Planes and Surfaces

Definition - Plane in \( \mathbb{R}^3 \)

Given a fixed point \( P_0 \) and a nonzero normal vector \( \mathbf{n} \), the set of points \( P \) in \( \mathbb{R}^3 \) for which \( P_0 \mathbf{P} \) is orthogonal to \( \mathbf{n} \) is called a plane.

\[
P_0 \mathbf{P} = \langle x-x_0, y-y_0, z-z_0 \rangle \text{ is orthogonal to } \mathbf{n}
\]

when

\[
\mathbf{n} \cdot P_0 \mathbf{P} = 0
\]

Thus

\[
\langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0
\]

\[
a(x-x_0) + b(y-y_0) + c(z-z_0) = 0
\]

\[
ax + by + cz = d \quad \text{(where } d = ax_0 + by_0 + cz_0) \]

General Equation of a Plane in \( \mathbb{R}^3 \)

The plane passing through (or containing) the point \( (x_0, y_0, z_0) \) with a nonzero normal vector \( \mathbf{n} = \langle a, b, c \rangle \) is described by the equation

\[
a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \quad \text{or} \quad ax + by + cz = d
\]

where \( d = ax_0 + by_0 + cz_0 \)

Ex/ The plane containing \( (2, -1, 5) \) with normal vector \( \mathbf{n} = \langle 0, 1, 0 \rangle \),

\[
\langle 0, 1, 0 \rangle \cdot \langle x-2, y+1, z-5 \rangle = 0
\]

\[
0 + y + 1 + 0 = 0 \quad \text{or} \quad y = -1
\]
Ex/ The plane containing the points \((0,0,3)\), \((2,1,1)\), \((-1,0,4)\).

Find \(\overrightarrow{PQ} \times \overrightarrow{PR}\) is normal to the plane?

\[\overrightarrow{PQ} = <2-0, 1-0, 1-3> = <2, 1, -2>\]
\[\overrightarrow{PR} = <-1-0, 0-0, 4-3> = <-1, 0, 1>\]

\[\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} = <1, 0, 1>\]

Then...

using \((0,0,3)\)

\[<1, 0, 1> \cdot <x-0, y-0, z-3> = 0\]

\[x + 0 + z - 3 = 0 \quad \text{or} \quad [x+z=3] \text{ check it?}\]

Could also use \((2,1,1)\), or \((-1,0,4)\) Try it?

Could also use any nonzero vector parallel to \(<1, 0, 1>\) for the normal vector. Try it?

Ex/ What is the intersection of the plane \(x - 2y + z = 4\) with the...

**xy-plane? (when \(z=0\))**

\[x - 2y + (0) = 4\]
\[y = \frac{1}{2}x - 2\]

**yz-plane? (when \(x=0\))**

\[-2y + z = 4\]
\[z = 2y + 4\]

**xz-plane? (when \(y=0\))**

\[x - 2(0) + z = 4\]
\[z = 4 - x\]
What is a normal vector to the plane \( x - 2y + z = 4 \)?

\[ \vec{n} = \langle 1, -2, 1 \rangle \]

**Definition - Parallel and Orthogonal Planes**

Two distinct planes are parallel if their normal vectors are parallel.

Two planes are orthogonal if their normal vectors are orthogonal.

**Example**

Find an equation for the plane containing \((2, -3, 1)\) that is parallel to the plane \(-x + 4y = 0\).

\[ \vec{n}_1 = \langle -1, 4, 0 \rangle \]

So, use \( \vec{n}_2 = \vec{n}_1 = \langle -1, 4, 0 \rangle \)

\[ \langle -1, 4, 0 \rangle \cdot \langle x - 2, y + 3, z - 1 \rangle = 0 \]

\[ -x + 2 + 4y + 12 + 0 = 0 \]

\[ -x + 4y = -14 \] (check it?)

**Example**

Are the planes \( x - y + 3z = 2 \) and \( 2x - 4y - 2z = 7 \) orthogonal?

\[ \vec{n}_1 = \langle 1, -1, 3 \rangle \quad \vec{n}_2 = \langle 2, -4, -2 \rangle \]

\[ \vec{n}_1 \cdot \vec{n}_2 = 2 + 4 - 6 = 0 \] yes √
Ex/ Find an equation for the line of intersection of the planes
\[3x - y + z = 6 \quad \text{and} \quad 2x + y - 4z = 0.\]

\[\vec{\nu}_1 = \langle 3, -1, 1 \rangle \quad \vec{\nu}_2 = \langle 2, 1, -4 \rangle\]

Note: \(\vec{\nu}_1, \vec{\nu}_2\) are not parallel (why?) so the planes intersect.

Line? Need point & direction.

Direction:

Use \(\vec{\nu}_1 \times \vec{\nu}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 2 & 1 & -4 \end{vmatrix} = \langle 3, 14, 5 \rangle\)

Point:

solve \[\begin{align*}
3x - y + z &= 6 \\
2x + y - 4z &= 0
\end{align*}\]

\[\rightarrow y = 3x + z - 6\]

Sub

\[2x + (3x + z - 6) - 4z = 0\]

\[5x - 3z = 6 \quad \text{(many solutions)}\]

choose \(x = 0\)

Then \(-3z = 6 \quad z = -2\)

So, \(y = 3(0) + (-2) - 6 = -8\)

\((0, -8, -2)\)

Line:

\[\vec{r}(t) = \langle 0, -8, -2 \rangle + t \langle 3, 14, 5 \rangle\]

or \[\vec{r}(t) = \langle 3t, 14t - 8, 5t - 2 \rangle \quad \text{check it!}\]

Definition - Cylinder

Given a curve \(C\) in a plane \(P\) and a line \(l\) not in \(P\), a cylinder is the surface consisting of all lines parallel to \(l\) that pass through \(C\).
Note: If \( x = a \) and \( z = b \) is a solution to \( z = \sin(x) \) then \( (a, y, b) \) is a solution to \( z = \sin(x) \) for ANY \( y \).

**Definition - Trace**
A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

**Example**
\[ x^2 - y + 4z^2 = 0 \]

- **xy-trace** \( (z=0) \):
  \[ x^2 - y + 0 = 0 \]
  \[ y = x^2 \]

- **xz-trace** \( (y=0) \):
  \[ x^2 + 4z^2 = 0 \]
  \[ (0, 0) \]

- **yz-trace** \( (x=0) \):
  \[ 0 - y + 4z^2 = 0 \]
  \[ y = 4z^2 \]

- **z=1**:
  \[ x^2 - y + 4 = 0 \]
  \[ y = x^2 + 4 \]

- **y=1**:
  \[ x^2 - 1 + 4z^2 = 0 \]
  \[ x^2 + 4z^2 = 1 \]

- **x=1**:
  \[ 1 - y + 4z^2 = 0 \]
  \[ y = 4z^2 + 1 \]

"Elliptic paraboloid"
Quadratic Surfaces
General quadratic (2nd degree) in three variables

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \]

Ex/ Sphere  \[ x^2 + y^2 + z^2 = 4 \]

For sketching:
- Fluid intercepts:
  - x-int (when \( y, z = 0 \))
  - y-int (when \( x, z = 0 \))
  - z-int (when \( x, y = 0 \))

Ex/ Ellipsoid  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Fluid traces:
- xy-trace
- xz-trace
- yz-trace
- others \( x = k, y = k, z = k \)...

Ex/ Elliptic Cone  \[ y^2 = x^2 + \frac{z^2}{4} \]

Intercepts?
- Only \( (0, 0, 0) \) Try it?

Trace
- \( y = 1 \) \( (\pm 1)^2 = x^2 + \frac{z^2}{4} \)
- \( y = -1 \)

xy-trace?
- \( z = 0 \)
- \( y^2 = x^2 \)
- or \( y = \pm |x| \)

x\(\bar{e}\)-trace?
- \( y = 0 \)
- \( 0 = x^2 + \frac{z^2}{4} \)