Limits and Continuity

Definition - Limit of a Function of Two Variables

The function \( f \) has the limit \( L \) as \((x, y)\) approaches \((a, b)\), written

\[
\lim_{(x, y) \to (a, b)} f(x, y) = L
\]

if given any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
|f(x, y) - L| < \varepsilon
\]

whenever \((x, y)\) is in the domain of \( f \) and

\[
0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta
\]

Note: \( \sqrt{(x-a)^2 + (y-b)^2} \) is the distance between \((x, y)\) and \((a, b)\).

\[\{(x, y) \mid \sqrt{(x-a)^2 + (y-b)^2} < \delta\}\]

The smaller \( \delta \) is, the closer to \((a, b)\) the points \((x, y)\) (in the set) must be.

Theorem -

Let \( a, b, \) and \( c \) be real numbers.

1) Constant function \( f(x, y) = c \)
\[
\lim_{(x, y) \to (a, b)} c = c
\]

2) \( f(x, y) = x \)
\[
\lim_{(x, y) \to (a, b)} x = a
\]

3) \( f(x, y) = y \)
\[
\lim_{(x, y) \to (a, b)} y = b
\]
Proof of \( \lim_{(x,y) \to (a,b)} x = a \)

Let \( \varepsilon > 0 \) be arbitrary. Fix \( \varepsilon \).

Choose \( \delta = \varepsilon \).

Then, for any point \((x, y)\),

if \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \)

then \( \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon \) (since \( \delta = \varepsilon \)).

Moreover

\[
|\ x - a \ | = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2}
\]

So,

\[
|x - a| < \varepsilon .
\]

Therefore, \( |x - a| < \varepsilon \) whenever \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \).

So,

\[
\lim_{(x,y) \to (a,b)} x = a
\]

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Theorem -

Let \( L \) and \( M \) be real numbers and suppose that

\[
\lim_{(x,y) \to (a,b)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \to (a,b)} g(x, y) = M.
\]

Assume \( c \) is a constant, and \( m \) and \( n \) are integers.

1. Sum

\[
\lim_{(x,y) \to (a,b)} (f(x, y) + g(x, y)) = L + M
\]

2. Difference

\[
\lim_{(x,y) \to (a,b)} (f(x, y) - g(x, y)) = L - M
\]

3. Constant Multiple

\[
\lim_{(x,y) \to (a,b)} cf(x, y) = cl
\]

4. Product

\[
\lim_{(x,y) \to (a,b)} f(x, y)g(x, y) = LM
\]

5. Quotient

\[
\lim_{(x,y) \to (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad \text{provided} \ M \neq 0
\]
(6) Power \[ \lim_{(x,y) \to (a,b)} (f(x,y))^n = L^n \]

7) m/n power. If m and n have no common factors and \( n \neq 0 \), then
\[ \lim_{(x,y) \to (a,b)} (f(x,y))^{m/n} = L^{m/n} \] (we assume \( L > 0 \) if n is even)

Example: Evaluate \( \lim_{(x,y) \to (-1,4)} \frac{3x^2 - \sqrt{y}}{x^2y} \)

\[ = \frac{\lim_{(x,y) \to (-1,4)} (3x^2 - y^{1/2})}{\lim_{(x,y) \to (-1,4)} x^2y} = \frac{\lim_{(x,y) \to (-1,4)} 3x^2 - \lim_{(x,y) \to (-1,4)} y^{1/2}}{\lim_{(x,y) \to (-1,4)} x^2 \lim_{(x,y) \to (-1,4)} y} \]

\[ = \frac{3\left(\lim_{(x,y) \to (-1,4)} x^2\right) - \left(\lim_{(x,y) \to (-1,4)} y^{1/2}\right)}{\left(\lim_{(x,y) \to (-1,4)} x\right)^2 \left(\lim_{(x,y) \to (-1,4)} y\right)} = \frac{3\left(\lim_{(x,y) \to (-1,4)} x^2\right) - \left(\lim_{(x,y) \to (-1,4)} y^{1/2}\right)}{\left(\lim_{(x,y) \to (-1,4)} x\right)^2 \left(\lim_{(x,y) \to (-1,4)} y\right)} \]

\[ = \frac{3(-1)^2 - (4)^{1/2}}{(-1)^2(4)} = \frac{3 - 2}{4} = \frac{1}{4} \]

Note: The domain of \( f(x,y) = \frac{3x^2 - \sqrt{y}}{x^2y} \) is \( \{(x,y) \mid x \neq 0 \text{ and } y > 0\} \)

What does it mean to say "\((x,y)\) approaches \((-1,4)\)?"

As \(\delta\) decreases to 0
Definition - Interior and Boundary Points

Let $R$ be a region in $\mathbb{R}^2$. An interior point $P$ of $R$ is a point that lies entirely within $R$, which means it is possible to find a disk centered at $P$ that contains only points of $R$.

A boundary point $Q$ of $R$ lies on the "edge" of $R$ in the sense that every disk centered at $Q$ contains at least one point in $R$ and at least one point not in $R$.

Example: $R = \{(x,y) \mid 1 \leq x \leq 3 \text{ and } 1 \leq y \leq 2\}$

- C is an interior point
- A, B, and D are boundary points
- E is neither interior nor boundary point (an exterior point)

Definition - Open and Closed Sets

A region $R$ is open if every point in $R$ is an interior point.
A region $R$ is closed if it contains all of its boundary points.
Example:

\[ \{ (x, y) \mid |x| \leq 1, |y| \leq 1 \} \]

Closed set

\[ \{ (x, y) \mid |x| < 1, |y| < 1 \} \]

Open set

\[ \{ (x, y) \mid -1 \leq x < 1, -1 \leq y \leq 1 \} \]

Neither open nor closed

An open disk \[ \{ (x, y) \mid (x-a)^2 + (y-b)^2 < r^2 \} \]

A closed disk \[ \{ (x, y) \mid (x-a)^2 + (y-b)^2 \leq r^2 \} \]

Recall definition of limit:

\[ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \]

Open disk, centered at \((a, b)\),

with the point \((a, b)\) removed, and with

radius \(\delta\).

"\((x, y)\) approaches \((a, b)\)"

Consider only \((x, y)\) in domain of function—an important

detail when \((a, b)\) is

a boundary point of

the domain.
Ex. \( \lim_{(x,y) \to (3,6)} \frac{4x^2 - y^2}{2x-y} \)

**Fix, y**

Domain: \( \{(x,y) \mid 2x-y \neq 0\} \)

But, \((3,6)\) is a boundary point of the domain.

**Note:** Plugging in yields

\[
\frac{4(3)^2 - (6)^2}{2(3) - (6)} = \frac{0}{0} \quad \text{indeterminant?}
\]

Simplify

\[
\frac{4x^2 - y^2}{2x-y} = \frac{(2x-y)(2x+y)}{2x-y} = 2x+y
\]

Then, \( \lim_{(x,y) \to (3,6)} \frac{4x^2 - y^2}{2x-y} = \lim_{(x,y) \to (3,6)} (2x+y) = 2(3) + (6) = 12 \)

\((x,y)\) approaches \((3,6)\) along "all possible paths" that lie in the domain.

Ex. \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \)

Domain: All \(\mathbb{R}^2\) except \((0,0)\)

Plugging in yields \( \frac{0}{0} \)?

Simplify. How?"

We can try looking at specific paths for \((x,y)\) to approach \((0,0)\).
Along the line $y = 0$
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2+0} = 0
\]

Along the line $x = 0$
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0
\]

Hmm?

Does this mean the limit is $0$? No!

We need to know what happens along EVERY PATH.

Along $y = x$?
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to0} \frac{x^2}{2x^2} = \frac{1}{2}
\]

Procedure - Two-Path Test for Nonexistence of Limits
If $f(x,y)$ approaches two different values as $(x,y)$ approaches $(a,b)$ along two different paths in the domain of $f$, then
\[
\lim_{(x,y)\to(a,b)} f(x,y) \text{ does not exist.}
\]

Thus,
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ DNE?}
\]

Ex: \[
\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}
\]

0/0?? Simplify?? Try some paths...
Along \( y = kx \)

\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,kx) \to (0,0)} \frac{x^2k^2}{x^2+(kx)^2} = \lim_{x \to 0} \frac{kx^3}{x(x+k^2)} = \lim_{x \to 0} \frac{kx}{x^2+k^2} = 0
\]

Along \( x = 0 \)

\[
\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} = \lim_{(0,y) \to (0,0)} \frac{0}{0+y^2} = 0
\]

So... is the limit 0?

We still do not know!

**EVERY PATH**

Note: Along \( y = kx^2 \)

\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,kx^2) \to (0,0)} \frac{x^2(kx^2)}{x^2+(kx^2)^2} = \lim_{x \to 0} \frac{kx^4}{x^2(1+k^2)} = \frac{k}{1+k^2}
\]

Many different values (depends on \( k \))

Along \( y = x^2 \)

\[
\frac{1}{1+1} = \frac{1}{2}
\]

Along \( y = 2x^2 \)

\[
\frac{2}{1+2^2} = \frac{2}{5}
\]

e.t.s...

Thus,

\[
\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^2+y^2} \text{ DNE.}
\]

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**Definition - Continuity**

The function \( f \) is continuous at the point \((a,b)\) provided

1) \( f \) is defined at \((a,b)\)

2) \( \lim_{(x,y) \to (a,b)} f(x,y) \) exists

3) \( \lim_{(x,y) \to (a,b)} f(x,y) = f(a,b) \)
Note: Polynomials, rational functions, trigonometric, logarithmic, and exponential functions are continuous on their domains.

**Theorem**

If \( u = g(x, y) \) is continuous at \((a, b)\) and \( z = f(u) \) is continuous at \( g(a, b) \), then the composite function \( z = f(g(x, y)) \) is continuous at \((a, b)\).

**Ex/** \[ f(x, y) = \sqrt{9 - x^2 - y^2} \]

Let \( g(x, y) = 9 - x^2 - y^2 \)  Polynomial. Continuous on all of \( \mathbb{R}^2 \)

Let \( h(u) = \sqrt{u} \). Continuous for \( u \geq 0 \)

Then \( f(x, y) = h(g(x, y)) \)

So, \( f(x, y) \) is continuous for all \((x, y)\) such that \( 9 - x^2 - y^2 \geq 0 \).

\( f \) is continuous on \( \{ (x, y) \mid x^2 + y^2 \leq 9 \} \)

**Ex/** \[ f(x, y, z) = \sin \left( \frac{x^2 y}{z^2 + 1} \right) \]

Domain \( \mathbb{E}(x, y, z) \mid z \neq -1 \)

Continuous on its domain.

\[ \lim_{(x, y, z) \to (-1, 0, 1)} f(x, y, z) = \sin \left( \frac{(-1)^2(0)}{(-1)^2 + 1} \right) = \sin \left( \frac{0}{2} \right) = 1 \]