Tangent Planes and Linear Approximation

Suppose \( w = F(x, y, z) \) is differentiable.

Note, \( F(x, y, z) = k \) is a level surface (\( k \) is a constant)

Suppose \( C \) is a smooth curve given by \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) that lies entirely in the level surface \( w = k \).

Thus,

\[
F(x(t), y(t), z(t)) = k \quad \text{for all } t.
\]

Suppose \( \vec{v}(t_0) = \langle a, b, c \rangle \)

Find \( \frac{dF}{dt} \bigg|_{t = t_0} \)

Differentiate both sides of \( F(x, y, z) = k \) with respect to \( t \).

\[
\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0
\]

\[
\frac{dF(x(t), y(t), z(t))}{dt} = 0
\]

\[
\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle x'(t), y'(t), z'(t) \right\rangle = 0
\]

\[
\nabla F(x, y, z) \cdot \vec{v}'(t) = 0
\]

Thus, at \( t = t_0 \), \( x = a, y = b, z = c \), we have

\[
\nabla F(a, b, c) \cdot \vec{v}'(t_0) = 0
\]

\( \nabla F(a, b, c) \) is orthogonal to \( \vec{r}'(t_0) \) (provided \( \nabla F(a, b, c) \neq \vec{0} \)).

This argument will work for ANY smooth curve \( C \) that lies in the level surface and passes through \( (a, b, c) \).
\( \nabla F(a, b, c) \)

any tangent vector \((\mathbf{r}'(t))\) to a curve \(C\) that lies in the level surface passing through \((a, b, c)\) is orthogonal to \(\nabla F(a, b, c)\).

So, all of these tangent vectors lie in the same plane.

**Definition:**
Suppose \(F\) is differentiable at the point \(P_0(a, b, c)\) with \(\nabla F(a, b, c) \neq \mathbf{0}\). The plane tangent to the surface \(F(x, y, z) = k\) at \(P_0\), called the tangent plane, is the plane passing through \(P_0\) with normal vector \(\nabla F(a, b, c)\).

\[
\nabla F(a, b, c) \cdot <x-a, y-b, z-c> = 0 \\
\text{or} \\
F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0
\]

Note: \((a, b, c)\) lies on the level surface \(F(x, y, z) = k\)
so \(F(a, b, c) = k\).

**EX:** \(F(x, y, z) = x^2 + y^2 + z^2\)

Consider the level surface containing the point \((-2, 1, 2)\).

\(F(-2, 1, 2) = (-2)^2 + (1)^2 + (2)^2 = 9\)

Level surface: \(x^2 + y^2 + z^2 = 9\)

Find an equation of the plane tangent to the level surface at the point \((-2, 1, 2)\).

\(F_x = 2x, \ F_y = 2y, \ F_z = 2z\)

\(\nabla F(x, y, z) = <2x, 2y, 2z>\)
\[ DF(-2, 1, 2) \cdot \langle x-(-2), y-(1), z-(2) \rangle = 0 \]
\[ \langle 2(-2), 2(1), 2(2) \rangle \cdot \langle x+2, y-1, z-2 \rangle = 0 \]
\[ -4(x+2) + 2(y-1) + 4(z-2) = 0 \]

or
\[ -4x + 2y + 4z = 18 \]

Are there any horizontal tangent planes to the sphere \( x^2 + y^2 + z^2 = 9 \)?

\[ \vec{n} = \langle 0, 0, c \rangle \]

Since \( DF \) is normal to tangent planes, we want to find where \( DF = \vec{n} \).

Solve \( \langle 2x, 2y, 2z \rangle = \langle 0, 0, c \rangle \)

\[
\begin{align*}
2x &= 0 & \Rightarrow & x &= 0 \\
2y &= 0 & \Rightarrow & y &= 0 \\
2z &= c & \Rightarrow & z &= \frac{c}{2} \\
& \text{or} & & \sqrt{\frac{c}{2}} \\
& x^2 + y^2 + z^2 = 9 & \Rightarrow & (0)^2 + (0)^2 + \left(\frac{c}{2}\right)^2 = 9 \\
& z &= \pm 3
\end{align*}
\]

At \((0, 0, -3)\) and at \((0, 0, 3)\).

Are the any vertical tangent planes?

\[ \vec{n} = \langle a, b, 0 \rangle \]

Solve \[ \begin{align*}
2x &= a \\
2y &= b \\
2z &= 0 \Rightarrow z &= 0 \\
x^2 + y^2 + z^2 = 9 & \Rightarrow x^2 + y^2 + (0)^2 = 9 \\
& x^2 + y^2 = 9
\end{align*}\]

At any point \((x, y, 0)\) such that \(x^2 + y^2 = 9\).
Ex/ Consider \( z = f(x, y) = 6 + x - y^2 \)

Look at \( f(3, 2) = 6 + (3) - (2)^2 = 5 \). Tangent plane at \((3, 2, 5)\)?

The graph of \( z = f(x, y) \) is a surface.

We can view it as a level surface as follows:

\[
\begin{align*}
z &= f(x, y) = 0 \\
F(x, y, z) &= 0 \\
F_x &= -f_x, \quad F_y = -f_y, \quad F_z = 1
\end{align*}
\]

Then \( \nabla F(x, y, z) = \langle -f_x(x, y), -f_y(x, y), 1 \rangle \)

For our problem, we have

\[
\begin{align*}
z &= (6 + x - y^2) = 0 \quad \text{(level surface)} \\
F(x, y, z) &= 0, \quad F_x = -1, \quad F_y = 2y, \quad F_z = 1
\end{align*}
\]

\( \nabla F(x, y, z) = \langle -1, 2y, 1 \rangle \)

At \((3, 2, 5)\), \( \nabla F(3, 2, 5) = \langle -1, 2(2), 1 \rangle = \langle -1, 4, 1 \rangle \)

So, tangent plane ... 

\[
\nabla F(3, 2, 5) \cdot \langle x-3, y-2, z-5 \rangle = 0
\]

\( \langle -1, 4, 1 \rangle \cdot \langle x-3, y-2, z-5 \rangle = 0 \)

\[-(x-3) + 4(y-2) + (z-5) = 0 \]

or, \( z = (x-3) - 4(y-2) + 5 \)
In general...

Tangent Plane for $z = f(x,y)$

Suppose $f$ is differentiable at the point $(a, b)$.
An equation of the plane tangent to the surface $z = f(x,y)$
at the point $(a, b, f(a,b))$ is

$$\langle -f_x(a,b), -f_y(a,b), 1 \rangle \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0$$

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z-f(a,b)) = 0$$

or

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Ex/ $f(x,y) = \sqrt{x^2 + 3y^2 + 3}$

Find an equation for the plane tangent to the surface at
$(1, 2, 4)$.

Is $f$ differentiable at $(1, 2)$?

$$f_x(x,y) = \frac{3}{2}(x^2 + 3y^2 + 3)^{-\frac{1}{2}} = \frac{3}{2}(x^2 + 3y^2 + 3)^{-\frac{1}{2}} = \frac{3}{2(x^2 + 3y^2 + 3)}$$

$$f_y(x,y) = \frac{3y}{\sqrt{x^2 + 3y^2 + 3}}$$

Then, $f_x(1, 2) = \frac{3}{\sqrt{16}} = \frac{3}{4}$

$$f_y(1, 2) = \frac{3}{4}$$

So,

$$\langle -\frac{1}{4}, -\frac{3}{4}, 1 \rangle \cdot \langle x-1, y-2, z-4 \rangle = 0$$

$$-\frac{1}{4}(x-1) - \frac{3}{4}(y-2) + (z-4) = 0$$

or

$$z = \frac{1}{4}(x-1) + \frac{3}{4}(y-2) + 4$$

Tangent Plane at $(1, 2, 4)$

Note: We can use the $z$ values of the tangent plane
to approximate values of $f(x,y)$ for $x$ close to 1
and $y$ close to 2.
Recall:
\[ y = f(x) \]
\[ f(x) - f(a) = f'(a)(x-a) \]
\[ f(x) = f(a) + f'(a)(x-a) \]

Similarly, we can use the tangent plane to approximate \( f(x,y) \).
\[ f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]

**Definition - Linear Approximation**

Suppose \( f \) is differentiable at \((a,b)\). The linear approximation to the surface \( z = f(x,y) \) at the point \((a,b,f(a,b))\) is the tangent plane at that point given by
\[ L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]

Note: \( L(x,y) \approx f(x,y) \) for \((x,y)\) close to \((a,b)\).

**Example**
\[ f(x,y) = \sqrt{x^2 + 3y^2 + 3} \]

Use a linear approximation to estimate \( f(0.96,2.08) \).

We would like to use \((a,b)\) close to \((0.96, 2.08)\) such that \( f(a,b) \) is "easy" to compute.

We saw before that \( f(1,2) = \sqrt{(1)^2 + 3(2)^2 + 3} = \sqrt{16} = 4 \)
So, we can use the tangent plane at (1, 2, 4).

We found this earlier...

\[ L(x, y) = \frac{1}{4}(x-1) + \frac{3}{4}(y-2) + 4 \]

Then

\[ f(0.96, 2.08) \approx L(0.96, 2.08) \]

\[ = \frac{1}{4}(0.96 - 1) + \frac{3}{4}(2.08 - 2) + 4 \]

\[ = -0.04 + \frac{3(0.08)}{4} + 4 = -0.01 + 0.06 + 4 \]

\[ = 4.05 \]

In other words

\[ \sqrt{(0.96)^2 + 3(2.08)^2 + 3} \approx 4.05 \]

Ex/ Find the linear approximation to the surface \( z = x^3 - 2xy^2 \) at the point \((2, -1, 4)\)

\[ \frac{\partial z}{\partial x} = 3x^2 - 2y^2 \quad \frac{\partial z}{\partial y} = -4xy \]

\[ \left. \frac{\partial z}{\partial x} \right|_{(2, -1)} = 3(2)^2 - 2(-1)^2 = 10 \quad \left. \frac{\partial z}{\partial y} \right|_{(2, -1)} = -4(2)(-1) = 8 \]

Thus,

\[ L(x, y) = 4 + 10(x - 2) + 8(y + 1) \]

[Note: \( L(2, -1) = 4 \) why?]
\[ \Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) \]

point on tangent plane \((a+\Delta x, b+\Delta y, f(a+\Delta x, b+\Delta y))\)

\[ \Delta z = f_x(a,b) \Delta x + f_y(a,b) \Delta y \]

let \(\Delta x = dx, \Delta y = dy\) and we get...

\[ \Delta z \approx dz = f_x(a,b) dx + f_y(a,b) dy \]

**Definition**: The differential \(dz\)

\[ dz = f_x(x,y) dx + f_y(x,y) dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \]

Note: We require \(f\) to be differentiable to define \(dz\).

Note: The change in \(z = f(x,y)\) from \((a,b)\) to \((a+\Delta x, b+\Delta y)\) is denoted \(\Delta z\) and it is approximated by \(dz\).

Note: The linear approx and the differential can be extended to functions of more than 2 variables.

**Ex:** Surface area of a can.

\[ S(r,h) = 2\pi r^2 + 2\pi rh \]

Find \(dS\)

\[ \frac{dS}{dr} = 4\pi r + 2\pi h, \quad \frac{dS}{dh} = 0 + 2\pi r \]

Thus,

\[ dS = (4\pi r + 2\pi h) dr + (2\pi r) dh \]
Ex/ Area of Rectangle \[ A(x, y) = xy \]
\[ \frac{dA}{dx} = y, \quad \frac{dA}{dy} = x \]
\[ dA = y \, dx + x \, dy \]

Ex/ Volume of a Cone \[ V = \frac{1}{3} \pi r^2 h \]
\[ \frac{dV}{dr} = \frac{2}{3} \pi rh, \quad \frac{dV}{dh} = \frac{1}{3} \pi r^2 \]
\[ dV = \frac{2}{3} \pi rh \, dr + \frac{1}{3} \pi r^2 \, dh \]

Say that a cone changes from \( r_0 = 3 \) in, \( h_0 = 5 \) in.

\[ \text{to } r_1 = 4 \text{ in, } h_1 = 4 \text{ in.} \]

Approximate the change in volume.

\[ dr = \Delta r = r_1 - r_0 = 4 - 3 = 1 \]
\[ dh = \Delta h = h_1 - h_0 = 4 - 5 = -1 \]

Then, \[ dV = \frac{2}{3} \pi (3)(5)(1) + \frac{1}{3} \pi (3)^2 (-1) \]
\[ = 10 \pi - 3 \pi = 7 \pi \text{ in}^3 \]

[what is the actual change \( dV \)?]

Say that you increase the radius by 2% and decrease the height by 6%. Approximate the change in volume.

original \( r_0 \) \( h_0 \)

new \( 1.02 r_0 \) \( 0.94 h_0 \)

\( \text{So, } dr = 0.02 r_0 \)
\( dh = -0.06 h_0 \)

\[ dV = \frac{2}{3} \pi r_0^2 h_0 (0.02 r_0) + \frac{1}{3} \pi r_0^2 (-0.06 h_0) = -0.02 \pi r_0^2 h_0 \]

Note, originally \( V = \frac{1}{3} \pi r_0^2 h_0 \)

\( \text{So, } dV = -0.02 V \)

The volume decreased by approx. 2%. 