac{b}{a}, \text{ thru } (x_0, y_0)
\end{align*} \]

If \( a = 0 \), and \( b \neq 0 \), then \( x = x_0 \) (constant)

and \( y = y_0 + bt \)

(Vertical Line) \( x = x_0 \).

**Ex/** Find a parametric representation of the line \( 2x - 3y = 5 \).

One Method: slope? \( y = \left( \frac{2}{3} \right) \left( x - \frac{5}{3} \right) \) \( m = \frac{2}{3} \)

point on line? \( (0, -\frac{5}{3}) \) [or others...]

Then \( y - \left( -\frac{5}{3} \right) = \frac{2}{3} (x - (0)) \)

\( y_0 = \frac{b}{a} = \frac{2}{3} \) (let \( a = 3 \), \( b = 2 \)

\[ \begin{align*}
x &= 3t, \quad y = -\frac{5}{3} + 2t; \quad -\infty < t < \infty \\
(\text{check it?})
\end{align*} \]

Another Method: \( [\text{Given } y = \text{eqn}, \text{ let } x = t, \text{ then } z(t) = \text{eqn}] \)

let \( x = t \). Then \( z(t) = \text{eqn} = 5 \)
So, \[ 3y = 2t - 5 \]
\[ y = \frac{2}{3}t - \frac{5}{3} \]
\[ x = t, \quad y = \frac{2}{3}t - \frac{5}{3} \quad j - \infty < t < \infty \]
(check it ?)

Another: Let \( y = t \), then \[ 2x - 3t = 5 \]
\[ 2x = 3t + 5 \quad x = \frac{3}{2}t + \frac{5}{2} \]
\[ x = \frac{3}{2}t + \frac{5}{2}, \quad y = t \quad j - \infty < t < \infty \] (check it !)

There are many way to represent this graph with a parametric curve

Check that those also work:
\[ x = t^3, \quad y = -\frac{5}{3} + \frac{2}{3}t^3 \quad j - \infty < t < \infty \]

\[ x = 4t - 5, \quad y = -\frac{5}{3} + \frac{2}{3}(4t-5) \quad j - \infty < t < \infty \]

\[ x = \tan t, \quad y = -\frac{5}{3} + \frac{2}{3}\tan t \quad j - \frac{\pi}{2} < t < \frac{\pi}{2} \]

Why does this not represent the graph?
\[ x = t^2, \quad y = -\frac{5}{3} + \frac{2}{3}t^2 \quad j - \infty < t < \infty \]
Ex/ \[ y^2 = 4 - x \] for \(-1 \leq y \leq 5\)

\[ x = 4 - y^2 \]

Let \( y = t \), then \( x = 4 - t^2 \)

\[ -1 \leq y \leq 5 \rightarrow -1 \leq t \leq 5 \]

\[ x = 4 - t^2, \quad y = t, \quad -1 \leq t \leq 5 \]

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Ex/ \( x = 1 + \frac{1}{t}, \quad y = 1 - \frac{1}{t}; \quad t \geq 1 \)

Note: \( x + y = (1 + \frac{1}{t}) + (1 - \frac{1}{t}) = 2 \)

So, \( x + y = 2 \quad y = -x + 2 \)

\[ t \geq 1 \rightarrow 0 < \frac{1}{t} \leq 1 \]

So, \( 1 < 1 + \frac{1}{t} \leq 2 \)

So, \( 1 < x \leq 2 \)

\[ (1, 1) \text{ NOT ON CURVE} \]
Theorem Derivative for Parametric Curves

Let \( x = g(t) \) and \( y = h(t) \), where \( g \) and \( h \) are differentiable on an interval \([a, b]\).

Then \[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\text{or } \frac{h''(t)}{g''(t)}, \, \text{or } \frac{y''(t)}{x''(t)})
\]

provided \( \frac{dx}{dt} \neq 0 \)

Based on the chain rule, \[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}
\]

Note: \( \frac{dy}{dx} \) change in \( y \) over change in \( x \) (instantaneous), slope of \( y \) vs. \( x \) graph

\( \frac{dy}{dt}, \frac{dx}{dt} \) are not "slopes" of a graph of \( y \) vs. \( x \).

We can think of these as "velocities" in the \( y \) and \( x \) directions.

Ex/ \( x = \cos t, \ y = \sin t \); \( 0 \leq t \leq 2\pi \)

\[
\frac{dy}{dx}?
\]

\[
\frac{dy}{dt} = \frac{d}{dt} \sin t = \cos t, \quad \frac{dx}{dt} = \frac{d}{dt} \cos t = -\sin t
\]

So, \[
\frac{dy}{dx} = -\frac{\cos t}{\sin t} = -\cot t, \quad 0 < t < 2\pi, \ t \neq \pi
\]

\( \frac{dy}{dx} \) at the point \((0, 1)\)?

\( t = \pi \)

\( \cos t = 0, \ \sin t = 1 \rightarrow t = \frac{\pi}{2} + 2k\pi \)

Then, \[
\frac{dy}{dx} \bigg|_{x=0} = -\cot \left( \frac{\pi}{2} \right) = 0
\]
\[ \frac{dy}{dx} \text{ at } \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right) \]

\[ \cos t = -\frac{\sqrt{3}}{2}, \quad \sin t = \frac{1}{2} \quad \rightarrow \quad t = \frac{\pi}{6} + 2k\pi \]

Thus, \[ \frac{dy}{dx} \bigg|_{t=\frac{\pi}{6}} = -\cot \left( \frac{\pi}{6} \right) = -\left( -\sqrt{3} \right) = \sqrt{3} \]

Ex/ \quad x = e^{2t} - 1, \quad y = e^t + 1 \quad i - \infty < t < \infty

\[ \frac{dy}{dx} ? \]

\[ \frac{dy}{dt} = e^t, \quad \frac{dx}{dt} = 2e^{2t} \]

So, \[ \frac{dy}{dx} = \frac{e^t}{2e^{2t}} = \frac{1}{2e^t} = \frac{1}{2}e^{-t} \]

Find \[ \frac{dy}{dx} \text{ at } (0, 2) \]

\[ t ? \]

\[ e^{2t} - 1 = 0 \rightarrow e^{2t} = 1 \rightarrow 2t = 0 \rightarrow t = 0 \]

\[ e^t + 1 = 2 \rightarrow e^t = 1 \rightarrow t = 0 \]

\[ \frac{dy}{dx} \bigg|_{t=0} = \frac{1}{2}e^{(0)} = \frac{1}{2} \]

Note: \[ e^t = y - 1 \rightarrow x = (y - 1)^2 - 1 \]

\[ x = y^2 - 2y, \quad \frac{dy}{dx} ? \]

Try it.
Ex/ \quad x = 3 \cos(2t), \quad y = 4 \sin(2t); \quad 0 \leq t \leq \pi

Note: \quad \frac{x}{3} = \cos(2t), \quad \frac{y}{4} = \sin(2t)

\Rightarrow \quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \quad \text{ellipse}

Find all points on the curve where the slope is \(-\frac{4}{3}\).

\frac{dy}{dt} = 8 \cos(2t), \quad \frac{dx}{dt} = -6 \sin(2t)

\frac{dy}{dx} = \frac{8 \cos(2t)}{-6 \sin(2t)}

Then, we want \(-\frac{4}{3} \cot(2t) = -\frac{4}{3}\) solve for \(t\).

\cot(2t) = 1 \quad 2t = \cot^{-1}(1)

2t = \frac{\pi}{2} + k\pi

t = \frac{\pi}{4} + \frac{1}{2}k\pi

(t = \ldots, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \ldots)

\begin{align*}
\frac{\pi}{8} & \Rightarrow \quad x = 3 \cos\left(2 \cdot \frac{\pi}{8}\right) = \frac{3\sqrt{2}}{2} \\
y = 4 \sin\left(2 \cdot \frac{\pi}{8}\right) = 2\sqrt{2} \\
& \quad \boxed{\left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)}
\end{align*}

\begin{align*}
\frac{5\pi}{8} & \Rightarrow \quad x = 3 \cos\left(2 \cdot \frac{5\pi}{8}\right) = -\frac{3\sqrt{2}}{2} \\
y = 4 \sin\left(2 \cdot \frac{5\pi}{8}\right) = -2\sqrt{2} \\
& \quad \boxed{\left(-\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right)}
\end{align*}
Line Segments

\[ P(x_1, y_1) \quad \text{From } P \text{ to } Q \quad Q(x_2, y_2) \]  
(Neon vertical)

Let \( 0 \leq t \leq 1 \)

Then when \( t = 0 \) we want \( x = x_1, y = y_1 \)
And when \( t = 1 \) we want \( x = x_2, y = y_2 \)

Also, our points must be on the line

\[
(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)
\]

Solution:

\[
\begin{align*}
x &= x_1 + (x_2 - x_1) t \\
y &= y_1 + (y_2 - y_1) t \\
0 \leq t \leq 1
\end{align*}
\]

Vertical Line

\[
\begin{align*}
(x_1, y_2) \quad \text{Q} \\
(x_1, y_1) \quad \text{P}
\end{align*}
\]

\[
\begin{align*}
x &= x_1 \quad \text{(constant)} \\
y &= y_1 + (y_2 - y_1) t \\
0 \leq t \leq 1
\end{align*}
\]

(check it!)