Maximum/Minimum Problems

**Definition**
A function $f$ has a local maximum value at $(a,b)$ if $f(x,y) \leq f(a,b)$
for all $(x,y)$ in the domain of $f$ in some open disk centered at $(a,b)$.

A function $f$ has a local minimum value at $(a,b)$ if $f(x,y) \geq f(a,b)$
for all $(x,y)$ in the domain of $f$ in some open disk centered at $(a,b)$.

Local maximum and local minimum values are also called local extreme values or local extrema.

**Note:** If $f$ has a local extrema at $(a,b)$, then
the value is $f(a,b)$ ["how much", "how high/low"],
and it occurs at $(a,b)$ ["where", "when"]

**Theorem**
If $f$ has a local extreme value at $(a,b)$ and if the partial
derivatives $f_x$ and $f_y$ exist at $(a,b)$, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$
[or simply $Df(a,b) = 0$]

**Note:** If $f$ has a local extreme at $(a,b)$ and $f$ is differentiable
at $(a,b)$, then (since $Df(a,b) = 0$) the tangent plane at $(a,b)$
is $z = f(a,b)$ (constant) which is a horizontal plane.

**Definition - Critical Point**
An interior point $(a,b)$ in the domain of $f$ is a critical point of $f$
if either

1. $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [or $Df(a,b) = 0$]

    OR

2. one (or both) of $f_x$ and $f_y$ does not exist at $(a,b)$

**Note:** Local extreme at $(a,b) \Rightarrow (a,b) \text{ is a critical point}$

$(a,b) \text{ is a critical point} \Rightarrow$ Local extreme at $(a,b)$

$(a,b) \text{ is NOT a critical point} \Rightarrow \text{No local extreme at } (a,b)$
Finding critical points of $F(x,y)$.

Find all points $(x,y)$ (in the interior of the domain of $F$) that satisfy $\nabla F(x,y) = \vec{0}$, or where $f_x$ or $f_y$ DNE.

\[ \begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \text{ involves solving a system of equations} \]

Ex: Find the critical points of $F(x,y) = (x^2-2x-8)(y+3)$

\[ f_x(x,y) = (2x-2)(y+3), \quad f_y(x,y) = x^2-2x-8 \]

Solve $DF = \vec{0}$

1. $(2x-2)(y+3) = 0 \implies x = 1 \text{ or } y = -3$

2. \[ x^2-2x-8 = 0 \]

Cases

\[ x = 1 \text{ Sub into 2} \]

\[ (1)^2-2(1)-8 = 0 \]

\[ -9 = 0 \text{ No solution} \]

\[ y = -3 \text{ Sub into 2} \]

\[ x^2-2x-8 = 0 \]

\[ (x-4)(x+2) = 0 \]

\[ x = 4 \text{ or } x = -2 \]

Two solutions

\[ (-2,-3) \text{ and } (4,-3) \]

Solve $DF$ DNE

$f_x$ and $f_y$ are never undefined. No critical pts

Critical Points $(-2,-3)$ and $(4,-3)$
Ex/ Find the critical points of \( g(x,y) = \sqrt{3x^2 + y^2} \)

\[
\begin{align*}
    g_x(x,y) &= \frac{3x}{\sqrt{3x^2 + y^2}} \quad & g_y(x,y) &= \frac{y}{\sqrt{3x^2 + y^2}}
\end{align*}
\]

Solve \( \nabla g = \mathbf{0} \):

\[
\begin{align*}
    \frac{3x}{\sqrt{3x^2 + y^2}} &= 0 & \rightarrow & x = 0 \\
    \frac{y}{\sqrt{3x^2 + y^2}} &= 0 & \rightarrow & y = 0
\end{align*}
\]

But \((0,0)\) does not solve the system. Why? No solutions.

Solve \( \nabla g \text{ DNE} \):

\( g_x \) (and \( g_y \)) are undefined at \((0,0)\).

Note: \((0,0)\) is in the domain of \( g \)

Critical points \((0,0)\)

Ex/ Find the critical points of \( f(x,y) = \frac{1}{x^2 + y^2} \)

\[
\begin{align*}
    f_x(x,y) &= -\frac{2x}{(x^2+y^2)^2} \quad & f_y(x,y) &= -\frac{2y}{(x^2+y^2)^2}
\end{align*}
\]

Solve \( \nabla f = \mathbf{0} \):

No solution

Solve \( \nabla f \text{ DNE} \):

\((0,0)\)

But, \((0,0)\) is NOT in the domain of \( f \).

No critical points
Ex/
\[ f(x, y) = x^2 - y^2 \]

\[ f_x(x, y) = 2x, \quad f_y(x, y) = 2y \]
\[ \sum 2x = 0 \quad \text{try it.} \]

Critical Points: \((0, 0)\)

Note: There is NO local extreme at \((0, 0)\)

for \( f(x, y) = x^2 - y^2 \)

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Definition-

A function \( f \) has a saddle point at a critical point \((a, b)\)
if, in every open disk centered at \((a, b)\), there are points \((x, y)\)
for which \( f(x, y) > f(a, b) \) and points for which \( f(x, y) < f(a, b) \).

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Theorem - Second Derivative Test

Suppose that the second partial derivatives of \( f \) are continuous
throughout an open disk centered at the point \((a, b)\), where \( f_x(a, b) = 0 \)
and \( f_y(a, b) = 0 \). Let \( D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 \).

1) If \( D(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \), then \( f \) has a local maximum
value at \((a, b)\).
2) If \( D(a, b) > 0 \) and \( f_{xx}(a, b) > 0 \), then \( f \) has a local minimum
value at \((a, b)\).
3) If \( D(a, b) < 0 \), then \( f \) has a saddle point at \((a, b)\).
4) If \( D(a, b) = 0 \), then the test is inconclusive.

---

Informal Proof

Consider the "concavity" of the surface at \((a, b)\) in the
direction of a unit vector \( \mathbf{u} = (\cos \theta, \sin \theta) \).
\[
\begin{align*}
D_\theta u(x,y) &= f_x(x,y) \cos \theta + f_y(x,y) \sin \theta \\
\text{"slope" in direction of } \vec{u} \\
D_\theta [D_\theta f(x,y)] &= (f_{xx}(x,y) \cos \theta + f_{yx}(x,y) \sin \theta) \cos \theta \\
&\quad + (f_{yy}(x,y) \cos \theta + f_{yx}(x,y) \sin \theta) \sin \theta \\
\text{(Assuming } f_{xy}, f_{yx} \text{ are continuous ...)} \\
D_\theta^2 f(x,y) &= f_{xx}(x,y) \cos^2 \theta + f_{xy}(x,y) \sin \theta \cos \theta + f_{yy}(x,y) \sin^2 \theta + 2f_{xy}(x,y) \cos \theta \sin \theta \\
\text{"concavity" in direction of } \vec{u} \\
\text{If } \sin \theta = 0 \text{ then } D_\theta^2 f &= f_{xx}(x,y) \cos^2 \theta \\
\text{so } f_{xx} > 0 \text{ or } f_{xx} < 0 \text{ determines concavity.} \\
\text{If } \sin \theta \neq 0 \text{ then ...} \\
D_\theta^2 f(x,y) &= f_{xx}(x,y) \cot^2 \theta + f_{yy}(x,y) + 2f_{xy}(x,y) \cot \theta \\
&= A \cot^2 \theta + 2B \cot \theta + C \\
&= Aw^2 + 2Bw + C \\
Aw^2 + 2Bw + C &> 0 \text{ for all } w \text{ when } 4B^2 - 4AC < 0 \text{ and } A > 0 \\
Aw^2 + 2Bw + C &< 0 \text{ for all } w \text{ when } 4B^2 - 4AC < 0 \text{ and } A < 0 \\
Aw^2 + 2Bw + C &> 0 \text{ for some } w \text{ and } Aw^2 + 2Bw + C < 0 \text{ for some } w \text{ when } 4B^2 - 4AC > 0 \\
Aw^2 + 2Bw + C &\geq 0 \text{ when } 4B^2 - 4AC = 0
\end{align*}
\]

(Note: $4B^2 - 4AC > 0$ if and only if $B^2 - AC > 0$)
Ex/ \[ P(x,y) = (x^2 - 2x - 8)(y + 3) \]

(we found the critical points earlier)

\[ f_x = (2x - 2)(y + 3), \quad f_y = x^2 - 2x - 8 \]

Crit. Pts: \((-2, -3), (4, -3)\)

\[ f_{xx} = 2y + 6, \quad f_{xy} = 2x - 2, \quad f_{yy} = 0 \]

\[ D(x,y) = (2y + 6)(0) - [2x - 2]^2 = -(2x - 2)^2 \]

At \((-2, -3)\)

\[ D(-2, -3) = -(2(-2) - 2)^2 < 0 \]

So, saddle point at \((-2, -3)\).

At \((4, -3)\)

\[ D(4, -3) = -(2(4) - 2)^2 < 0 \]

Saddle point at \((4, -3)\).

Ex/ \[ g(x,y) = (x-3)^2 + (y+1)^2 \]

\[ g_x = 2(x-3), \quad g_y = 2(y+1) \]

Crit. Pts: \((3, -1)\)

\[ g_{xx} = 2, \quad g_{xy} = 0, \quad g_{yy} = 2 \]

\[ D(x,y) = (2)(2) - [0]^2 = 4 \]

At \((3, -1)\)

\[ D(3, -1) = 4 > 0 \]

And \[ g_{xx}(3, -1) = 2 > 0 \]

So, \(g\) has a local minimum at \((3, -1)\).
Ex/ \( f(x, y) = x^4 + y^4 \)

\[ f_x = 4x^3, \quad f_y = 4y^3 \]

Crit Pts: (0, 0)

\[ f_{xx} = 12x^2, \quad f_{xy} = 0, \quad f_{yy} = 12y^2 \]

\[ D(x, y) = (12x^2)(12y^2) - [0]^2 = 144x^2y^2 \]

At (0, 0)

\[ D(0, 0) = 144(0)^2(0)^2 = 0 \]

Inconclusive

Ex/ \( f(x, y) = x^4y \)

\[ f_x = 4x^3y, \quad f_y = x^4 \]

Crit Pts: (0, b) for any b

\[ f_{xx} = 12x^2y, \quad f_{xy} = 4x^3, \quad f_{yy} = 0 \]

\[ D(x, y) = (12x^2y)(0) - [4x^3]^2 = -16x^6 \]

At (0, b)

\[ D(0, b) = -16(0)^6 = 0 \] Inconclusive

Definition -

If \( f(x, y) \leq f(a, b) \) for all \((x, y)\) in the domain of \( f \), then \( f \) has an absolute maximum value at \((a, b)\). If \( f(x, y) \geq f(a, b) \) for all \((x, y)\) in the domain of \( f \), then \( f \) has an absolute minimum value at \((a, b)\).

Theorem - Extreme Value Theorem for Functions of Two Variables

If \( f \) is continuous on a closed, bounded set \( R \) in \( \mathbb{R}^2 \), then \( f \) attains an absolute maximum value \( f(x_1, y_1) \) and an absolute minimum value \( f(x_2, y_2) \) at some points \((x_1, y_1)\) and \((x_2, y_2)\) in \( R \).
Procedure - Finding Absolute Extrema on a Closed, Bounded Set

Let \( f \) be continuous on a closed, bounded set \( R \) in \( \mathbb{R}^2 \).

To find the absolute maximum and minimum values of \( f \) on \( R \):

1) Determine the values of \( f \) at all critical points in \( R \).
2) Find the extreme values of \( f \) on the boundary of \( R \).
3) The greatest value found in steps 1 and 2 is the absolute maximum value of \( f \) on \( R \). The least value found in steps 1 and 2 is the absolute minimum value of \( f \) on \( R \).

Ex/ Find the absolute maximum and minimum values of

\[
 f(x,y) = (x-2)^2 + (y-1)^2 + 3 \quad \text{on the set } R = \{ (x,y) : 0 \leq x \leq 4, -1 \leq y \leq 3 \} 
\]

1) Critical Points of \( f \) in \( R \):

\[
 \nabla f = \left< 2(x-2), 2(y-1) \right> \]

Solve \( \nabla f = \vec{0} \) and \( \nabla f \) DNE ...

Critical Point: \( (2,1) \) in \( R \) \( \checkmark \) yes

\[
 f(2,1) = (2-2)^2 + (1-1)^2 + 3 = 3
\]

2) Boundary of \( R \) (a curve in the \( xy \)-plane; try to write \( f(x,y) \) as a function of one variable on a closed interval)

\[
 C_1 : x=4, -1 \leq y \leq 3 \\
 C_2 : y=3, 0 \leq x \leq 4 \\
 C_3 : x=0, -1 \leq y \leq 3 \\
 C_4 : y=-1, 0 \leq x \leq 4 
\]
On $C_1$, \[ x = 4 \text{ and } -1 \leq y \leq 3 \]

\[ f(x, y) = f(4, y) = (4 - 2)^2 + (y - 1)^2 + 3 = (y - 1)^2 + 7 \]

and $-1 \leq y \leq 3$.

Let $g_1(y) = (y - 1)^2 + 7$ on $[-1, 3]$.

$g'_1(y) = 2(y - 1) \rightarrow$ crit number $y = 1$

Test $g_1(y)$ at end points $y = -1, y = 3$

and crit number $y = 1$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$g_1(y)$</th>
<th>$(x, y)$</th>
<th>$f(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>11</td>
<td>$(4, -1)$</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>$(4, 3)$</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>$(4, 1)$</td>
<td>7</td>
</tr>
</tbody>
</table>

extreme values of $f$ on $C_1$.

On $C_2$, \[ y = 3 \text{ and } 0 \leq x \leq 4 \]

\[ f(x, y) = f(x, 3) = (x - 2)^2 + (3 - 1)^2 + 3 = (x - 2)^2 + 7 \]

and $0 \leq x \leq 4$.

Let $g_2(x) = (x - 2)^2 + 7$

$g'_2(x) = 2(x - 2)$, ...

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g_2(x)$</th>
<th>$(x, y)$</th>
<th>$f(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11</td>
<td>$(0, 3)$</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>$(4, 3)$</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>$(2, 3)$</td>
<td>7</td>
</tr>
</tbody>
</table>

On $C_2$, ...

<table>
<thead>
<tr>
<th>$y$</th>
<th>$g_3(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

On $C_4$, ...

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g_4(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>
Step 3: 

<table>
<thead>
<tr>
<th>Inside $R$</th>
<th>$(x, y)$</th>
<th>$p(x,y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 1)</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Boundary</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, -1)</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>(4, 1)</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>(0, 3)</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>(2, 3)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>(0, 1)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>(0, -1)</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>(2, -1)</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Absolute Max value is 11
Absolute Min value is 3

Ex/ Find the absolute max and min values of $f(x,y) = (x-2)^2 + (y-1)^2 + 3$ on the set $R = \{ (x,y) : x^2 + y^2 \leq 9 \}$

1) Crit pts of $f$ in $R$?

(as before...)

(2, 1) In $R$? $(2)^2 + (1)^2 \leq 9$ \checkmark

Yes.

$f(2,1) = 3$

2) Boundary of $R$

Parametrize $x = 3\cos t, y = 3\sin t, 0 \leq t \leq 2\pi$

\[ g(t) = f(3\cos t, 3\sin t) = (3\cos t - 2)^2 + (3\sin t - 1)^2 + 3 \]
\[ g'(t) = 2(3\cos t - 2)(-3\sin t) + 2(3\sin t - 1)(3\cos t) \]

\[ = 12\sin t - 6\cos t \]

Solve \( g'(t) = 0 \) and \( g'(t) \text{ DNE} \ldots \)

\[ 12\sin t - 6\cos t = 0 \]
\[ 12\sin t = 6\cos t \]
\[ \tan t = \frac{1}{2} \]
\[ t = \tan^{-1} \frac{1}{2} \quad (\text{also } \pi + \tan^{-1} \frac{1}{2}) \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-5)</td>
</tr>
<tr>
<td>( \tan^{-1} \frac{1}{2} )</td>
<td>(17 - 12 \cdot \frac{2}{5} - 6 \cdot \frac{1}{5} = 17 - \frac{30}{5} = \frac{17 - 6\sqrt{5}}{5})</td>
</tr>
<tr>
<td>( \pi + \tan^{-1} \frac{1}{2} )</td>
<td>(17 - 12 \left(\frac{2}{5}\right) - 6 \left(\frac{1}{5}\right) = 17 + \frac{30}{5} = \frac{17 + 6\sqrt{5}}{5})</td>
</tr>
<tr>
<td>2\pi</td>
<td>5 (&lt;\text{ not extreme value on boundary})</td>
</tr>
</tbody>
</table>

Step 3:

<table>
<thead>
<tr>
<th>Inside ( R )</th>
<th>(2, 1)</th>
<th>( f(2, 1) )</th>
<th>3</th>
<th>\text{Abs Min Value}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary</td>
<td>(3, 0)</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{6}{5}, \frac{3}{15})</td>
<td>17 - 6\sqrt{5}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{-6}{5}, \frac{-3}{15})</td>
<td>17 + 6\sqrt{5}</td>
<td>\text{Abs Max Value}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[Try it again using \( y = \sqrt{9 - x^2} \) and \( y = -\sqrt{9 - x^2} \) to describe the boundary.]
Ex/ Find extreme values of \( f(x,y) = x^2 + y^2 \) on the set \( R = \{(x,y) : x^2 + y^2 < 4\} \)

\[ f(0,0) = 0 \]

Absolute Max Value? None! why?

Absolute Min Value is 0 at (0,0)

[Try using the Second Derivative Test to show it is a Local minimum.]

Ex/ \( f(x,y) = x^2 + y^2 \) on \( R = \{(x,y) : x, y > 0\} \)

\[ \text{NOT closed, NOT bounded} \]

No critical points in R. (why?)

No Abs Max
No Abs Min.