Physical Applications of Integration

Density and Mass

An object with uniform density satisfies

\[ \text{mass} = \text{density} \cdot \text{volume} \]

We will work with "thin objects" - one-dimensional (a line segment).

In this case, we use "linear density" (mass per length) instead of density (mass per volume).

\[ \begin{align*}
\text{uniform linear density, } \rho \\
\text{length, } l \\
\text{mass} = \rho \cdot l
\end{align*} \]

What if the density (linear) is NOT uniform?

Use slices:

\[ p(x), \text{ density depends on } x. \]

\[ \text{density } p(x) \]

\[ \Delta x \]

mass of slice \( \approx p(x) \Delta x \)

Then total mass of bar \( \approx \sum p(x) \Delta x \)
Mass of a One-Dimensional Object

The mass is

\[ m = \int_{a}^{b} \rho(x) \, dx \]

EX/ Given a thin bar located on \( 0 \leq x \leq 6 \) with the density function \( \rho(x) = x^2 + 1 \) [units: \( x \) meters, \( \rho \) \( \text{kg/m} \)]

1) Find the mass of the bar,

\[ m = \int_{0}^{6} (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_0^6 = \frac{6^3}{3} + 6 = 78 \text{ kg} \]

2) Find the mass of the left half of the bar,

\[ m_e = \int_{0}^{3} (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_0^3 = \frac{3^3}{3} + 3 = 12 \text{ kg} \]

3) Find the mass of the right half of the bar,

\[ m_r = \int_{3}^{6} (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_3^6 = \left( \frac{6^3}{3} + 6 \right) - \left( \frac{3^3}{3} + 3 \right) = 46 \text{ kg} \]

4) Find the position on the bar where half of the mass will lie to the left.
\[ c \]

Call it \( c \)

\[ \text{Total mass} \rightarrow \int_0^c (x^2 + 1) \, dx = \frac{1}{2} (78) \]

\[ \int_0^c (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_0^c = \frac{c^3}{3} + c \]

So, we want \( \frac{c^3}{3} + c = \frac{1}{2} (78) \)

\[ 2c^3 + 6c - 234 = 0 \]

Cubic \( \approx \); approximate answer on calculator

\[ c \approx 4.687 \text{ meters (from the left)} \]

Ex/ Given a thin bar on \( 0 \leq x \leq 4 \) with the density function

\[ \rho(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 3 \\ x & \text{if } 3 < x \leq 4 \end{cases} \]

Find the mass.

\[ m = \int_0^4 \rho(x) \, dx = \int_0^3 \rho(x) \, dx + \int_3^4 \rho(x) \, dx \]

depends on \( x \) interval

\[ \rho(x) = 2 \quad \text{for } 0 \leq x < 3 \]

\[ \rho(x) = x \quad \text{for } 3 < x < 4 \]
So, \[ m = \int_0^3 2x \, dx + \int_3^4 x \, dx = \left(2x\right)\bigg|_0^3 + \left(\frac{x^2}{2}\right)\bigg|_3^4 \]
\[ = 6 + \left(\frac{4^2}{2} - \frac{3^2}{2}\right) = 6 + \frac{7}{2} = \frac{19}{2} \text{ (mass units)} \]

**WORK** (The physics definition of work?)

\[ W = F \cdot d \]

Work = force \cdot distance

\( \checkmark \) This is valid when the force is constant and the distance \( d \) is in the direction of the force.

**Variable Force**

Given an object moving in a straight line (say the x-axis) and acted upon by a force directed along that same line (direction is important) and the force varies as the object's x position.

Let \( F(x) \) = force on the object at \( x \)

at \( x = x_k \) \[ \rightarrow F(x_k) \] is the force

at \( x = x_{k+1} \) \[ \rightarrow F(x_{k+1}) \] is the force

Use a "slice" over the interval \([x_k, x_{k+1}]\) (small interval?) the force will be approximately constant.
So, the work over one "slice" is approximately

$$\Delta W = F(x) \cdot \Delta x$$

Then we add up the works of each slice.

$$\sum F(x) \Delta x$$

Finally, we get...

**Work**

The work done by a variable force $F(x)$ in moving an object along a line from $x=a$ to $x=b$ in the direction of the force is

$$W = \int_{a}^{b} F(x) \, dx$$

**Ex:** How much work is required to move an object from $x=2$ to $x=5$ (meters) in the presence of a force (Newtons) given by $F(x) = x+1$ acting in the positive $x$ direction.

![Diagram of motion and force](motion_force_diagram)
\[ W = \int_2^5 (x+1) \, dx = \left( \frac{x^2}{2} + x \right) \bigg|_2^5 = \left( \frac{25}{2} + 5 \right) - \left( 2 + 2 \right) = \frac{27}{2} \, \text{N} \cdot \text{m} \]

or \( \frac{27}{2} \, \text{Joules} \)

Units of Work, Force, etc...

| SI units (metric) | US
|-------------------|-----
| length            | ft  |
| mass              | slug|
| Force             | \( \text{N} = \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \) | \( \text{lb} = \frac{\text{slug} \cdot \text{ft}}{\text{s}^2} \)
| work              | \( \text{J} = \text{N} \cdot \text{m} \) | \( \text{ft} \cdot \text{lb} \)

The full definition of Work involves a dot product of the vectors \( \vec{F} \) (force) and \( \vec{d} \) (displacement).

\[ W = \vec{F} \cdot \vec{d} \]

For the examples we are working here, we need only understand that

1) If \( \vec{F} \) and \( \vec{d} \) are in the same direction, \( W = Fd \)
2) If \( \vec{F} \) and \( \vec{d} \) are opposite in direction, \( W = -Fd \)

Keep this in mind when setting up your coordinate system.

Make an axis that runs parallel to the direction of the force.
Choose a convenient position for 0 on your axis.
Choose which direction is the positive direction.
Once these are set, you must adhere to them.
Springs

\[ \rightarrow \pm x \text{ direction} \]

\[ x = 0 \iff \text{for convenience} \]

\[ \begin{array}{c}
\text{Stretched} \\
\rightarrow F > 0 \\
\end{array} \]

\[ x > 0 \]

\[ \begin{array}{c}
\text{Compressed} \\
\leftarrow F < 0 \\
\end{array} \]

\[ x < 0 \]

Hooke's Law \[ F(x) = kx \quad (k > 0 \text{ is the spring constant}) \]

The force required to keep the spring in position \( x \).

Note: The force from the spring will be equal and opposite.

\[ F_{\text{spring}} = -kx \]

Ex/ Find the spring constant given a force of 8N is required to stretch the spring 0.2m.

Use \( F = kx \)

\[ 8 = k(0.2) \quad k = 40 \frac{N}{m} \]

So, \( F(x) = 40x \).

Find the work needed to stretch the spring 0.4m from equilibrium.

\( (\text{when } x = 0) \)
Since \( x > 0 \) on \([0, 0.4]\) \( F > 0 \). So force and displacement are in the same direction.

\[
\Delta W = F(x) \Delta x
\]

\[
W = \int_0^{0.4} F(x) \, dx = \int_0^{0.4} 40x \, dx = 20x^2 \bigg|_0^{0.4} = 3.2 \text{ N} \cdot \text{m} \text{ (or J)}
\]

Find the work needed to compress the spring 0.3 m from equilibrium.

\( F < 0 \) (but going from 0 to -0.3 we have \( \Delta x < 0 \))

\[
\Delta W = F(x) \Delta x
\]

\[
W = \int_0^{-0.3} 40x \, dx = 20x^2 \bigg|_0^{-0.3} = 1.8 \text{ N} \cdot \text{m} \text{ (or J)}
\]

Example: It takes 40 J of work to stretch a spring 0.2 meters from its equilibrium. How much work is required to stretch it an additional 0.2 meters?

1st: Find spring constant \( k \). We do not have a set of values for \( F \) and \( x \) to use.?

We do know \( W = \int_0^{0.2} kx \, dx = 40 \text{ J} \), given
Simplify integral, \( k \cdot \frac{1}{2} x^2 \)

Thus \( 0.02k = 40 \)

So \( k = 2000 \frac{N}{m} \)

2nd - Compute work in stretching from \( 0.2 \) to \( 0.4 \)

\[
W = \int_{0.2}^{0.4} 2000x \, dx = 1000 \cdot x^2 \bigg|_{0.2}^{0.4} = 120 \text{ N}\cdot\text{m} \text{ (or J)}
\]

Work involving a constant force acting on an object whose "pieces" move different distances.

To find Work by Force

This slice, we need

- Mass of slice: \( (\text{Volume}) \cdot (\text{Density}) \)
  \( \Rightarrow (Ay \cdot Dy) \cdot \rho \)
  \( \Rightarrow \text{cross sectional area} \cdot \text{thickness} \)
Then

\[ F = mg = A(y) \, dy \, \rho \, g \]

So

\[ \Delta W = A(y) \, dy \, \rho \, g \cdot \text{(displacement)} \]

Thus

\[ \Delta W = A(y) \, dy \, \rho \, g \, (h - y) \]

Add up work of each slice,

\[ \sum A(y) \rho g (h - y) \, dy \]

Finally,

\[ W = \int_{a}^{b} A(y) \rho g (h - y) \, dy \]

From slice \( y = a \) to slice \( y = b \)

\[ W = \int_{a}^{b} A(y) \rho g (h - y) \, dy \]

**Ex:** A 10 meter chain hangs vertically with a density of 3 kg/m. How much work is required to wind the entire chain onto a winch at the top of the chain?

**Note:** All "pieces" of the chain will end up at the same height as the top of the original chain position.
\[ \text{slice} = \pi \Delta y \text{ meters} \]

mass of slice: \((3 \frac{\text{kg}}{\text{m}}) \cdot \Delta y (\text{m}) = 3 \Delta y \text{ kg} \]

\([A(y)] \text{ is not needed, why?} \]

So, \( F = 3 \Delta y \cdot (9.8) \text{ Newtons} \)

and \( \Delta y = 10 - y \)

So, \( w = \int_0^{10} 3(9.8)(10-y) \, dy = \ldots \)

Try setting up with different choices for the "0" on y-axis.

Such as:

\[ w = \int_{-10}^{0} 3(9.8)(0-y) \, dy = \ldots \]

Ex/ A tank is formed by revolving \( y = x^2 \) on \([0, 4]\) (in meters) about the y-axis.

The tank is filled with water (density \( \rho = 1000 \text{ kg/m}^3 \)) up to a height of 10 meters.

Find the work needed to pump all of the water to an outflow pipe at the top of the tank.
\[ D(y) = 16 - y \quad (\text{m}) \]

\[ A(y) = \pi \left( \sqrt{y} - 0 \right)^2 = \pi y \quad (\text{m}^2) \]

\[ r(y) \text{ radius in terms of } y \]
\[ r(y) = \sqrt{y} - 0 \]

Mass of slice: \[ \frac{A(y) \, dy \cdot \rho}{\text{Vol}} = \pi y \, dy \cdot 1000 \quad (\text{kg}) \]

Force on slice: \[ m \cdot g = \pi y \, dy \cdot 1000 \cdot 9.8 \quad (\text{N}) \]

\[ W = \int_0^{10} A(y) \rho g D(y) \, dy \]

\[ = \int_0^{10} \pi y \cdot 1000 \cdot 9.8 \cdot (16 - y) \, dy \]

\[ = \pi 1000 (9.8) \left[ 16y - y^2 \right]_0^5 \]

\[ = \pi 1000 (9.8) \left( 8y^2 - \frac{y^3}{3} \right) \]

\[ \approx 14,367,550.4 \quad \text{N} \cdot \text{m} \quad \text{(or J)} \]
Useful Geometry Review

Cone

\[ \frac{r}{y} = \frac{D/2}{H} \]
\[ \therefore r = \frac{D}{2} \cdot \frac{y}{H} \]
\[ A(y) = \pi r^2 = \ldots \]

Sphere

\[ x^2 + y^2 = R^2 \]

Right Triangle

\[ r^2 + (0-y)^2 = R^2 \]
\[ \therefore r = \sqrt{R^2 - y^2} \]
\[ A(y) = \pi r^2 = \ldots \]

Similar Triangles

\[ \frac{w}{y} = \frac{D}{H} \]
\[ \therefore w = \frac{D \cdot y}{H} \]
\[ A(y) = L \cdot w = \ldots \]